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Constructing a Maximally Entangled Seven-Qubit State via Orthogonal Arrays

Xin-Wei Zha(查新未), Min-Rui Wang(王敏锐)*, and Ruo-Xu Jiang(姜若虚)

School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, China

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Huber *et al.* [Phys. Rev. Lett. 118 (2017) 200502] have proved that a seven-qubit state whose three-body marginal states are all maximally mixed does not exist. Here, we propose a method to build a maximally entangled state based on orthogonal arrays to construct maximally entangled seven-qubit states. Using this method, we not only determine that a seven-qubit state whose three-body marginals are all maximally mixed does not exist, but also find the condition for maximally entangled seven-qubit states. We consider that $\pi_{\rm ME} = 19/140$ is a condition for maximally entangled seven-qubit states. Furthermore, we derive three forms of maximally entangled seven-qubit states via orthogonal arrays.

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Entanglement is considered as a significant key resource for quantum information and computation.[1−4] In particular, the search for maximally entangled states has attracted great deal of attention.[5−14] With regard to an n-qubit pure state $|\psi\rangle$, if all k-qubit reduced states are maximally mixed, it is referred to as a k-uniform state.[15,16] It is particularly interesting that these n-qubit states are $[n/2]$-uniform. These states are also known as absolutely maximally entangled (AME) states. In fact, it has been shown that these states exist only for special values of n ($n = 2, 3, 5, 6$). For instance, there is no AME state for four qubits,[17] seven qubits,[18] or for eight or more qubits.[13,19] It is therefore relevant to ask: which states are maximally entangled?

Those states whose entanglement is maximal for every (balanced) bipartition are maximally multi-qubit entangled states (MMESs).[13] Therefore, we can determine whether a state is an MMES by determining whether the averaged purity over all $[n/2]$-qubit subsystems reaches its minimum.[12] The averaged subsystem purity for a state is specified as:

$$\pi_{\rm ME} = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A, \tag{1}$$

where $n_A = [n/2]$ ($[n/2]$ takes the integer of $n/2$); $\pi_A = {\rm Tr}_A\, \rho_A^2$, and $\rho_A = {\rm Tr}_{\overline{A}} |\psi\rangle\langle\psi|$, used for calculating π_A, is the result after a partial trace operation over the complementary subsystem of A, i.e., $\overline{A}$, in the summation. It is obvious that for qubit cases, $\frac{1}{N_A} \leqslant \pi_{\rm ME} \leqslant 1$, where $N_A = 2^{n_A}$. A maximally multi-qubit entangled state is a minimizer of $\pi_{\rm ME}$.[20]

Although Huber *et al.*[18] have proved that no AME state exists for the seven-qubit case, they have not given a criterion for the identification of a maximally entangled seven-qubit state. In Ref. [21], we find a maximally entangled seven-qubit state. However, for seven-qubit entangled states, a general theoretical criterion for a maximally entangled seven-qubit state has not yet been found. In 2014, Goyeneche and Zyczkowski established a link between the combinatorial notions of orthogonal arrays.[15] In 2018, we proposed a method of adding phase parameters,[22] and were therefore able to construct four-qubit quantum states via an orthogonal array. In this study, using phase parameters and the minimum of $\pi_{\rm ME}$, we propose a new method for constructing a maximal seven-qubit entangled state. We find that $\pi_{\rm ME} = 19/140$ is a condition for maximally entangled seven-qubit states.

Table 1. Orthogonal array of eight qubits.

A	B	C	D	E	F	G	H
0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1
0	0	1	1	1	1	0	0
0	1	0	1	0	1	0	1
0	1	0	1	1	0	1	0
0	1	1	0	0	1	1	0
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	0	1	1	0	0	1
1	0	1	0	0	1	0	1
1	0	1	0	1	0	1	0
1	1	0	0	0	0	1	1
1	1	0	0	1	1	0	0
1	1	1	1	0	0	0	0
1	1	1	1	1	1	1	1

A pure quantum state of N subsystems with d levels may be referred to as a k-maximally entangled state, expressed as k-uniform, if every reduction to k qubits is maximally mixed. In addition, the connection between orthogonal arrays and k-uniform quantum states, as discussed in Ref. [15]. Goyeneche and Zyczkowski[15] gave the orthogonal array as listed in Table 1.

*Corresponding author. Email: 503989460@qq.com
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This array leads to the following 3-uniform state of eight qubits:

$$\begin{aligned}|\varphi_8\rangle = \frac{1}{4}(&|00000000\rangle + |00001111\rangle + |00110011\rangle \\ &+ |00111100\rangle + |01010101\rangle + |01011010\rangle \\ &+ |01100110\rangle + |01101001\rangle + |10010110\rangle \\ &+ |10011001\rangle + |10100101\rangle + |10101010\rangle \\ &+ |11000011\rangle + |11001100\rangle + |11110000\rangle \\ &+ |11111111\rangle)_{12345678}. \end{aligned} \tag{2}$$

It is easy to show

$$\pi_{ijk} = \mathrm{Tr}_{ijk}\,\rho^2_{ijk} = \frac{1}{8}, \quad ijk = 123, \ldots, 678. \tag{3}$$

Furthermore, we can show that

$$\pi_{\mathrm{ijkl}} = \mathrm{Tr}_{\mathrm{ijkl}}\,\rho^2_{\mathrm{ijkl}} = \frac{1}{16}, \quad ijkl = 1235, 1236, \ldots \tag{4}$$

except for

$$\begin{aligned}&\pi_{1234} = \frac{1}{4},\ \pi_{1256} = \frac{1}{4},\ \pi_{1278} = \frac{1}{4},\ \pi_{1357} = \frac{1}{4}, \\ &\pi_{1368} = \frac{1}{4},\ \pi_{1458} = \frac{1}{4},\ \pi_{1467} = \frac{1}{4}. \end{aligned} \tag{5}$$

Now, we remove the last column from Table 1, thereby obtaining the orthogonal array in Table 2.

Table 2. Orthogonal array of seven qubits.

A	B	C	D	E	F	G
0	0	0	0	0	0	0
0	0	0	0	1	1	1
0	0	1	1	0	0	1
0	0	1	1	1	1	0
0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	1	0	0	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	0	0	1	1	0	0
1	0	1	0	0	1	0
1	0	1	0	1	0	1
1	1	0	0	0	0	1
1	1	0	0	1	1	0
1	1	1	1	0	0	0
1	1	1	1	1	1	1

This array leads to the 2-uniform state of seven qubits:

$$\begin{aligned}|\varphi_7\rangle = \frac{1}{4}(&|0000000\rangle + |0000111\rangle + |0011001\rangle \\ &+ |0011110\rangle + |0101010\rangle + |0101101\rangle \\ &+ |0110011\rangle + |0110100\rangle + |1001011\rangle \\ &+ |1001100\rangle + |1010010\rangle + |1010101\rangle \\ &+ |1100001\rangle + |1100110\rangle + |1111000\rangle \\ &+ |1111111\rangle)_{1234567}. \end{aligned} \tag{6}$$

It is easy to show that

$$\pi_{ijk} = \mathrm{Tr}_{ijk}\,\rho^2_{ijk} = \frac{1}{8}, \quad ijk = 123, 124, \ldots, 467 \tag{7}$$

except for

$$\begin{aligned}&\pi_{127} = \frac{1}{4},\ \pi_{136} = \frac{1}{4},\ \pi_{145} = \frac{1}{4},\ \pi_{235} = \frac{1}{4}, \\ &\pi_{246} = \frac{1}{4},\ \pi_{347} = \frac{1}{4},\ \pi_{567} = \frac{1}{4}. \end{aligned} \tag{8}$$

As a result of

$$\pi_{\mathrm{ME}} = \frac{1}{35}(\pi_{123}+\pi_{124}+\pi_{125}+\cdots+\pi_{456}+\pi_{457}+\pi_{467}+\pi_{567}), \tag{9}$$

from Eqs. (7) and (8), we can obtain

$$\pi_{\mathrm{ME}} = \frac{1}{35}\left(\frac{1}{8}\times 28 + \frac{1}{4}\times 7\right) = \frac{3}{20}. \tag{10}$$

Now, let us assume that

$$\begin{aligned}|\varphi_7\rangle = \frac{1}{4}(&|0000000\rangle + e^{i\phi_7}|0000111\rangle + e^{i\phi_{25}}|0011001\rangle \\ &+ e^{i\phi_{30}}|0011110\rangle + e^{i\phi_{42}}|0101010\rangle \\ &+ e^{i\phi_{45}}|0101101\rangle + e^{i\phi_{51}}|0110011\rangle \\ &+ e^{i\phi_{52}}|0110100\rangle + e^{i\phi_{75}}|1001011\rangle \\ &+ e^{i\phi_{76}}|1001100\rangle + e^{i\phi_{82}}|1010010\rangle \\ &+ e^{i\phi_{85}}|1010101\rangle + e^{i\phi_{97}}|1100001\rangle \\ &+ e^{i\phi_{102}}|1100110\rangle + e^{i\phi_{120}}|1111000\rangle \\ &+ e^{i\phi_{127}}|1111111\rangle)_{1234567}, \end{aligned} \tag{11}$$

we can then obtain

$$\pi_{ijk} = \mathrm{Tr}_{ijk}\,\rho^2_{ijk} = \frac{1}{8}, \quad ijk = 123, 124, \ldots, 467 \tag{12}$$

except for

$$\begin{aligned}\pi_{127} = \frac{1}{64}[&12 + \cos(\varphi_{127} - \varphi_{30} - \varphi_{97}) \\ &+ \cos(\varphi_7 + \varphi_{120} - \varphi_{25} - \varphi_{102}) \\ &+ \cos(\varphi_{42} + \varphi_{85} - \varphi_{52} - \varphi_{75}) \\ &+ \cos(\varphi_{45} + \varphi_{82} - \varphi_{51} - \varphi_{76})], \end{aligned}$$

$$\begin{aligned}\pi_{136} = \frac{1}{64}[&12 + \cos(\varphi_{127} - \varphi_{45} - \varphi_{82}) \\ &+ \cos(\varphi_7 + \varphi_{120} - \varphi_{42} - \varphi_{85}) \\ &+ \cos(\varphi_{25} + \varphi_{102} - \varphi_{52} - \varphi_{75}) \\ &+ \cos(\varphi_{30} + \varphi_{97} - \varphi_{51} - \varphi_{76})], \end{aligned}$$

$$\begin{aligned}\pi_{145} = \frac{1}{64}[&12 + \cos(\varphi_{127} - \varphi_{51} - \varphi_{76}) \\ &+ \cos(\varphi_7 + \varphi_{120} - \varphi_{52} - \varphi_{75}) \\ &+ \cos(\varphi_{25} + \varphi_{102} - \varphi_{42} - \varphi_{85}) \\ &+ \cos(\varphi_{30} + \varphi_{97} - \varphi_{45} - \varphi_{82})], \end{aligned}$$

$$\begin{aligned}\pi_{235} = \frac{1}{64}[&12 + \cos(\varphi_{127} - \varphi_{52} - \varphi_{75}) \\ &+ \cos(\varphi_7 + \varphi_{120} - \varphi_{51} - \varphi_{76}) \\ &+ \cos(\varphi_{25} + \varphi_{102} - \varphi_{45} - \varphi_{82}) \\ &+ \cos(\varphi_{30} + \varphi_{97} - \varphi_{42} - \varphi_{85})], \end{aligned}$$

$$\pi_{246} = \frac{1}{64}[12 + \cos(\varphi_{127} - \varphi_{42} - \varphi_{85}) + \cos(\varphi_7 + \varphi_{120} - \varphi_{45} - \varphi_{82}) + \cos(\varphi_{25} + \varphi_{102} - \varphi_{51} - \varphi_{76}) + \cos(\varphi_{30} + \varphi_{97} - \varphi_{52} - \varphi_{75})],$$

$$\pi_{347} = \frac{1}{64}[12 + \cos(\varphi_{127} - \varphi_{25} - \varphi_{102}) + \cos(\varphi_7 + \varphi_{120} - \varphi_{30} - \varphi_{97}) + \cos(\varphi_{42} + \varphi_{85} - \varphi_{51} - \varphi_{76}) + \cos(\varphi_{45} + \varphi_{82} - \varphi_{52} - \varphi_{75})],$$

$$\pi_{567} = \frac{1}{64}[12 + \cos(\varphi_{127} - \varphi_7 - \varphi_{120}) + \cos(\varphi_{25} + \varphi_{102} - \varphi_{30} - \varphi_{97}) + \cos(\varphi_{42} + \varphi_{85} - \varphi_{45} - \varphi_{82}) + \cos(\varphi_{51} + \varphi_{76} - \varphi_{52} - \varphi_{75})]. \quad (13)$$

Given that

$$\pi_{\rm ME} = \frac{1}{35}(\pi_{123} + \pi_{124} + \pi_{125} + \cdots + \pi_{456} + \pi_{457} + \pi_{467} + \pi_{567}), \quad (14)$$

based on Eqs. (12) and (13), we then obtain

$$\pi_{\rm ME} = \frac{1}{35}\Big[\frac{1}{8} \times 28 + \frac{1}{64}[84 + f(\phi_{7,}\phi_{25,\ldots})]\Big], \quad (15)$$

where

$$\begin{aligned} f(\phi_{7,}\phi_{25,}\cdots) = {} & \cos(\phi_{127} - \phi_{30} - \phi_{97}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{25} - \phi_{102}) \\ & + \cos(\phi_{42} + \phi_{85} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_{45} + \phi_{82} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_{127} - \phi_{45} - \phi_{82}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{42} - \phi_{85}) \\ & + \cos(\phi_{25} + \phi_{102} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_{30} + \phi_{97} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_{127} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_{25} + \phi_{102} - \phi_{42} - \phi_{85}) \\ & + \cos(\phi_{30} + \phi_{97} - \phi_{45} - \phi_{82}) \\ & + \cos(\phi_{127} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_{25} + \phi_{102} - \phi_{45} - \phi_{82}) \\ & + \cos(\phi_{30} + \phi_{97} - \phi_{42} - \phi_{85}) \\ & + \cos(\phi_{127} - \phi_{42} - \phi_{85}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{45} - \phi_{82}) \\ & + \cos(\phi_{25} + \phi_{102} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_{30} + \phi_{97} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_{127} - \phi_{25} - \phi_{102}) \\ & + \cos(\phi_7 + \phi_{120} - \phi_{30} - \phi_{97}) \\ & + \cos(\phi_{42} + \phi_{85} - \phi_{51} - \phi_{76}) \\ & + \cos(\phi_{45} + \phi_{82} - \phi_{52} - \phi_{75}) \\ & + \cos(\phi_{127} - \phi_7 - \phi_{120}) \\ & + \cos(\phi_{25} + \phi_{102} - \phi_{30} - \phi_{97}) \\ & + \cos(\phi_{42} + \phi_{85} - \phi_{45} - \phi_{82}) \\ & + \cos(\phi_{51} + \phi_{76} - \phi_{52} - \phi_{75}). \end{aligned} \quad (16)$$

For seven-qubit maximally entangled states, the function $f(\varphi_7, \varphi_{25}, \ldots)$ must reach its minimum. We can show that the minimum of $f(\varphi_7, \varphi_{25}, \ldots)$ is -4, thereby obtaining $\pi_{\rm ME} = 19/140$. Therefore, we determine that $\pi_{\rm ME} = 19/140$ is a condition for maximally entangled seven-qubit states.

We now know that for maximally seven-qubit entangled states, it is definite that the function $f(\varphi_7, \varphi_{25}, \ldots)$ reaches its minimum, and the minimum of $f(\varphi_7, \varphi_{25}, \ldots)$ is -4.

For example: (1) When $\varphi_7 = \varphi_{25} = \varphi_{30} = \varphi_{42} = \varphi_{45} = \varphi_{51} = \varphi_{52} = \varphi_{97} = \varphi_{102} = \varphi_{120} = \varphi_{127} = 0$ and $\varphi_{75} = \varphi_{76} = \varphi_{82} = \varphi_{85} = \pi$, we obtain the following seven-qubit maximally entangled state:

$$\begin{aligned} |\varphi_m^1\rangle = {} & \frac{1}{4}(|0000000\rangle + |0000111\rangle + |0011001\rangle \\ & + |0011110\rangle + |0101010\rangle + |0101101\rangle \\ & + |0110011\rangle + |0110100\rangle - |1001011\rangle \\ & - |1001100\rangle - |1010010\rangle - |1010101\rangle \\ & + |1100001\rangle + |1100110\rangle + |1111000\rangle \\ & + |1111111\rangle)_{1234567} \\ = {} & \frac{1}{4}[(|000\rangle + |111\rangle)_{127}(|0000\rangle + |1111\rangle)_{3456} \\ & + (|001\rangle + |110\rangle)_{127}(|0011\rangle + |1100\rangle)_{3456} \\ & + (|010\rangle - |101\rangle)_{127}(|0101\rangle + |1010\rangle)_{3456} \\ & + (|011\rangle - |100\rangle)_{127}(|0110\rangle + |1001\rangle)_{3456}]. \end{aligned} \quad (17)$$

We can also obtain

$$\pi_{ijk} = {\rm Tr}_{ijk}\,\rho_{ijk}^2 = \frac{1}{8}, \quad ijk = 123, 124, \ldots, 467 \quad (18)$$

except for

$$\pi_{127} = \frac{1}{4}, \quad \pi_{347} = \frac{1}{4}, \quad \pi_{567} = \frac{1}{4}. \quad (19)$$

(2) When $\varphi_{75} = \varphi_{76} = \varphi_{82} = \varphi_{85} = \varphi_{97} = \varphi_{102} = \varphi_{120} = \varphi_{127} = 0$ and $\varphi_7 = \frac{\pi}{4}$, $\varphi_{25} = \frac{2\pi}{4}$, $\varphi_{30} = \frac{3\pi}{4}$, $\varphi_{42} = \frac{4\pi}{4}$, $\varphi_{45} = \frac{5\pi}{4}$, $\varphi_{51} = \frac{6\pi}{4}$, $\varphi_{52} = \frac{7\pi}{4}$, we obtain the following seven-qubit maximally entangled state:

$$\begin{aligned} |\varphi_m^2\rangle = {} & \frac{1}{4}\big(|0000000\rangle + e^{i\frac{\pi}{4}}|0000111\rangle \\ & + e^{i\frac{2\pi}{4}}|0011001\rangle + e^{i\frac{3\pi}{4}}|0011110\rangle \\ & + e^{i\frac{4\pi}{4}}|0101010\rangle + e^{i\frac{5\pi}{4}}|0101101\rangle \\ & + e^{i\frac{6\pi}{4}}|0110011\rangle + e^{i\frac{7\pi}{4}}|0110100\rangle \\ & + |1001011\rangle + |1001100\rangle + |1010010\rangle \\ & + |1010101\rangle + |1100001\rangle + |1100110\rangle \\ & + |1111000\rangle + |1111111\rangle\big)_{1234567}. \end{aligned} \quad (20)$$

We can also obtain

$$\pi_{ijk} = \mathrm{Tr}_{ijk}\,\rho_{ijk}^2 = \frac{1}{8}, \quad ijk = 123, 124, \ldots, 467 \quad (21)$$

except for

$$\pi_{127} = \frac{3}{16}, \quad \pi_{136} = \frac{1}{32}(6-\sqrt{2}), \quad \pi_{145} = \frac{3}{16},$$
$$\pi_{235} = \frac{3}{16}, \quad \pi_{347} = \frac{3}{16}, \quad \pi_{567} = \frac{1}{32}(6+\sqrt{2}). \quad (22)$$

(3) When $\varphi_7 = \varphi_{25} = \varphi_{30} = \varphi_{42} = \varphi_{45} = \varphi_{51} = \varphi_{52} = 0$ and $\varphi_{75} = \varphi_{120} = \frac{\pi}{2}$, $\varphi_{82} = \varphi_{97} = \pi$, $\varphi_{85} = \varphi_{102} = \frac{3\pi}{2}$, we obtain the following seven-qubit maximally entangled state:

$$\begin{aligned}|\varphi_m^3\rangle =& \frac{1}{4}(|0000000\rangle + |0000111\rangle + |0011001\rangle \\ &+ |0011110\rangle + |0101010\rangle + |0101101\rangle \\ &+ |0110011\rangle + |0110100\rangle + i|1001011\rangle \\ &+ |1001100\rangle - |1010010\rangle - i|1010101\rangle \\ &- |1100001\rangle - i|1100110\rangle + i|1111000\rangle \\ &+ |1111111\rangle)_{1234567} \\ =& \frac{1}{4}[(|000\rangle + |111\rangle)_{145}(|0000\rangle + |1111\rangle)_{2367} \\ &+ (|001\rangle + i|110\rangle)_{145}(|0011\rangle + |1100\rangle)_{2367} \\ &+ (|010\rangle - i|101\rangle)_{145}(|0101\rangle + |1010\rangle)_{2367} \\ &+ (|011\rangle - |100\rangle)_{145}(|0110\rangle + |1001\rangle)_{2367}]. \end{aligned} \quad (23)$$

We can also obtain

$$\pi_{ijk} = \mathrm{Tr}_{ijk}\rho_{ijk}^2 = \frac{1}{8}, ijk = 123, 124, \ldots, 467 \quad (24)$$

except for

$$\pi_{145} = \frac{1}{4}, \quad \pi_{235} = \frac{3}{16}, \quad \pi_{246} = \frac{3}{16},$$
$$\pi_{347} = \frac{3}{16}, \quad \pi_{567} = \frac{3}{16}. \quad (25)$$

In summary, based on previous study of k-uniform states and orthogonal arrays, we have proposed a new method for constructing seven-qubit maximally entangled states. For seven-qubit entangled states, a condition for a maximally entangled seven-qubit state has been found. We find that the maximally entangled seven-qubit states satisfy $\pi_{\mathrm{ME}} = 19/140$. We also find some new forms of maximally entangled seven-qubit states. We believe that this new method can in theory be useful in the discovery of multi-qubit maximally entangled states.

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