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Abundant Traveling Wave Structures of (1+1)-Dimensional Sawada–Kotera Equation: Few Cycle Solitons and Soliton Molecules

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Traveling wave solutions have been well studied for various nonlinear systems. However, for high order nonlinear physical models, there still exist various open problems. Here, travelling wave solutions to the well-known fifth-order nonlinear physical model, the Sawada–Kotera equation, are revisited. Abundant travelling wave structures including soliton molecules, soliton lattice, kink-antikink molecules, peak-plateau soliton molecules, few-cycle-pulse solitons, double-peaked and triple-peaked solitons are unearthed.

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In the recent three decades, travelling wave solutions are well studied for various nonlinear systems. Especially, for lower order nonlinear systems, all the travelling waves can be simply obtained by some elliptical integrals. For instance, all travelling wave solutions to the nonlinear equation

$$u_t + 6auu_x + 6bu^2u_x + u_{xxx} = 0 \tag{1}$$

can be derived by a combination of the Korteweg-de Vries (KdV, $b = 0, a = 1$) and the modified KdV (mKdV, $a = 0, b = 1$) equations, expressed by the following elliptic integral

$$\pm \int \frac{\mathrm{d}u}{\sqrt{c_0 + c_1 u - cu^2 - 2au^3 - bu^4}} = x + ct - x_0 \tag{2}$$

with three integral constants c_0, c_1 and x_0.

However, for higher order nonlinear equations, e.g., the Sawada–Kotera [SK, also named Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK)] equation

$$u_t + u_{xxxxx} + 15uu_{xxx} + 45u^2u_x + 15u_xu_{xx} = 0, \tag{3}$$

only some special travelling waves are known. Some types of singular and nonsingular travelling wave solutions to the SK equation have been studied by many researchers.[1,2]

The SK equation has been derived from different physical fields in its general extended forms.[3–7] The SK equation appeared and was firstly introduced as an integrable system independently in Refs. [8,9]. Many interesting aspects related to integrability and exact solutions to Eq. (3) have been investigated before. For instance, the recursion operator of the SK equation was given by Fuchssteiner and Oevel.[10] The infinitely many conservation laws were studied by Satsuma and Kaup.[11] The multiple soliton solutions and inverse scattering form of an extension of the SK equation were given by Ito.[12] The existence of multi-periodic wave solutions was proved by Hirota and Ito.[13] The Darboux and/or Darboux-Levi transformations have been studied by authors in Refs. [14,15]. The (2+1)-dimensional SK equation, discrete SK equation and super symmetric SK equation were also studied in Refs. [16–20] The inverse recursion operator and twelve sets of infinitely many local and nonlocal symmetries were given by Lou in Ref. [21]. The localization of a special nonlocal symmetry was studied in Ref. [22]. Some types of nonlocal Alice-Bob SK models were also introduced in Ref. [23].

Soliton molecules, bound states of solitons, have luckily been observed experimentally in optics[24–28] and predicted numerically in Bose–Einstein condensates.[29] It is interesting that some types of single soliton molecules (SM) exhibit the same property of travelling wave.[17,18,30–33] Nevertheless, it is not easy to find SM solutions directly via the travelling wave approach.

In the (1+1)-dimensional case, a general travelling wave possesses the form

$$u = U(z), \quad z = kx + \omega t, \tag{4}$$

which moves along the x-axis direction with velocity $-\omega/k$.

Substituting Eq. (4) into Eq. (3) and integrating the result equation with respect to z once, we can see that the general travelling wave of the SK equation

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satisfies the following fourth-order ordinary differential equation (ODE):

$$k^5U_{zzzz}+15k^3UU_{zz}+\omega U+15kU^3+C=0, \quad (5)$$

with constants k,ω and an integral constant C. Equation (5) is equivalent to the following variable coefficient third-order ODE:

$$k^5\Big(W^3W_{UU}+\frac{1}{2}W^2W_U^2\Big)_U+15k^3UWW_U +15kU^3+\omega U+C=0,\ W=W(U)=U_z. \quad (6)$$

The general solution to the travelling wave Eq. (5), or equivalently Eq. (6), is still open. The known special solutions to Eq. (5) are the special cases of

$$\pm\int\frac{6k^2dU}{\sqrt{6k(-12kU^3\pm\sqrt{-k\omega}U^2+C)}}=z-z_0. \quad (7)$$

To find more significant travelling wave solutions to the SK equation, we transform the travelling wave equation into a bilinear form

$$[k^5D_z^6+15k^3u_0D_z^4+(\omega+45ku_0^2)]F\cdot F=0 \quad (8)$$

using the transformation

$$u=u_0+2[\ln(F)]_{xx}, \quad F=F(kx+\omega t)=F(z). \quad (9)$$

The bilinear operator D_z is defined by

$$D_z^nF\cdot G=(\partial_z-\partial_{z'})^nF(z)G(z')\big|_{z'=z}.$$

To solve the bilinear Eq. (8), we assume

$$F(z)=G(z)+H(z). \quad (10)$$

Under the assumption (10), the bilinear form (8) becomes

$$[k^5D_z^6+15k^3u_0D_z^4+(\omega+45ku_0^2)](G\cdot G+2G\cdot H+H\cdot H)=0. \quad (11)$$

To solve Eq. (11), we introduce the following restrictions

$$G_z^2=a_0+a_1G+a_2G^2, \quad (12)$$
$$H_z^2=b_0+b_1H+b_2H^2, \quad (13)$$

with undetermined constants a_i,b_i $(i=0,1,2)$.

Substituting Eqs. (12) and (13) into Eq. (11) and vanishing the coefficients of the different differential power polynomials of G and H respectively, we obtain the following five constraints:

$$\omega=-45ku_0^2-30k^3u_0(a_2+b_2)-k^5(a_2+3b_2)(b_2+3b_2), \quad (14)$$
$$15(a_2-b_2)^2u_0+2k^2(a_2^2-b_2^2)=0, \quad (15)$$
$$15u_0[a_1(a_2-4b_2)-b_1(b_2+2a_2)]+k^2a_1(2a_2+3b_2)\cdot(a_2-4b_2)-k^2b_1(3a_2^3+10a_2b_2+2b_2^2)=0, \quad (16)$$
$$15u_0[a_1(a_2+2b_2)+b_1(4a_2-b_2)]+k^2a_1(2a_2^2+10a_2b_2+3b_2^2)+k^2b_1(3a_2+2b_2)(4a_2-b_2)=0, \quad (17)$$
$$[120u_0(a_0-b_0)+4k^2a_0(13a_2+3b_2)-4k^2b_0(3a_2+13b_2)](a_2-b_2)-45u_0(a_1+b_1)^2-15k^2(a_1+b_1)(a_1b_1+b_1b_2)=0, \quad (18)$$

on nine parameters ω,k,u_0,a_i,b_i $(i=0,1,2)$. To write down the conditions (15)–(18), we have applied Eq. (14). After some detailed calculations, we obtain four independent solutions to Eqs. (14)–(18).

Case 1. First type of solitons and soliton molecules induced/excited by negative background $(u_0\to -ak^2<0)$:

$$a_2=b_2=-3k^{-2}u_0=3a>0, \quad \omega=-9a^2k^5,$$
$$G(z)=\frac{k}{6u_0}\Big(a_1k\pm\sqrt{a_1^2k^2+12a_0u_0}\cosh[\sqrt{3a}k^{-1}(z-z_1)]\Big),$$
$$H(z)=\frac{k}{6u_0}\Big(b_1k\pm\sqrt{b_1^2k^2+12b_0u_0}\cosh[\sqrt{3a}k^{-1}(z-z_2)]\Big).$$

In this case, the final solution for the field u can be rewritten as

$$u=-ak^2+6ac\frac{c+\cosh[\sqrt{3a}k(x-9a^2k^4t-x_0)]}{[c\cosh[\sqrt{3a}k(x-9a^2k^4t-x_0)]+1]^2} \quad (19)$$

after using some redefinitions of constants, where $a>0,c,k$ and x_0 are arbitrary constants. After analysis, the analytic condition of solution (19) is filed out to be $c>0$ or $c<-1$. It is clear that when the background is ruled out $(a=0)$, solution (19) becomes a trivial vacuum solution $u=0$. Thus, we call this kind of soliton as the background induced/excited soliton. From the solution expression (19), we know that this type of soliton molecule can only move to right because $\omega/k=-9a^2k^4<0$.

Solution (19) possesses several different types of structures under different selections of the constant c. Figure 1 displays four standard structures, the soliton molecule (SM) for $0<c\ll 1/2$ [Fig. 1(a)], the M-shape double-peak soliton for $c<1/2$ [Fig. 1(b)], the kink-antikink molecule (KAKM) or plateau soliton with $c=1/2$ [Fig. 1(c)] and the single-peak soliton for $c>1/2$ [Fig. 1(d)]. For the plateau-like soliton, one can readily find $u_{x'}=u_{x'x'}=u_{x'x'x'}=0$ for $x=x'=9a^2k^4t+x_0$, which means that the soliton solution (19) is quite flat at the center of the plateau.

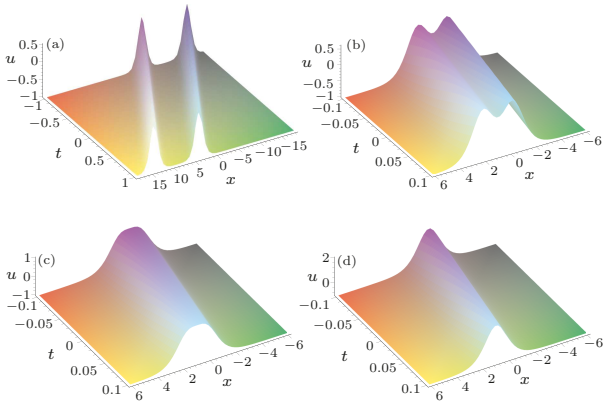

Fig. 1. First type of background induced soliton and soliton molecules described by solution (19). (a) Soliton molecule (19) with $c = 1/4000$, $k = a = 1$ and $x_0 = 0$. (b) M-shape soliton (19) with $c = 1/4$, $k = a = 1$ and $x_0 = 0$. (c) Kink-antikink molecule (19) with $c = 1/2$, $k = a = 1$ and $x_0 = 0$. (d) Single-peak soliton (19) with $c = 1$, $k = a = 1$ and $x_0 = 0$.

Case 2. Periodic waves induced/excited by positive background ($u_0 \to ak^2 > 0$):

$$a_2 = b_2 = -3k^{-2}u_0 = -3a < 0, \quad \omega = -9a^2k^5.$$

In this case, the final solution for the field u can be simplified to

$$u = ak^2 - 6ca\frac{c + \cos[\sqrt{3a}k(x - 9a^2k^4\,t - x_0)]}{\left[c\cos[\sqrt{3a}k(x - 9a^2k^4\,t - x_0)] + 1\right]^2} \tag{20}$$

with arbitrary constants $a > 0, c, k$ and x_0.

The analytic condition for the periodic wave is $|c| < 1$. For $c \geq 1$, the singular solution (20) can be called the complexiton solution.[34]

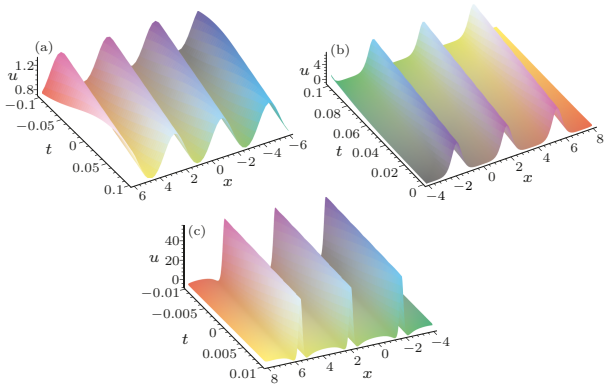

Fig. 2. Periodic waves induced by positive background described by the solution (20). (a) Periodic wave (20) with $c = 1/20$, $k = a = 1$ and $x_0 = 0$. (b) Soliton lattice (20) having U-shape bottom with $c = 1/2$, $k = a = 1$ and $x_0 = 0$. (c) Soliton lattice (20) having W-shape bottom with $c = 0.9$, $k = a = 1$ and $x_0 = 0$.

Similar to Case 1, the analytic periodic wave (20) with $|c| < 1$ is induced/excited by background $ak^2 \geq 0$, which also possesses several different types of structures according to different selections of the constant c. Figure 2 depicts some typical structures, i.e., the usual periodic waves for $|c| \ll 1/2$ [Fig. 2(a)], the soliton lattice with U-shape bottom for $|c| \leq 1/2$ [Fig. 2(b) for $c = 1/2$] and the soliton lattice with W-shape bottom for $1/2 < |c| < 1$ [Fig. 2(c)]. In the critical case for $c = 1/2$, the centers between two solitons located at $x = y_n = 9a^2k^4t + x_0 + 2n\pi (n = 0, \pm1, \pm2, \ldots)$ are quite flat with the property $u_{y_n} = u_{y_n y_n} = u_{y_n y_n y_n} = 0$.

Case 3. Standard solitons with and without background ($u_0 \to ak^2$):

$$a_1 = -b_1, \ a_2 = b_2, \ \omega = -16b_2^2k^5 - 16b_2k^3u_0 - 45ku_0^2,$$
$$G(z) = \frac{b_1}{2b_2} \pm \frac{\sqrt{b_1^2 - 4a_0b_2}}{2b_2}\cosh[\sqrt{b_2}(z - z_1)],$$
$$H(z) = -\frac{b_1}{2b_2} \pm \frac{\sqrt{b_1^2 - 4b_0b_2}}{2b_2}\cosh[\sqrt{b_2}(z - z_2)].$$

In this case the final solution for the field u can be rewritten as

$$u = u_0 + 2k^2\mathrm{sech}^2[kx - k(16k^4 + 60k^2u_0 + 45u_0^2)t - z_0], \tag{21}$$

which is deduced after finishing some redefinitions of the arbitrary parameters.

From the solution expression (21), we realize that it is just the standard soliton solution known before that it exists with and without background. The positive background ($u_0 > 0$) always accelerates solitons. The negative background ($u_0 < 0$) accelerates the long waves with $1/k > \sqrt{\frac{-4}{3u_0}}$ and decelerates the short waves with $1/k < \sqrt{\frac{-4}{3u_0}}$.

Case 4. Second type of solitons and soliton molecules:

$$b_0 = \frac{9a_1(a_1^2 - b_1^2)}{4a_2(a_1 + 9b_1)} - a_0\frac{9a_1 + b_1}{a_1 + 9b_1}, \ b_2 = -\frac{a_2b_1}{a_1},$$
$$u_0 = \frac{2a_2k^2}{15a_1}(b_1 - a_1), \ \omega = \frac{a_2^2k^5}{5a_1^2}(a_1^2 + b_1^2 + 18a_1b_1),$$
$$G(z) = -\frac{a_1}{2a_2} \pm \frac{\sqrt{a_1^2 - 4a_0a_2}}{2a_2}\cosh\left(\sqrt{a_2}(z - z_1)\right),$$
$$H(z) = \frac{a_1}{2a_2} \pm \frac{\sqrt{a_1^2 - 4a_0a_2}}{2a_2}\sqrt{\frac{a_1(9a_1 + b_1)}{b_1(a_1 + 9b_1)}}$$
$$\cdot \cosh\left(\sqrt{\frac{-a_2b_1}{a_1}}(z - z_1)\right).$$

In this case, the final result for the field u reads

$$\begin{aligned} u &= u_0 + 2\big[\ln[b\sqrt{a^2 + 9b^2}\cosh(a\xi) \\ &\quad \pm a\sqrt{b^2 + 9a^2}\cos(b\eta)]\big]_{xx} \\ &= u_0 + 2abk^2\,A, \end{aligned} \tag{22}$$

where

$$A = \{8ab(b^2-a^2) \pm \sqrt{(a^2+9b^2)(b^2+9a^2)}[(a^2-b^2) \cdot \cosh(a\xi)\cos(b\eta) + 2ab\sinh(a\xi)\sin(b\eta)]\} \cdot \{[b\sqrt{a^2+9b^2}\cosh(a\xi) \pm a\sqrt{b^2+9a^2}\cos(b\eta)]^2\}^{-1},$$
$$\xi = kx+\omega t-\xi_0, \quad \eta = kx+\omega t-\eta_0,$$
$$u_0 = \frac{2k^2}{15}(b^2-a^2), \quad \omega = \frac{k^5}{5}(a^4+b^4+18a^2b^2).$$

For real parameters a and b, the analytical conditions for the solution (22) read

$$|b\sqrt{a^2+9b^2}| > |a\sqrt{b^2+9a^2}|, \ a \neq b, \tag{23}$$
$$\xi_0 - \eta_0 \neq (2n+1)\pi, \ n = 0, \pm1, \pm2, \ldots, \ a = b. \tag{24}$$

Under the nonsingular condition (23) or (24), the solution (22) reveals the so-called few-cycle-pulse soliton structure. Few-cycle-pulse solitons have been discovered in different physical fields especially in nonlinear optics.[35–40]

The few-cycle-pulse soliton (22) under the condition (23) is a soliton with a nonzero background because of $a \neq b$. Under the zero background condition $a = b$ the few-cycle-pulse soliton (22) under the condition (24) can be simplified to

$$u = \frac{4k^2\sinh(kx+4k^5t-\xi_0)\sin(kx+4k^5t-\eta_0)}{[\cosh(kx+4k^5t-\xi_0)+\cos(kx+4k^5t-\eta_0)]^2}. \tag{25}$$

Figure 3 displays four special few-cycle-pulse soliton structures described by solution (25) under special parameter selections: (a) $\{k=1, \xi_0=\eta_0=0\}$, (b) $\{k=1, \xi_0=0, \eta_0=-0.25\pi\}$, (c) $\{k=1, \xi_0=0, \eta_0=0.5\pi\}$ and (d) $\{k=1, \xi_0=0, \eta_0=0.9\pi\}$.

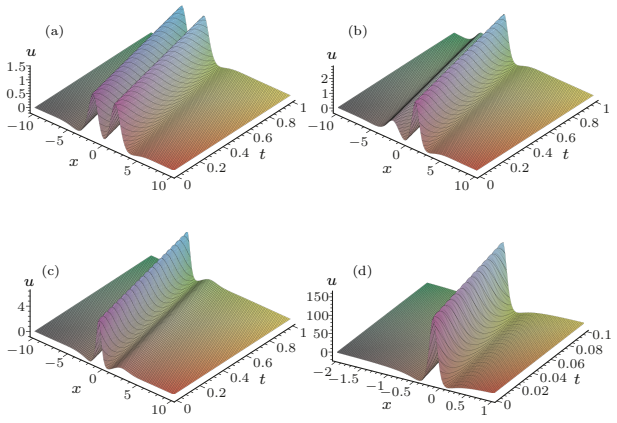

Fig. 3. Few-cycle-pulse soliton structures without background described by solution (25) with different parameter selections for (a) $\{k=1, \xi_0=\eta_0=0\}$, (b) $\{k=1, \xi_0=0, \eta_0=-0.25\pi\}$, (c) $\{k=1, \xi_0=0, \eta_0=0.5\pi\}$ and (d) $\{k=1, \xi_0=0, \eta_0=0.9\pi\}$.

If the parameter b is taken as an imaginary number, say, $b=\sqrt{-1}c$, the solution (22) changes to

$$u = -\frac{2k^2}{15}(a^2+c^2) + 2\big[\ln\big(c\sqrt{9c^2-a^2}\cosh(a\xi) \pm a\sqrt{9a^2-c^2}\cosh(c\eta)\big)\big]_{xx}, \tag{26}$$
$$\xi = kx+\omega t-\xi_0, \quad \eta = kx+\omega t-\eta_0,$$
$$\omega = \frac{k^5}{5}(a^4+c^4-18a^2c^2).$$

From the solution expression (26), we know that this type of soliton molecule is also excited/induced by background. However, this type of soliton molecule is different from the first type of soliton molecule described by (19) because it possesses completely different dispersion relations. The soliton molecule (19) is constituted only by right moving solitons. However, the soliton molecule (26) may be right moving for $9c^2-4\sqrt{5}c_2^2 < a^2 < 9c^2+4\sqrt{5}c_2^2$ and left moving for $a^2 > 9c^2+4\sqrt{5}c_2^2$ or $a^2 < 9c^2-4\sqrt{5}c_2^2$. Figure 4 shows some special plots of three right moving solitons and/or soliton molecules described by (26) with different parameters: (a) $\{k=c=1, \xi_0=-\eta_0=3, a=2.9\}$, (b) $\{k=c=1, \xi_0=0, \eta_0=0.67, a=2.99\}$ and (c) $\{k=c=1, \xi_0=\eta_0=0, a=2.99\}$.

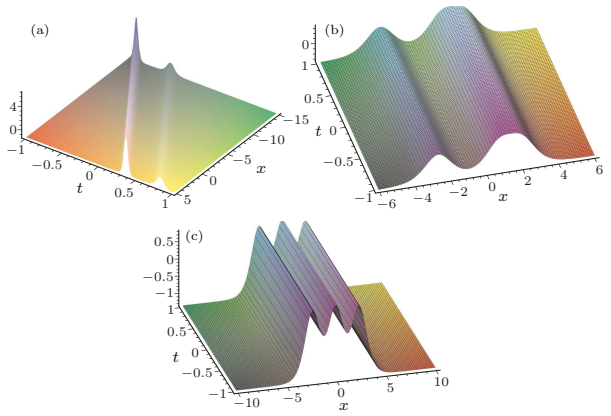

Fig. 4. Second type of soliton and soliton molecules (26) induced by background. (a) Soliton molecule (26) with $k=c=1$, $\xi_0=-\eta_0=3$ and $a=2.9$. (b) Peak-plateau soliton molecule (26) with $k=c=1$, $\xi_0=0$, $\eta_0=0.67$ and $a=2.99$. (c) Three peaked soliton (26) with $k=c=1$, $\xi_0=\eta_0=0$ and $a=2.99$.

In summary, for high-order nonlinear systems like the SK equation, there are abundant nonlinear wave structures. For the SK equation, in addition to the standard one-soliton solution (21), there are many kinds of solitons such as the few-cycle-pulse solitons, the double-peaked solitons, triple-peaked solitons, plateau solitons and soliton molecules. It should be emphasized that although we have obtained many new travelling wave structures for the SK Eq. (3), the general travelling wave solution of the model is still open.

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