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Lax Pairs of Integrable Systems in Bidifferential Graded Algebras *

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Lax pairs regarded as foundations of the inverse scattering methods play an important role in integrable systems. In the framework of bidifferential graded algebras, we propose a straightforward approach to constructing the Lax pairs of integrable systems in functional environment. Some continuous equations and discrete equations are presented.

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Integrable systems have been regarded as the models to describe nonlinear phenomena widely in physics and developed rapidly. Since the turn of this century some celebrated models of partial differential or difference equations (PDDEs) including the self-dual Yang–Mills equation[1] have been given based on the framework of bidifferential graded algebra (or bidifferential calculus).[2–7] The framework consists of a graded algebra, which two anti-commuting graded derivations act on. As a strengthening of bicomplex calculus,[8] bidifferential graded algebras are equipped with the graded Leibniz rule, which provides calculation rules to explore the characters generally in the framework. Many efficient solution generating methods[6,7] are established through bidifferential calculus. Moreover the relations with Darboux transformations and binary Darboux transformations have been subsequently clarified in Refs. [9,10]. Also in this framework, self-consistent sources for several integrable equations are derived in Ref. [11] by employing the deformations of binary Darboux transformations. It is interesting that similar to the Cauchy matrix approach,[12–14] an infinite system of equations in bidifferential calculus with solutions is presented in Ref. [15], generated by the binary Darboux transformation. As a whole, much work in this framework has been carried out recently and satisfactory results are derived to investigate integrable systems.

In terms of integrability, the existence of an infinite set of conserved currents[16,17] in several completely integrable classical models has been traced back to a simple construction of an infinite chain of closed 1-forms in the bidifferential calculus.[3] Considering the Lax integrability, there is no straightforward approach for constructing Lax pairs so far, since the linear equations generated by the known approach always contain operators rather than pure functions and we have no universal way to transfer them to functional environment. Hence, the target of this study is to clearly establish the Lax pairs of integrable systems in a functional environment. If it is clarified in general, then every time that we construct this frame, we can directly derive its corresponding Lax pairs, exact solutions and the other characters. In turn, the form of the Lax pair can inspire us to know to what extent this covers the existing variety of integrable systems.

In this Letter, we first introduce the framework of bidifferential calculus, we then construct its connection with Lax integrability by assuming certain forms of linear maps and give some examples involving a class of discrete equations. Finally, some discussions and conclusions are presented.

First, we briefly recall the basic theory of bidifferential calculus. We refer the reader to Ref. [3] for more details.

Suppose that $\mathcal{A}$ is an associative algebra over $\mathcal{C}$. A graded algebra over $\mathcal{A}$ is a $\mathbb{N}_0$-graded associative algebra $\Omega(\mathcal{A}) = \bigoplus_{s\in\mathbb{N}_0}\Omega^s$, where $\Omega^0(\mathcal{A}) = \mathcal{A}$. A bidifferential graded algebra or bidifferential calculus is a graded algebra Ω equipped with two linear maps d and $\bar{d}$: $\Omega^s \to \Omega^{s+1}$, with the properties

$$d^2 = 0, \quad \bar{d}^2 = 0, \quad d\bar{d} + \bar{d}d = 0, \tag{1}$$

and the graded Leibniz rule

$$\begin{aligned} d(\omega\omega') &= (d\omega)\omega' + (-1)^s\omega d\omega', \\ \bar{d}(\omega\omega') &= (\bar{d}\omega)\omega' + (-1)^s\omega\bar{d}\omega', \end{aligned} \tag{2}$$

for all $\omega \in \Omega^s$ and $\omega' \in \Omega$.

For any $\mathcal{A}$, the corresponding graded algebra is given by

$$\Omega = \mathcal{A} \otimes \wedge\left(\mathbb{C}^N\right),$$

where $\wedge(\mathbb{C}^N)$ denotes the exterior algebra of $\mathbb{C}^N (N > 1)$ and further the max grade is N. Defining graded derivations d and $\bar{d}$ on $\mathcal{A}$, they extend in an obvious way to Ω such that the graded Leibniz rule holds. Here, element ζ_i is an $\mathcal{A}$ left-module basis in $\wedge(\mathbb{C}^N)$ and treated as constants with respect to d and $\bar{d}$, i.e. $d\zeta_i = \bar{d}\zeta_i = 0$. The relations between basis and elements in $\mathcal{A}$ are specified by $\zeta_i\zeta_j + \zeta_j\zeta_i = 0$, $[\zeta_i, f] = 0$, $f \in \mathcal{A}$. In this study, we only consider the common case $N = 2$.

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To derive integrable equations, one can employ a dressing transformation which maps the bidifferential calculus $(\Omega, d, \bar{d})$ to $(\Omega, d, \bar{D})$ admitting

$$\bar{D}^2 = 0, \quad d\bar{D} + \bar{D}d = 0. \tag{3}$$

Introducing the relations $\bar{D} = \bar{d} - d\phi, \quad \phi \in \mathcal{A}$ and $\bar{D} = (\bar{d}g)g^{-1}$, invertible $g \in \mathcal{A}$ we obtain the remain condition

$$\bar{d}d\phi = d\phi d\phi, \tag{4}$$
$$d\left[(\bar{d}g)g^{-1}\right] = 0, \tag{5}$$

which turns out many integrable equations by suitably choosing d and $\bar{d}$. Meanwhile

$$(\bar{d}g)g^{-1} = d\phi \tag{6}$$

can be viewed as a Miura transformation between Eqs. (4) and (5).

In the following, we consider Lax pairs for integrable equations expressed in the forms of (4) or (5). As far as we know, Eq. (4) gives the integrability condition of the linear equation

$$\bar{d}\,\Psi = (d\phi)\,\Psi + \lambda d\,\Psi, \tag{7}$$

which reduces to the Miura transformation (6) if $\lambda = 0$, while Eq. (5) is the integrability condition of the linear equation

$$\bar{d}\,\Psi = [(\bar{d}g)g^{-1}]\,\Psi + \lambda d\,\Psi.$$

However, the linear equations usually involve operators, Lax pairs cannot be derived immediately and we should transform operator Ψ to a function form.

Taking the KdV equation as an example, Let $\mathcal{A}_0$ be the space of smooth complex functions, and $\mathcal{A}$ the noncommutative algebra generated by the elements of $\mathcal{A}_0$ and the partial derivative ∂_x, ∂_t. Then in the frame of bidifferential calculus, $\Omega = \mathcal{A}\otimes\wedge\left(\mathbb{C}^2\right)$, where d and $\bar{d}$ are defined as

$$df = -[\partial_x, f]\zeta_1 - [3\partial_x^2, f]\zeta_2,$$
$$\bar{d}f = [\partial_x^2, f]\zeta_1 + [\partial_t + 4\partial_x^3, f]\zeta_2.$$

Equation (4) becomes the celebrated KdV equation

$$u_t + u_{xxx} + 6uu_x = 0$$

with the substitution $u = \phi_x$. The linear system (7) reads

$$\Psi_{xx} + 2\,\Psi_x\partial_x + u\,\Psi + \lambda\,\Psi_x = 0,$$
$$\Psi_t + 4\,\Psi_{xxx} + 6u\,\Psi_x + 3u_x\,\Psi + 3\lambda(\,\Psi_{xx} + 2\,\Psi_x\partial_x)$$
$$+ (12\,\Psi_{xx} + 6u\,\Psi + 12\,\Psi_x\partial_x)\partial_x = 0,$$

which is difficultly used to derive the Lax pair.

Therefore, it makes sense to connect Eqs. (4) and (5) in bidifferential calculus with its Lax pair directly in functional environment.

First, we construct the following d and $\bar{d}$. For simplicity, we introduce that $\mathrm{Mat}(N_1, N_2, \mathcal{C})$ denotes the set of $N_1 \times N_2$ matrices over $\mathcal{C}$, and $\mathrm{Mat}_{N_0}(\mathcal{C})$ denotes the set of $\oplus_{N_1,N_2\geq N_0}\mathrm{Mat}(N_1, N_2, \mathcal{C})$. More concretely, supposing that $\mathcal{B}_0$ is an algebra consisting of defined functions, we extend $\mathcal{B}_0$ to algebra $\mathcal{B} = \mathcal{L}(\mathcal{B}_0)$ by adding related partial derivatives or shifts. Hence we regard $\mathcal{B}_0$ as a subalgebra of $\mathcal{B}$. Further, to consider bidifferential calculations in functional environment we introduce forms of $L \circ \Psi$ to denote operator L acting on Ψ and distinguish it from the operator calculation. For example, $\partial_x \circ \Psi = \Psi_x$ while $\partial_x\,\Psi = \Psi_x + \Psi\partial_x$.

Theorem 1. Define $\mathcal{A}_0 = Mat_{N_0}(\mathcal{B}_0)$ for some $N_0 \in \mathbb{N}$. Suppose $\Omega = \mathcal{A}\otimes\wedge\left(\mathbb{C}^2\right)$ and $\mathcal{A} = \mathcal{L}(\mathcal{A}_0) = Mat_{N_0}(\mathcal{B})$. For any $f \in \Omega$ we assume

$$df = [A, f]\zeta_1 + [B, f]\zeta_2, \quad \bar{d}f = [C, f]\zeta_1 + [D, f]\zeta_2, \tag{8}$$

where A, B, C, D are linear operators independent of variables such that d and $\bar{d}$ preserve the size of matrices. When the dimensions for matrices do not match, we set the matrix product $XY = 0$. Then $(\Omega, d, \bar{d})$ is a bidifferential graded algebra if and only if

$$[A, B] = 0, \quad [C, D] = 0, \quad [A, D] = [B, C].$$

Proof. First, it is easy to detect that d and $\bar{d}$ satisfy the graded Leibniz rule. In respect of bidifferential graded algebra $(\Omega, d, \bar{d})$, for any $f \in \mathcal{A}$ we have $d^2 f = [[A, B], f]\zeta_1\zeta_2 = 0$, which infers $[A, B] = kI$ with constant k. Since the elements in A, B are commutative, we have $\mathrm{tr}(kI) = \mathrm{tr}([A, B]) = 0$, thus k should be zero. As a consequence

$$d^2 = 0 \Longleftrightarrow [A, B] = 0, \tag{9a}$$

and in the same way it follows that

$$\bar{d}^2 = 0 \Longleftrightarrow [C, D] = 0, \tag{9b}$$
$$d\bar{d} + \bar{d}d = 0 \Longleftrightarrow [A, D] = [B, C]. \tag{9c}$$

Theorem 2. For a bidifferential graded algebra $(\Omega, d, \bar{d})$ defined in theorem 1 with linear maps (8), the equation $\bar{d}d\phi = d\phi d\phi, \ \phi \in \mathcal{A}$, has the following Lax pair

$$C \circ \Psi = ([A, \phi] + \lambda + \nu A) \circ \Psi,$$
$$D \circ \Psi = ([B, \phi] + \mu + \nu B) \circ \Psi, \tag{10}$$

meanwhile the equation $d\left[(\bar{d}g)g^{-1}\right] = 0$ with invertible matrix $g \in \mathcal{A}$ has the following Lax pair

$$(gCg^{-1}) \circ \Psi = (\lambda + \nu A) \circ \Psi,$$
$$(gDg^{-1}) \circ \Psi = (\mu + \nu B) \circ \Psi. \tag{11}$$

Here λ, μ, ν are arbitrary constants, $\Psi \in \mathcal{A}_0$ is a matrix (or scalar) of functions.

Proof. By making use of (9), in a simple way we

can calculate

$$\begin{aligned}&\bar{d}d\phi = d\phi d\phi\\ \longrightarrow&[[A,\phi],[B,\phi]] = [C,[B,\phi]] - [D,[A,\phi]]\\ \xrightarrow{(9b)}&[[A,\phi]-C,[B,\phi]-D] = 0\\ \xrightarrow{(9a)(9c)}&[[A,\phi]-C+\lambda+\nu A,[B,\phi]\\ &-D+\mu+\nu B] = 0.\end{aligned} \tag{12}$$

Therefore, Eq. (4) is equivalent to Eq. (12), and it is easy to note that (12) is the integrability condition of linear system (10).

From Eq. (12), it is apparent that if we redefine C and D as $C+\nu A$ and $D+\nu B$, the resulting equation remains invariant.

By a similar calculation, it turns out that

$$\begin{aligned}&d\left[(\bar{d}g)g^{-1}\right] = 0\\ \longrightarrow&[C,[B,g]] - [D,[A,g]]\\ &= [C,g]g^{-1}[B,g] - [D,g]g^{-1}[A,g]\\ \xrightarrow{(9c)}&AgD - BgC + gCg^{-1}Bg - gDg^{-1}Ag = 0\\ \longrightarrow&[A,gDg^{-1}] = [B,gCg^{-1}]\\ \xrightarrow{(9a)(9b)}&[\nu A - gCg^{-1} + \lambda, \nu B - gDg^{-1} + \mu] = 0.\end{aligned} \tag{13}$$

Obviously Eq. (13) is the integrability condition of linear system (11).

We can also obtain (11) from Miura transformation $(\bar{d}g)g^{-1} = d\phi$. Substituting

$$[A,\phi] = [C,g]g^{-1}, \quad [B,\phi] = [D,g]g^{-1}, \tag{14}$$

into system (10), we have

$$\begin{aligned}C\circ\Psi &= \left([C,g]g^{-1} + \lambda + \nu A\right)\circ\Psi,\\ D\circ\Psi &= \left([D,g]g^{-1} + \mu + \nu B\right)\circ\Psi,\end{aligned}$$

which is equivalent to system (11).

As a whole, once Eq. (4) or (5) is solvable equipped with a bidifferential graded algebra $(\Omega, d, \bar{d})$ in the form of (8), one is ready to give the corresponding Lax pair (10) or (11). Turning back to the previous example of the KdV equation, substituting

$$A = -\partial_x, \quad B = -3\partial_x^2, \quad C = \partial_x^2, \quad D = \partial_t + 4\partial_x^3,$$

into Eq. (10), we have the Lax pair

$$\begin{aligned}&\Psi_{xx} = (\lambda - u)\Psi - \nu\Psi_x,\\ &\Psi_t = -4\Psi_{xxx} - 6u\Psi_x - 3u_x\Psi + \mu\Psi - 3\nu\Psi_{xx}.\end{aligned}$$

Furthermore, if we choose

$$\begin{aligned}&A = \frac{3}{2}(\partial_y + \partial_x^2), \quad B = \partial_x,\\ &C = \partial_t - \partial_x^3, \quad D = \frac{1}{2}(\partial_y - \partial_x^2),\end{aligned}$$

Eq. (4) reduces to the KP equation

$$\left(u_t - \frac{1}{4}u_{xxx} - 3uu_x\right)_x - \frac{3}{4}u_{yy} = 0$$

with the substitution $u = \phi_x$. Then the corresponding Lax pair reads

$$\begin{aligned}&\Psi_y = \Psi_{xx} + 2(u+\lambda)\Psi + 2\nu\Psi_x,\\ &\Psi_t = \Psi_{xxx} + 3u\Psi_x + \frac{3}{2}(u_x + (\partial_x^{-1}u_y) + \mu)\Psi\\ &\quad + \frac{3}{2}\nu(\Psi_y + \Psi_{xx}).\end{aligned}$$

Assume

$$A = P, \quad B = \partial_t, \quad C = \partial_x, \quad D = P,$$

where

$$P = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

Eq. (5) reads

$$g^{-1}PgP - Pg^{-1}Pg + \left(g^{-1}g_t\right)_x = 0.$$

If we take

$$g = \begin{pmatrix}\cos\left(\frac{u}{2}\right) & -\sin\left(\frac{u}{2}\right)\\ \sin\left(\frac{u}{2}\right) & \cos\left(\frac{u}{2}\right)\end{pmatrix},$$

with $u \in \mathcal{B}_0$, the equation is transformed to the sine-Gordon equation

$$u_{xt} = \sin(u).$$

Consequently using theorem 2, the corresponding Lax pair takes the form

$$\begin{aligned}\Psi_x &= \left(g_x g^{-1} + \lambda + \nu P\right)\Psi = \begin{pmatrix}\lambda+\nu & -\frac{1}{2}u_x\\ \frac{1}{2}u_x & \lambda\end{pmatrix}\Psi,\\ \Psi_t &= \frac{1}{\nu}\left(gPg^{-1} - \mu\right)\Psi\\ &= \frac{1}{2\nu}\begin{pmatrix}\cos(u)+1-2\mu & \sin(u)\\ \sin(u) & -\cos(u)+1-2\mu\end{pmatrix}\Psi.\end{aligned}$$

If we let $\nu = -2\lambda$ and $\mu = \frac{1}{2}$, it can be reduced to the familiar Lax pair.

In fully discrete cases, suppose that linear operator $S_i (1 \le i \le 4)$ consists of shift operators or identity admitting the property $S_i f S_i^{-1} = S_i \circ f$ and $S_i S_j = S_j S_i (1 \le i, j \le 4)$. Here we set $f_{i^+} = S_i \circ f$ and $f_{i^-} = S_i^{-1} \circ f$. On the basis of theorem 2, the linear operators are usually in the following forms

$$A = P_1 S_1, \quad B = P_2 S_2, \quad C = P_3 S_3, \quad D = P_4 S_4, \tag{15}$$

where $S_1 S_4 = S_2 S_3$ and P_i $(1 \le i \le 4)$ is a scalar or constant square matrix satisfying

$$[P_1, P_2] = [P_3, P_4] = [P_1, P_4] - [P_2, P_3] = 0. \tag{16}$$

Equation (4) and (5) can be rewritten as

$$\begin{aligned}&(P_1\varphi_{1^+} - \varphi P_1 - P_3)(P_2\varphi_{2^+3^+} - \varphi_{3^+}P_2 - P_4)\\ &= (P_2\varphi_{2^+} - \varphi P_2 - P_4)(P_1\varphi_{1^+4^+} - \varphi_{4^+}P_1 - P_3)\end{aligned} \tag{17}$$

by transformation $\phi = \varphi S_1^{-1} S_3$ and

$$P_1 g_{1^+} P_4 - P_2 g_{2^+} P_3 + g(P_3 g_{3^+}^{-1} P_2 - P_4 g_{4^+}^{-1} P_1) g_{2^+3^+} = 0 \tag{18}$$

respectively. Upon theorem 2, Eq. (17) has Lax pair

$$\begin{aligned} &(P_1\varphi_{1^+} - \varphi P_1 - P_3)\,\Psi_{3^+} + \lambda\Psi + \nu P_1\,\Psi_{1^+} = 0, \\ &(P_2\varphi_{2^+} - \varphi P_2 - P_4)\,\Psi_{4^+} + \mu\Psi + \nu P_2\,\Psi_{2^+} = 0, \end{aligned} \tag{19}$$

while Eq. (18) has Lax pair

$$\begin{aligned} g P_3 g_{3^+}^{-1}\,\Psi_{3^+} &= \lambda\Psi + \nu P_1\,\Psi_{1^+}, \\ g P_4 g_{4^+}^{-1}\,\Psi_{4^+} &= \mu\Psi + \nu P_2\,\Psi_{2^+}. \end{aligned} \tag{20}$$

By choosing suitable S_i and P_i, Eq. (17) reduces to the discrete KP equation[11,18] and the discrete NLS system.[6] Furthermore by making use of Miwa shift operators, AKNS hierarchies[5] can also be derived from Eq. (17).

In summary, the Lax pair of equations in bidifferential calculus has been constructed directly in functional environment, which clearly explains the connection between bidifferential calculus and Lax integrability. Theorem 2 claims once the linear operators A, B, C, D are defined, one can immediately give the corresponding Lax pair of the Eqs. (4) and (5). That is to say, we can use bidifferential calculus to find new integrable equations. However these linear operators are difficult to choose such that the operator Eq. (4) or (5) can be shifted to the respective PDDEs by certain transformations and we even wonder the existence of C, D for any given A, B. Hence how to choose the valid operators in the framework has always been the biggest hurdle and attracts us a great deal of interest. This letter proposes a class of discrete Eqs. (17) and (18) involving the discrete NLS equation and discrete KP equation. It is noted that for shift operators $\mathbb{S}$ and $\mathbb{T}$, the product $S = \mathbb{S}\mathbb{T}$ satisfies $SfS^{-1} = S \circ f$, while $S = \mathbb{S} + \mathbb{T}$ does not possess this property. We wonder if there are any other discrete equations if we extend the linear operators to polynomial of shift operators. Dealing with continuous cases, we have $[\partial_x^n, f] = \sum_{i=1}^{n} C_n^i f_{ix} \partial_x^{n-i}$, where $C_n^i = \frac{n!}{(n-i)!i!}$ and f_{ix} denotes i-th order derivative of f, and it is much more complicated to reduce operator equations in continuous cases. In the future, we will discuss low order cases in two dimensional evolution equations $u_t = F(\partial_x^{-1}u, u, u_x, \ldots, u_{nx})$ and make classifications in the form

$$\begin{aligned} df &= \Big[\sum_{i=1}^{n_1} a_i \partial_x^i, f\Big]\zeta_1 + \big[b_1\partial_x, f\big]\zeta_2, \\ \bar{d}f &= \Big[\partial_t + \sum_{i=1}^{n} c_i \partial_x^i, f\Big]\zeta_1 + \Big[\sum_{i=1}^{n-n_1} d_i \partial_x^i, f\Big]\zeta_2, \end{aligned}$$

with a determined number n.

Actually, not all the integrable systems can be inserted into the frame of bidifferential calculus. For almost all of the known integrable systems, the corresponding Lax pair is known and once it is presented in the form (10) or (11), which should be described by bidifferential calculus. As a whole, our aim is to search for more models that are applicable to bidifferential calculus and we obtain more characters in this framework.

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