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$\mathcal{PT}$ Symmetry of a Square-Wave Modulated Two-Level System

Liwei Duan(段立伟)[1], Yan-Zhi Wang(汪延志)[1], and Qing-Hu Chen(陈庆虎)[1,2*]

[1]*Department of Physics and Zhejiang Province Key Laboratory of Quantum Technology and Device, Zhejiang University, Hangzhou 310027, China*

[2]*Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China*

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We study a non-Hermitian two-level system with square-wave modulated dissipation and coupling. Based on the Floquet theory, we achieve an effective Hamiltonian from which the boundaries of the $\mathcal{PT}$ phase diagram are captured exactly. Two kinds of $\mathcal{PT}$ symmetry broken phases are found, whose effective Hamiltonians differ by a constant $\omega/2$. For the time-periodic dissipation, a vanishingly small dissipation strength can lead to the $\mathcal{PT}$ symmetry breaking in the $(2k-1)$-photon resonance ($\Delta = (2k-1)\omega$), with $k = 1, 2, 3\ldots$ It is worth noting that such a phenomenon can also happen in $2k$-photon resonance ($\Delta = 2k\omega$), as long as the dissipation strengths or the driving times are imbalanced, namely $\gamma_0 \neq -\gamma_1$ or $T_0 \neq T_1$. For the time-periodic coupling, the weak dissipation induced $\mathcal{PT}$ symmetry breaking occurs at $\Delta_{\text{eff}} = k\omega$, where $\Delta_{\text{eff}} = (\Delta_0 T_0 + \Delta_1 T_1)/T$. In the high frequency limit, the phase boundary is given by a simple relation $\gamma_{\text{eff}} = \pm\Delta_{\text{eff}}$.

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A non-Hermitian Hamiltonian is a natural extension of the conventional Hermitian one to describe the open quantum system. The discovery of the real spectra in non-Hermitian Hamiltonians by Bender and Boettcher[1] has stimulated enormous interest in the systems with parity-time ($\mathcal{PT}$) symmetry.[2–5] Early theoretical and experimental explorations of the non-Hermitian systems with $\mathcal{PT}$ symmetry mainly focus on the optics and photonics.[6–17] Feng *et al.* realized the nonreciprocal light propagation in a silicon photonic circuit which provides a way to chip-scale optical isolators for optical communications and computing.[12] Hodaei *et al.* stabilized single-longitudinal mode operation in a system of coupled mirroring lasers by harnessing notions from $\mathcal{PT}$ symmetry, which provides the possibilities to develop optical devices with enhanced functionality.[14] Xiao *et al.* achieved the first experimental characterization of critical phenomena in $\mathcal{PT}$-symmetric nonunitary quantum dynamics.[17] Recent experiments have realized the non-Hermitian magnon-polaritons systems, and higher-order exceptional points were observed, which can be used to measure the output spectrum of the cavity.[18–20] The anomalous edge state in a non-Hermitian lattice[21] has intrigued persistent attention to the combination of the non-Hermiticity and the topological phase.[22–35] The non-Bloch band theory has been developed to describe the non-Hermitian lattice systems.[22,29,32,33] Kawabata *et al.* established a fundamental symmetry principle in non-Hermitian physics, which paved the way towards a unified framework for non-equilibrium topological phase.[25,26] Yao *et al.* studied the bulk-boundary correspondence in the non-Hermitian systems and found the non-Hermitian skin effect.[29,30] Xiao *et al.* observed the topological edge states in $\mathcal{PT}$-symmetric quantum walks.[31]

Recently, Joglekar *et al.* investigated a two-level system coupled to a sinusoidally varying gain-loss potential, namely, the non-Hermitian Rabi model with time-periodic dissipation.[36] They found that multiple frequency windows existed where $\mathcal{PT}$ symmetry was broken and restored. The non-Hermitian Rabi model has drawn growing attention due to its specially rich phenomena which are absent in the static counterparts.[37–46] Lee *et al.* found the $\mathcal{PT}$ symmetry breaking at the $(2k-1)$-photon resonance and derived the boundaries of the $\mathcal{PT}$ phase diagram by doing perturbation theory beyond rotating-wave approximation.[37] Gong *et al.* found that a periodic driving could stabilize the dynamics despite the loss and gain in the non-Hermitian system.[40,41] Xie *et al.* studied a non-Hermitian Rabi model with time-periodic coupling and found exact analytical results for certain exceptional points.[42] A synchronous modulation that combined the time-periodic dissipation and coupling was studied in Ref. [43], which provided an additional possibility for pulse manipulation and coherent control of the $\mathcal{PT}$-symmetric two-level systems. The experimental approach of the Floquet $\mathcal{PT}$-symmetric system has been proposed with two coupled high-frequency oscillators.[44] A $\mathcal{PT}$ symmetry breaking transition by engineering time-periodic dis-

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*Corresponding author. Email: qhchen@zju.edu.cn

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sipation and coupling has been realized through state-dependent atom loss in an optical dipole trap of ultracold ^{6}Li atoms.[45] They confirmed that a weak time-periodic dissipation could lead to $\mathcal{PT}$-symmetry breaking in $(2k-1)$-photon resonance. It should be noted that the $\mathcal{PT}$-symmetry breaking can occur in a finite non-Hermitian system, which is quite different from the quantum phase transition in the Hermitian system where the thermodynamic limit is needed.[45,47]

In this letter, we study the $\mathcal{PT}$ symmetry of a two-level system with time-periodic dissipation and coupling. Instead of the widely used sinusoidal modulation,[36,37,40–42] we consider a square-wave one, which is easier to implement in the experiment of ultracold atoms[45] and has analytical exact solutions based on the Floquet theory.[48,49] The square-wave modulation has a broad range of applications in the Hermitian system. It has been used to suppress the quantum dissipation in spin chains,[50] to generate many Majorana modes in a one-dimensional p-wave superconductor system,[51] to generate large-Chern-number topological phases,[52] and so on. The square-wave modulation has also been realized in the non-Hermitian systems.[45] In this study, firstly we describe the non-Hermitian Hamiltonian of the driving two-level system. Secondly we briefly introduce the Floquet theory and derive the effective static Hamiltonian. Thirdly, we achieve the $\mathcal{PT}$ phase diagram and analyze the influence of multiphoton resonance. An equivalent Hamiltonian is obtained in the high frequency limit. Finally we give some concluding remarks.

We consider a periodically driving two-level system $H(t)=H(t+T)$, with

$$H(t)=\frac{\Delta(t)}{2}\sigma_x+i\frac{\gamma(t)}{2}\sigma_z, \tag{1}$$

where $\sigma_{x,z}$ are the Pauli matrices, $T=T_0+T_1$ is the driving period, $\omega=2\pi/T$ is the driving frequency, $\Delta(t)$ is the time-periodic coupling strength, and $\gamma(t)$ is the dissipation strength which leads to the periodic gain and loss. Lee *et al.*[37] studied the $\mathcal{PT}$ phase diagram of the non-Hermitian two-level system by doing the perturbation theory, which corresponds to $\Delta(t)=\Delta$ and $\gamma(t)=4\lambda\cos(\omega t)$. Xie *et al.*[42] found the exact analytical results for certain exceptional points of the two-level system with time-periodic coupling, which corresponds to $\Delta(t)=v_0+v_1\cos(\omega t)$ and $\gamma(t)=\gamma$. Luo *et al.*[43] studied the analytical results of the non-Hermitian two-level systems with sinusoidal modulations of both $\Delta(t)$ and $\gamma(t)$. In order to obtain the exact analytical results without using perturbation theory, we consider a synchronous square-wave modulation of both dissipation and coupling. The corresponding time-periodic parameters are

$$f(t)=\begin{cases} f_0, & \text{if } \ mT-\frac{T_0}{2}\le t<mT+\frac{T_0}{2},\\ f_1, & \text{if } \ mT+\frac{T_0}{2}\le t<(m+1)T-\frac{T_0}{2}, \end{cases} \tag{2}$$

with $f=\Delta$, γ and $m=\ldots,-1,0,1,\ldots$ It is easy to confirm that the non-Hermitian Hamiltonian has a $\mathcal{PT}$ symmetry, namely $\hat{\mathcal{P}}H^\dagger(t)\hat{\mathcal{P}}=H(t)$, where $H^\dagger(t)$ is the Hermitian conjugate of $H(t)$ and $\hat{\mathcal{P}}=\hat{\mathcal{P}}^{-1}=\sigma_x$ is the parity operator.[2,5] This non-Hermitian system has been realized by Li *et al.* in the ultracold atom experiments.[45] However, they focused on a special case with only one time-periodic parameter (either dissipation or coupling), and $f_0=f$, $f_1=0$, $T_0=T_1=T/2$. We consider a more general case in which those constraints are relieved. Two time-independent Hamiltonians H_0 and H_1 appear alternately, with

$$H_j=\frac{\Delta_j}{2}\sigma_x+i\frac{\gamma_j}{2}\sigma_z,\quad j=0,1, \tag{3}$$

and the corresponding eigenenergies are $E_j^{\pm}=\pm h_j$, with

$$h_j=\frac{\sqrt{\Delta_j^2-\gamma_j^2}}{2}. \tag{4}$$

H_j is one of the simplest non-Hermitian systems with $\mathcal{PT}$ symmetry.[5] When $|\Delta_j|>|\gamma_j|$, the eigenenergy is real and it corresponds to the $\mathcal{PT}$-symmetric phase. When $|\Delta_j|<|\gamma_j|$, the eigenenergy is imaginary and the $\mathcal{PT}$ symmetry is broken. When $|\Delta_j|=|\gamma_j|$, an exceptional point (EP exists). The dynamics at each time domain is governed by the time evolution operator

$$\begin{aligned} U_j(T_j)&=\exp(-iH_jT_j)\\ &=\cos(h_jT_j)I-i\mathrm{sinc}(h_jT_j)T_jH_j, \end{aligned} \tag{5}$$

where I is a 2×2 identity matrix, and $\mathrm{sinc}(x)=\sin(x)/x$.

According to the Floquet theory,[48,49] we can define an effective Hamiltonian $H_{\rm eff}$, which satisfies the condition

$$U_{\rm eff}(T)=\exp(-iH_{\rm eff}T)=\mathcal{T}\exp\left[-i\int_{-\frac{T_0}{2}}^{T-\frac{T_0}{2}}dtH(t)\right]. \tag{6}$$

The eigenenergies of the effective Hamiltonian correspond to the Floquet quasi-energies. Due to the simplicity of the square-wave modulation, the time evolution operator in a period can be written as

$$\mathcal{T}\exp\left[-i\int_{-\frac{T_0}{2}}^{T-\frac{T_0}{2}}dtH(t)\right]=\exp(-iH_1T_1)\exp(-iH_0T_0). \tag{7}$$

Therefore,

$$U_{\rm eff}(T)=U_1(T_1)U_0(T_0). \tag{8}$$

From Eqs. (5) and (8), we can achieve the effective time evolution operator

$$\begin{aligned}U_{\rm eff}(T) &= [\cos(h_1T_1)\cos(h_0T_0)\\ &+\frac{1}{4}(\gamma_1\gamma_0-\Delta_1\Delta_0)T_1T_0{\rm sinc}(h_1T_1){\rm sinc}(h_0T_0)]I\\ &-i\frac{1}{2}[\Delta_0T_0\cos(h_1T_1){\rm sinc}(h_0T_0)\\ &+\Delta_1T_1\cos(h_0T_0){\rm sinc}(h_1T_1)]\sigma_x\\ &+\frac{1}{4}(\Delta_0\gamma_1-\Delta_1\gamma_0)T_1T_0{\rm sinc}(h_1T_1){\rm sinc}(h_0T_0)\sigma_y\\ &+\frac{1}{2}[\gamma_0T_0\cos(h_1T_1){\rm sinc}(h_0T_0)\\ &+\gamma_1T_1\cos(h_0T_0){\rm sinc}(h_1T_1)]\sigma_z. \end{aligned}\tag{9}$$

Since h_j can be either a pure real number in the $\mathcal{PT}$-symmetric phase or a pure imaginary one in the $\mathcal{PT}$ symmetry broken phase, both $\cos(h_jT_j)$ and ${\rm sinc}(h_jT_j)$ must be real numbers. Accordingly, the coefficients before I, σ_y and σ_z must be real, while those before σ_x must be imaginary. It is easy to confirm that the effective Hamiltonian can only be in the form of

$$H_{\rm eff}=\frac{J}{2}\sigma_x+i\Big(\frac{\Gamma_y}{2}\sigma_y+\frac{\Gamma_z}{2}\sigma_z\Big)+\frac{n\omega}{2}I,\tag{10}$$

with $n=0,1$. The eigenenergies of $H_{\rm eff}$, or the Floquet quasi-energies of $H(t)$ would be $E^{\pm}=\pm h+\frac{n\omega}{2}$, where

$$h=\frac{\sqrt{J^2-\Gamma_y^2-\Gamma_z^2}}{2}.\tag{11}$$

The effective time evolution operator can be rewritten as

$$\begin{aligned}U_{\rm eff}(T) &= \exp(-iH_{\rm eff}T)\\ &=(-1)^n\cos(hT)I\\ &\quad -i\frac{(-1)^nT}{2}{\rm sinc}(hT)J\sigma_x\\ &\quad +\frac{(-1)^nT}{2}{\rm sinc}(hT)(\Gamma_y\sigma_y+\Gamma_z\sigma_z).\end{aligned}\tag{12}$$

By comparing the coefficients before I, σ_x, σ_y, and σ_z in Eqs. (9) and (12), we can directly obtain

$$\begin{aligned}(-1)^n\cos(hT) &= \cos(h_1T_1)\cos(h_0T_0)\\ &+\frac{1}{4}(\gamma_1\gamma_0-\Delta_1\Delta_0)T_0T_1{\rm sinc}(h_1T_1){\rm sinc}(h_0T_0),\end{aligned}\tag{13}$$

$$\begin{aligned}J &= \frac{(-1)^n}{T{\rm sinc}(hT)}[\Delta_0T_0\cos(h_1T_1){\rm sinc}(h_0T_0)\\ &\quad +\Delta_1T_1\cos(h_0T_0){\rm sinc}(h_1T_1)],\end{aligned}\tag{14}$$

$$\begin{aligned}\Gamma_y &= \frac{(-1)^n}{2\,T{\rm sinc}(hT)}(\Delta_0\gamma_1-\Delta_1\gamma_0)T_0T_1\\ &\quad \cdot{\rm sinc}(h_1T_1){\rm sinc}(h_0T_0),\end{aligned}\tag{15}$$

$$\begin{aligned}\Gamma_z &= \frac{(-1)^n}{T{\rm sinc}(hT)}[\gamma_0T_0\cos(h_1T_1){\rm sinc}(h_0T_0)\\ &\quad +\gamma_1T_1\cos(h_0T_0){\rm sinc}(h_1T_1)].\end{aligned}\tag{16}$$

Once we obtain J, Γ_y, Γ_z and n, the effective Hamiltonian (10) will be finally determined.

The major differences of the effective Hamiltonian and the original one are the dissipation Γ_y in y-axis and the additional constant $\omega/2$. We will show later that the additional constant is closely related with the $\mathcal{PT}$ symmetry broken phases and the exceptional points. One can easily confirm that the effective Hamiltonian has a $\mathcal{PT}$ symmetry, namely $\hat{\mathcal{P}}H_{\rm eff}^{\dagger}\hat{\mathcal{P}}=H_{\rm eff}$, since $\hat{\mathcal{P}}\sigma_x\hat{\mathcal{P}}=\sigma_x$, $\hat{\mathcal{P}}\sigma_y\hat{\mathcal{P}}=-\sigma_y$ and $\hat{\mathcal{P}}\sigma_z\hat{\mathcal{P}}=-\sigma_z$. When $|\cos(hT)|<1$, h must be a real number and the $\mathcal{PT}$ symmetry is preserved. For the $\mathcal{PT}$-symmetric phase, we suppose that the eigenenergies are $E_{\pm}^{(n)}=\pm h^{(n)}+\frac{n\omega}{2}$. From Eq. (13), we can obtain $\cos(h^{(0)}T)=-\cos(h^{(1)}T)$. Then, $h^{(1)}T=h^{(0)}T+\pi$, which leads to $h^{(1)}=h^{(0)}+\frac{\omega}{2}$. Finally, $E_+^{(0)}=E_+^{(1)}+\omega$ and $E_-^{(0)}=E_-^{(1)}$. As is well known, the Floquet quasi-energies are periodic with period ω, and the total quasi-energies should be $E_{\pm}^{(n)}+l\omega$ with $l=0,\pm1,\pm2,\ldots$. Therefore, $E_{\pm}^{(0)}$ and $E_{\pm}^{(1)}$ are equivalent. From now on, we only consider $n=0$ in the $\mathcal{PT}$-symmetric phase.

When h is an imaginary number, $\cos(hT)>1$, it corresponds to the $\mathcal{PT}$ symmetry spontaneous breaking. There are two kinds of $\mathcal{PT}$ symmetry broken phases, and their effective Hamiltonians differ by a constant. For simplicity, we assign the right-hand side of Eq. (13) to $\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1)$, namely,

$$\begin{aligned}\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1) &= \cos(h_1T_1)\cos(h_0T_0)\\ &+\frac{1}{4}(\gamma_1\gamma_0-\Delta_1\Delta_0)T_0T_1{\rm sinc}(h_1T_1){\rm sinc}(h_0T_0).\end{aligned}\tag{17}$$

If $\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1)$ is greater than 1, then $n=0$. If $\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1)$ is less than -1, then $n=1$. The exceptional points correspond to $h=0$. From Eq. (13), we can easily find that the exceptional points occur when $\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1)=\pm1$, where $+$ $(-)$ corresponds to $n=0$ (1). Unlike the static Hamiltonians H_j whose eigenenergies can only be 0 in the exceptional points, the quasi-energies of the driven two-level system can be either 0 for $n=0$ or $\omega/2$ for $n=1$. Once the parameters Δ_j, γ_j, T_j of the driving two-level systems are obtained, we can calculate $\Pi(\Delta_0,\Delta_1,\gamma_0,\gamma_1,T_0,T_1)$, from which one can determine whether the $\mathcal{PT}$ symmetry is broken or not.

Next, we concentrate on multiphoton resonance. For the two-level system with square-wave modulated dissipation and time-independent coupling, the multiphoton resonance refers to the case in which the coupling strength Δ of the two-level system is an integral multiple of the driving frequency ω. A vanishingly small dissipation strength can lead to the $\mathcal{PT}$ symmetry spontaneous breaking in the $(2k-1)$-photon resonance case ($k=1,2\ldots$), which has been found

in the two-level system with sinusoidal[37] and square-wave[45] modulated dissipations.

For the two-level system with a square-wave modulated coupling, one may naively think that the necessary condition for the weak dissipation induced $\mathcal{PT}$ symmetry breaking is that both Δ_0 and Δ_1 are integral multiples of ω. However, it is not just the case. The $\mathcal{PT}$ phase transition induced by the weak dissipation in the multiphoton resonance indicates that $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ deviates from ± 1 once the dissipation occurs. We expect that the necessary condition is $\Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) = \pm 1$. From Eq. (17), we can reach

$$\begin{aligned}
&\Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) \\
&= \cos\left(\frac{\Delta_1 T_1}{2}\right)\cos\left(\frac{\Delta_0 T_0}{2}\right) - \sin\left(\frac{\Delta_1 T_1}{2}\right)\sin\left(\frac{\Delta_0 T_0}{2}\right) \\
&= \cos\left(\frac{\Delta_0 T_0 + \Delta_1 T_1}{2}\right) \\
&= \cos\left(\frac{\Delta_{\text{eff}} T}{2}\right),
\end{aligned}$$

where

$$\Delta_{\text{eff}} = \frac{\Delta_0 T_0 + \Delta_1 T_1}{T}. \tag{18}$$

Therefore, the necessary condition for the $\mathcal{PT}$ phase transition induced by the weak dissipation should be $\Delta_{\text{eff}} = k\omega$. In other words, the driving frequency should resonate with the effective coupling strength Δ_{eff}, rather than Δ_0 or Δ_1. When k is an even number, $\Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) = 1$. A weak dissipation can lead to $\Pi > 1$ which corresponds to the $\mathcal{PT}$ symmetry broken phase with $n = 0$, or $\Pi < 1$ which corresponds to the $\mathcal{PT}$-symmetric phase. Similarly, when k is an odd number, a weak dissipation can lead to $\Pi < -1$ which corresponds to the $\mathcal{PT}$ symmetry broken phase with $n = 1$, or $\Pi > -1$ which corresponds to the $\mathcal{PT}$-symmetric phase.

We firstly consider the two-level system with only square-wave modulated dissipation. The coupling strength is time-independent, namely $\Delta_0 = \Delta_1 = \Delta$, which leads to $\Delta_{\text{eff}} = \Delta$. According to the former analysis, we expect that the $\mathcal{PT}$ phase transition at weak dissipation occurs when $\Delta = k\omega$. However, Li *et al.* only showed the $\mathcal{PT}$-symmetry breaking in $(2k-1)$-photon resonance,[45] namely $\Delta = (2k-1)\omega$. In Fig. 1(a), we recover the $\mathcal{PT}$ phase diagram near the one-photon resonance in Ref. [45], by setting $T_0 = T_1 = T/2$, $\gamma_0 = \gamma$ and $\gamma_1 = 0$. The boundary of the phase diagram can be determined by either $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ (Fig. 1(b)) or the imaginary part of the quasi-energies (Fig. 1(c)). Near the one-photon resonance region, Π is less than -1 and the imaginary part of the quasi-energies is nonzero, which indicates that it corresponds to a $\mathcal{PT}$ symmetry broken phase with $n = 1$.

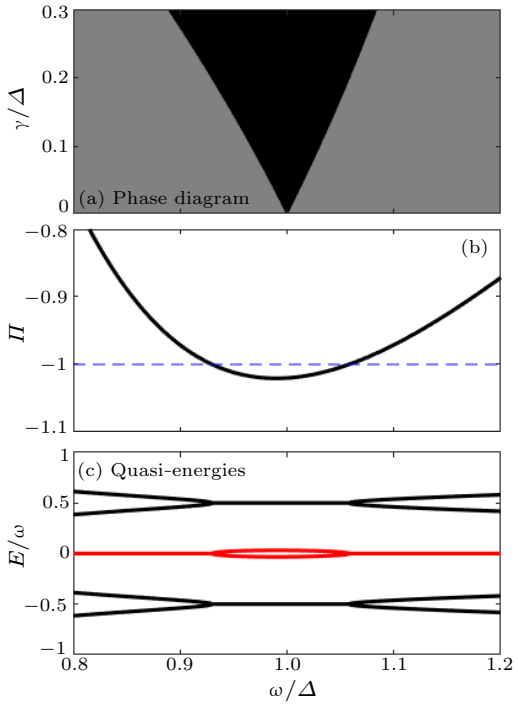

Fig. 1. (a) $\mathcal{PT}$ phase diagram near the one-photon resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 1$ (black). (b) Π (Eq. (17)) as a function of ω/Δ at $\gamma/\Delta = 0.2$. The dashed line represents $\Pi = -1$, below which corresponds to $\mathcal{PT}$ symmetry broken phase with $n = 1$. (c) Real (black lines) and imaginary (red lines) parts of the quasi-energies as a function of ω/Δ at $\gamma/\Delta = 0.2$. The other parameters are $\Delta_0 = \Delta_1 = \Delta = 1$, $T_0 = T_1 = T/2$, $\gamma_0 = \gamma$ and $\gamma_1 = 0$.

When we further decrease the driving frequency ω to the two-photon resonance region, we find that a weak dissipation can also lead to the $\mathcal{PT}$ symmetry breaking, which is not observed in Ref. [45]. As depicted in Fig. 2(a), the $\mathcal{PT}$ symmetry broken region is much narrower than that in the one-photon resonance case. In addition, the driving frequency ω at the phase boundary tends to decrease with increasing γ. Therefore, the $\mathcal{PT}$ symmetry breaking occurs at the region where ω is a bit less than $\Delta/2$. Near the two-photon resonance, Π is greater than 1 and the imaginary part of the quasi-energies is nonzero, which indicates that it corresponds to a $\mathcal{PT}$ symmetry broken phase with $n = 0$.

Figure 3(a) is a generalization of Figs. 1(a) and 2(a), which extends the range of ω. The driving two-level system has a much richer phase diagram than the static one. Clearly, a vanishingly small dissipation strength can lead to the $\mathcal{PT}$ symmetry spontaneous breaking in both $(2k-1)$- and $2k$-photon resonances, which is consistent with our criteria $\Delta = k\omega$. To explain the behavior of the $\mathcal{PT}$ symmetry breaking near the $2k$-photon resonance, we reexamine $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ in Eq. (17) in more detail. We suppose that $\gamma_0 = \gamma \ll \Delta$, $\gamma_1 = \lambda\gamma$, $T_0 = T_1 = T/2$, and $\Delta \simeq 2k\omega$. When γ tends to zero, $h_i T_i$ tends to $k\pi$. The first term on the right-

hand side of Eq. (17) tends to one while the second term tends to zero. If the second term is greater than zero, it can lead to $\Pi > 1$ and the $\mathcal{PT}$ symmetry broken phase with $n = 0$. Since $\left(\gamma_0\gamma_1 - \Delta^2\right) T_0T_1/4$ in the second term is less than zero, one needs that $\mathrm{sinc}(h_1T_1)\mathrm{sinc}(h_0T_0) < 0$, or $\sin(h_1T_1)\sin(h_0T_0) < 0$. Then, the condition for the occurrence of $\mathcal{PT}$ symmetry breaking is that one of h_iT_i should be less than $k\pi$, while the other one should be greater than $k\pi$. If Δ is a bit less than $2k\omega$, a finite γ will always decrease h_i, which leads to both $h_iT_i < \Delta T_i/2 < k\pi$ and $\Pi < 1$. Therefore, no $\mathcal{PT}$ symmetry breaking occurs when $\Delta < 2k\omega$. If Δ is a bit larger than $2k\omega$, one can always find certain γ which satisfies the condition for the occurrence of $\mathcal{PT}$ symmetry breaking, as long as $\lambda \neq \pm 1$. Figure 3(a) corresponds to $\lambda = 0$. Therefore, a finite γ can lead to the $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance.

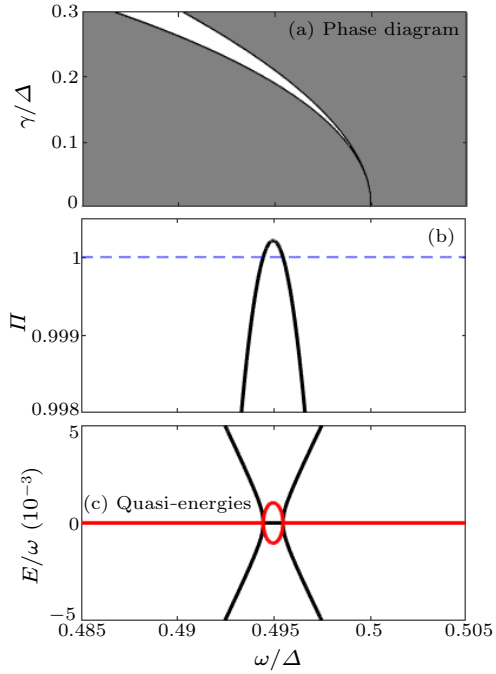

Fig. 2. (a) $\mathcal{PT}$ phase diagram near the two-photon resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phase with $n = 0$ (white). (b) Π (Eq. (17)) as a function of ω/Δ at $\gamma/\Delta = 0.2$. The dashed line represents $\Pi = 1$, above which corresponds to $\mathcal{PT}$ symmetry broken phase with $n = 0$. (c) Real (black lines) and imaginary (red lines) parts of the quasi-energies as a function of ω/Δ at $\gamma/\Delta = 0.2$. The other parameters are $\Delta_0 = \Delta_1 = \Delta = 1$, $T_0 = T_1 = T/2$, $\gamma_0 = \gamma$ and $\gamma_1 = 0$.

When $\lambda = +1$, namely $\gamma_0 = \gamma_1$, the Hamiltonian (1) becomes time-independent, which is trivial. When $\lambda = -1$, namely $\gamma_0 = -\gamma_1 = \gamma$, h_0 will equal $h_1.h_1T_1 = h_0T_0$ if $T_0 = T_1$, which leads to the $\mathcal{PT}$-symmetric phase with $\Pi < 1$ near the $2k$-photon resonance, as shown in Fig. 3(b). Following the above analysis, we can easily prove that an imbalanced driving time $T_0 \neq T_1$ can lead to the $\mathcal{PT}$ symmetry breaking when $\gamma_0 = -\gamma_1$, as depicted in Fig. 3(c). The $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance induced by the imbalanced driving time $T_0 \neq T_1$ is more obvious than that induced by $\gamma_0 \neq -\gamma_1$, when the dissipation strength is very weak. Therefore, the imbalanced driving time $T_0 \neq T_1$ is a more efficient method to access the $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance in the experiments. Figures 3(a) and 3(c) verify our conclusion that the $\mathcal{PT}$ symmetry breaking induced by weak dissipation generally occurs at both $2k$- and $(2k-1)$-photon resonances, namely $\Delta = k\omega$. The $\mathcal{PT}$ symmetry breaking at $2k$-photon resonance disappears only if $\gamma_0 = -\gamma_1$ and $T_0 = T_1$, as shown in Fig. 3(c).

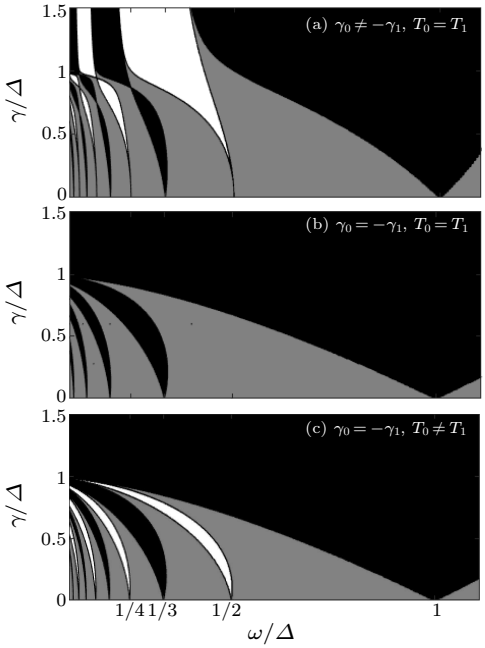

Fig. 3. $\mathcal{PT}$ phase diagram for time-periodic dissipation near the multiphoton resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 0$ (white) and $n = 1$ (black): (a) $\gamma_0 = \gamma$, $\gamma_1 = 0$, $T_0 = T_1$; (b) $\gamma_0 = -\gamma_1 = \gamma$, $T_0 = T_1$; (c) $\gamma_0 = -\gamma_1 = \gamma$, $T_0 = 0.55T$, $T_1 = 0.45T$.

For the two-level system with only square-wave modulated coupling, the dissipation strength is time-independent, namely $\gamma_0 = \gamma_1 = \gamma$. Figure 4 shows the $\mathcal{PT}$ phase diagram for time-periodic coupling near the multiphoton resonance. Li *et al.* studied the influence of the time-periodic coupling on the non-Hermitian two-level system based on a simpler model with $\Delta_0 = \Delta$, $\Delta_1 = 0$ and $T_0 = T_1 = T/2$,[45] which corresponds to Fig. 4(a). They concluded that the $\mathcal{PT}$ phase transition induced by the weak dissipation occurs at $\Delta = 2k\omega$, which is consistent with our results $\Delta_{\text{eff}} = k\omega$ due to $\Delta_{\text{eff}} = \Delta/2$. Figure 4(b) introduces a nonzero Δ_1 and imbalanced driving time $T_0 \neq T_1$, which cannot be explained by Ref. [45]. However, $\Delta_{\text{eff}} = k\omega$ can still provide the right condition under which the $\mathcal{PT}$ phase transitions occur. The $\mathcal{PT}$ symmetry broken phase with $n = 0$ (1) occurs when k

is even (odd), which is also consistent with the above analysis.

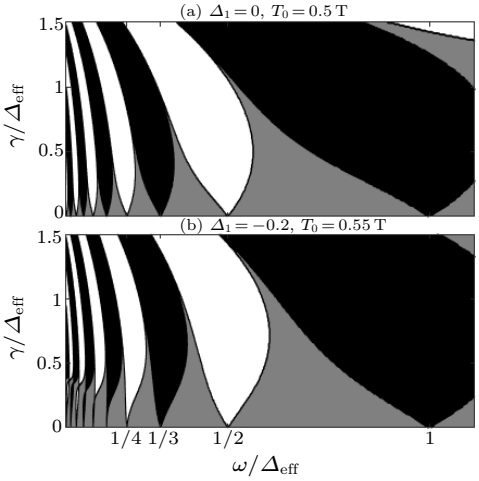

Fig. 4. $\mathcal{PT}$ phase diagram for time-periodic coupling near the multiphoton resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 0$ (white) and $n = 1$ (black). (a) $\Delta_0 = 1$, $\Delta_1 = 0$, $T_0 = 0.5T$. (b) $\Delta_0 = 1$, $\Delta_1 = -0.2$, $T_0 = 0.55T$.

If the driving frequency is very large, namely $\omega \gg \Delta_j, \gamma_j$, the period T tends to zero. We suppose that T_0 and T_1 are of the same order as T. Expanding $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ to the second order of T, we obtain

$$\begin{aligned}
&\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)\\
\simeq&\Big(1 - \frac{h_1^2 T_1^2}{2}\Big)\Big(1 - \frac{h_0^2 T_0^2}{2}\Big) + \frac{1}{4}(\gamma_1\gamma_0 - \Delta_1\Delta_0)T_1T_0\\
\simeq&1 + \frac{1}{4}[(\gamma_1\gamma_0 - \Delta_1\Delta_0)T_1T_0 - 2h_0^2T_0^2 - 2h_1^2T_1^2]\\
=&1 + \frac{1}{8}[(\gamma_0T_0 + \gamma_1T_1)^2 - (\Delta_0T_0 + \Delta_1T_1)^2]\\
=&1 + \frac{1}{8}(\gamma_{\rm eff}^2 - \Delta_{\rm eff}^2)T^2, \qquad (19)
\end{aligned}$$

where

$$\gamma_{\rm eff} = \frac{\gamma_0T_0 + \gamma_1T_1}{T}. \qquad (20)$$

Therefore, the exceptional points, as well as the $\mathcal{PT}$ phase boundary, are located at $\gamma_{\rm eff} = \pm\Delta_{\rm eff}$. If $|\gamma_{\rm eff}| < |\Delta_{\rm eff}|$, it will correspond to the $\mathcal{PT}$-symmetric phase. Otherwise, the $\mathcal{PT}$ symmetry will be broken with $n = 0$. Alternately, if we expand Eqs. (14)–(16) to the lowest order of T, we can find

$$J \simeq \Delta_{\rm eff}, \quad \Gamma_y \simeq 0, \quad \Gamma_z \simeq \gamma_{\rm eff}, \qquad (21)$$

which give rise to the effective Hamiltonian as follows:

$$H_{\rm eff} \simeq \frac{\Delta_{\rm eff}}{2}\sigma_x + i\frac{\gamma_{\rm eff}}{2}\sigma_z. \qquad (22)$$

It leads to the same $\mathcal{PT}$ phase boundary. In a word, we find that when the driving frequency is very large, the Floquet effective Hamiltonian is equivalent to a static one with time-averaged coupling and dissipation strength. When $\Delta_0 \simeq -\Delta_1$, $\Delta_{\rm eff}$ tends to zero and one can easily achieve the $\mathcal{PT}$ symmetry broken phase no matter how large Δ_j is. When $\gamma_0 \simeq -\gamma_1$, $\gamma_{\rm eff}$ tends to zero and one can easily preserve the $\mathcal{PT}$ symmetry no matter how large γ_j is.

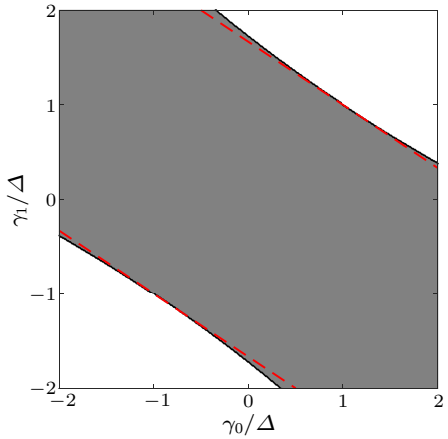

Fig. 5. $\mathcal{PT}$ phase diagram, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phase with $n = 0$ (white) at $\Delta_0 = \Delta_1 = \Delta = 1$, $\omega = 3$, $T_0 = 0.4T$, $T_1 = 0.6T$. The red dashed lines refer to the analytical results in the high frequency limit.

Figure 5 shows the $\mathcal{PT}$ phase diagram at $\Delta_0 = \Delta_1 = \Delta$, $\omega/\Delta = 3$ and $T_0/T_1 = 2/3$. The phase boundary $\gamma_{\rm eff} = \pm\Delta_{\rm eff}$ fits well with the exact results.

In summary, we have studied a non-Hermitian two-level system with square-wave modulated dissipation and coupling. Two time-independent Hamiltonians H_0 and H_1 appear alternately. Comparing with the formerly well-known sinusoidal modulation, the square-wave modulation has three advantages: firstly, exact analytical solutions can be achieved by employing the Floquet theory. Secondly, the $\mathcal{PT}$ phase diagram becomes richer. Thirdly, the square-wave modulation has been realized in the ultracold atoms experiment.[45]

Based on the Floquet theory, we achieve an effective Hamiltonian with $\mathcal{PT}$ symmetry. We define a parameter $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$, from which one can derive the boundaries of the $\mathcal{PT}$ phase diagram exactly. The driving two-level system has a much richer phase diagram than the static one. Two kinds of $\mathcal{PT}$ symmetry broken phases are found, whose effective Hamiltonians differ by a constant $\omega/2$. When $\Pi > 1$, the $\mathcal{PT}$ symmetry broken phase with $n = 0$ occurs. When $\Pi < -1$, the $\mathcal{PT}$ symmetry broken phase with $n = 1$ occurs. When $-1 < \Pi < 1$, the $\mathcal{PT}$ symmetry is preserved.

With the help of Π, we firstly study the $\mathcal{PT}$ phase transition with only square-wave modulated dissipation near multiphoton resonance. The coupling strength is time-independent with $\Delta_0 = \Delta_1 = \Delta$.

A weak dissipation can lead to the $\mathcal{PT}$ symmetry breaking near the $(2k-1)$-photon resonance ($\Delta = (2k-1)\omega$), which has been observed in the ultracold atom experiment.[45] We predict that the $\mathcal{PT}$ symmetry breaking near the $2k$-photon resonance ($\Delta = 2k\omega$) can also happen as long as the dissipation strengths or the driving times are imbalanced, with $\gamma_0 \neq -\gamma_1$ or $T_0 \neq T_1$. Our studies pave a way to access the $\mathcal{PT}$ symmetry broken phase near the $2k$-photon resonance in the experiments. For the $\mathcal{PT}$ phase transition with square-wave modulated coupling, we define an effective coupling strength $\Delta_{\text{eff}} = (\Delta_0 T_0 + \Delta_1 T_1)/T$. The weak dissipation induced $\mathcal{PT}$ symmetry breaking can occur only if $\Delta_{\text{eff}} = k\omega$.

In the high frequency limit, we achieve a simple relation $\gamma_{\text{eff}} = \pm\Delta_{\text{eff}}$, which gives the $\mathcal{PT}$ phase boundary. When $\Delta_0 \simeq -\Delta_1$, one can easily achieve the $\mathcal{PT}$ symmetry broken phase no matter how large the coupling strength $|\Delta_j|$ is. When $\gamma_0 \simeq -\gamma_1$, one can easily preserve the $\mathcal{PT}$ symmetry no matter how large the dissipation strength $|\gamma_j|$ is.

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