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Modulational Instability of Trapped Two-Component Bose–Einstein Condensates *

Jian-Wen Zhou(周建文), Xiao-Xun Li(李晓旬), Rui Gao(高瑞), Wen-Shan Qin(秦文山), Hao-Hao Jiang(蒋浩浩), Tao-Tao Li(李涛涛), Ju-Kui Xue(薛具奎)**

College of Physics and Electronics Engineering, Northwest Normal University, Lanzhou 730070

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The modulational instability of two-component Bose–Einstein condensates (BECs) under an external parabolic potential is discussed. Based on the trapped two-component Gross–Pitaevskill equations, a time-dependent dispersion relation is obtained analytically by means of the modified lens-type transformation and linear stability analysis. It is shown that a modulational unstable time scale exists for trapped two-component BECs. The modulational properties—which are determined by the wave number, external trapping parameter, intra- and inter-species atomic interactions—are modified significantly. The analytical results are confirmed by direct numerical simulation. Our results provide a criterion for judging the occurrence of instability of the trapped two-component BECs in experiment.

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Research of Bose–Einstein condensates (BECs) in weakly interacting systems is expected to reveal new macroscopic quantum phenomena that can be understood from first principles, and may also increase our understanding of superconductivity and superfluidity in more complex systems. The spatial contraction of wave packets and the formation of a singularity in finite time, wave collapse, or the blowup of the wave packets are basic phenomena in nonlinear physics of wave systems. In recent years, BECs of ultracold atomic gases have captured interests of atomic, molecular, optical physics communities, and nonlinear wave communities.[1–3] This is due to the effective nonlinearity introduced at the lowest-order mean-field theory in a significant degree, which leads to the Gross–Pitaevskii (GP) equation.[4] Modulational instability (MI) is a very common type of nonlinear physics phenomenon. This phenomenon can be formed spontaneously in homogeneous medium and inhomogeneous medium, such as plasma physics, hydrodynamics, and nonlinear optics.[5,6] The MI in single-component BECs has been investigated in many earlier works.[6–8] As in other physical systems,[6,9] the single-component MI is only possible for the self-focusing (attractive) sign of the nonlinearity.[10,11] In attractive BECs, the formation of soliton trains is initiated by phase fluctuations via the MI.[11,12] In two-component BECs, which were first considered by Goldstein and Meystre,[13] the MI is possible even for repulsive interactions,[14,15] and the result is similar to the result known in nonlinear optics.[16] If the cross-phase-modulation mediated repulsion between the components is stronger than the self phase modulation (self-repulsion) of each component, then the MI does not create trains of bright solitons but rather creates domain walls that realize the phase separation in the immiscible binary BEC.[14,15,17–23] In contact-interacting BECs, the MI has been recently reported at a negative scattering length.[24,25] The MI of contact-interacting BECs differs from that of dipolar BECs because all of the lowest-lying modes are unstable and the favored wavelength for the instability is set to the mode that has the highest growth rate.[24] The localized nonlinear matter waves in the two-component BECs with time- and space-modulated nonlinearities are investigated.[26] The dynamics of bright and dark solitons in BECs with the time-dependent interatomic interaction in an expulsive parabolic and complex potential and how to control interactions between solitons are discussed.[27] The MI of two-component spin-orbit-coupled (SOC) BECs in the framework of coupled GP equations is studied in free space without the external trapping potential. The effects of the interaction parameters, Rabi coupling, and SOC on the MI are investigated.[28] However, the previous studies on the MI of two-component BECs concentrated in free space. The MI of two-component BECs in the presence of external potential is still unclear. In this Letter, we discuss the MI of two-component BECs under the external parabolic potential.

Based on the trapped two-component GP equations, a time-dependent dispersion relation is obtained by means of the modified lens-type transformation and linear stability analysis. According to the dispersion relation, the unstable condition, the local-instability growth rate, and particularly, an instability time scale are obtained.

At low temperatures, the MI is well described by the nonlinear GP equation, which has played a cen-

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**Corresponding author. Email: xuejk@nwnu.edu.cn

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tral role in understanding of BECs.[29,30] For two-component BECs trapped in a one-dimensional (1D) harmonic potential in the presence of two-body interatomic interactions, the system can be expressed by the dimensionless GP equations

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} &= -\frac{\partial^2\psi_1}{\partial x^2}+V(x)\psi_1+g_1|\psi_1|^2\psi_1+g_{12}|\psi_2|^2\psi_1,\\ i\frac{\partial\psi_2}{\partial t} &= -\frac{\partial^2\psi_2}{\partial x^2}+V(x)\psi_2+g_2|\psi_2|^2\psi_2+g_{12}|\psi_1|^2\psi_2, \end{aligned} \tag{1}$$

where ψ_j $(j=1,2)$ is the normalized macroscopic wave function, g_1, g_2 and g_{12} are the intra- and inter-species interaction coefficients, positive for repulsive interatomic interactions and negative for attractive ones.[31] This equation serves as a mean-field model in description of dynamics of cigar-shaped BECs with repulsive interatomic interactions trapped in the potential $V(x)$. The external parabolic potential has the form

$$V(x)=\Omega x^2, \tag{2}$$

where Ω (a real constant) expresses the normalized trapping frequency in the x direction. Wave function ψ_j, time t, and variables x are, respectively, normalized to $\sqrt{\frac{8\pi a_s\hbar}{mw_\perp}}$, the oscillation period $w_\perp^{-1}$, the harmonic oscillator length $a_\perp=\sqrt{\frac{\hbar}{mw_\perp}}$, and the potential $V(x)$ is measured in units of $a_\perp^2\hbar^2/8m$, with a_s being the s-wave scattering length, $w_\perp$ the harmonic frequency corresponding to the strong confinement in the cross-section, and m the mass of the atom. In Eq. (1), both the focusing and defocusing characteristics of the nonlinearity (which represents the attractive and repulsive nature of the inter-atomic interactions) are considered.[31] To investigate the MI of BECs, let us first perform a modified lens-type transformation, we set

$$\psi_j(t,x)=l^{-1}\exp(if(t)x^2)\varphi_j(\xi,\tau), \tag{3}$$

where $f(t)=-\frac{1}{2}\sqrt{\Omega}\tan(2\sqrt{\Omega}t)$, $l(t)=\cos(2\sqrt{\Omega}t)$, $\tau(t)=\tan(2\sqrt{\Omega}t)/(2\sqrt{\Omega})$, and $\xi(t,x)=x\sec(2\sqrt{\Omega}t)$. Then, Eq. (1) becomes

$$\begin{aligned} &i\frac{\partial\varphi_1}{\partial\tau}+\frac{\partial^2\varphi_1}{\partial\xi^2}-g_1|\varphi_1|^2\varphi_1-g_{12}|\varphi_2|^2\varphi_1\\ &+i\lambda(\tau)\varphi_1=0,\\ &i\frac{\partial\varphi_2}{\partial\tau}+\frac{\partial^2\varphi_2}{\partial\xi^2}-g_2|\varphi_2|^2\varphi_2-g_{12}|\varphi_1|^2\varphi_2\\ &+i\lambda(\tau)\varphi_2=0, \end{aligned} \tag{4}$$

where $\lambda(\tau)=\frac{1}{2}\sqrt{\Omega}\sin(4\sqrt{\Omega}t)$. Note that when $t=t_s=\pi/4\sqrt{\Omega}$, the transformation of Eqs. (3) and (4) is singular. However, Eq. (4) is only used to predict the occurrence of MI. As we will show by Eq. (15) the critical time scale for occurrence of MI is much less than $t_s=\pi/4\sqrt{\Omega}$. Hence, our model given by Eqs. (4) and (5) is rational for prediction of the MI. To examine the MI of BECs described by Eq. (4), considering the development of the small modulation $\delta\varphi_j$, we use the ansatz

$$\varphi_j=(\delta\varphi_j(\tau,\xi)+\varphi_{0j})\exp(-i\int\Delta_j(\tau)d\tau-\int\lambda d\tau), \tag{5}$$

where $\Delta_j(\tau)$ is a real time-dependent function representing the nonlinear frequency shift, φ_{0j} is a real constant, and $\delta\varphi_j$ is the amplitude of the perturbation. Substituting Eq. (5) into Eq. (4) and collecting terms in the zeroth and first order (linearization), we obtain

$$\begin{aligned} \Delta_1 &= (g_1\varphi_{01}^2+g_{12}\varphi_{02}^2)\exp(-2\int\lambda d\tau),\\ \Delta_2 &= (g_2\varphi_{02}^2+g_{12}\varphi_{01}^2)\exp(-2\int\lambda d\tau), \end{aligned} \tag{6}$$

$$\begin{aligned} &i\frac{\partial\delta\varphi_1}{\partial\tau}+\frac{\partial^2\delta\varphi_1}{\partial\xi^2}-[g_1\varphi_{01}^2(\delta\varphi_1+\delta\varphi_1{}^*)+g_{12}\varphi_{01}\varphi_{02}\\ &(\delta\varphi_2+\delta\varphi_2{}^*)]\exp(-2\int\lambda d\tau)=0,\\ &i\frac{\partial\delta\varphi_2}{\partial\tau}+\frac{\partial^2\delta\varphi_2}{\partial\xi^2}-[g_2\varphi_{02}^2(\delta\varphi_2+\delta\varphi_2{}^*)+g_{12}\varphi_{02}\varphi_{01}\\ &(\delta\varphi_1+\delta\varphi_1{}^*)]\exp(-2\int\lambda d\tau)=0, \end{aligned} \tag{7}$$

where $\delta\varphi_j{}^*$ is the complex conjugate of $\delta\varphi_j$. Letting $\delta\varphi_j=u_j+iv_j$, and separating the real and imaginary parts, then Eq. (7) is transformed into the following four coupled equations

$$\begin{aligned} &\frac{\partial u_1}{\partial\tau}+\frac{\partial^2 v_1}{\partial\xi^2}=0,\\ &\frac{\partial v_1}{\partial\tau}-\frac{\partial^2 u_1}{\partial\xi^2}+[2g_1\varphi_{01}^2u_1+2g_{12}\varphi_{01}\varphi_{02}u_2]\\ &\cdot\exp(-2\int\lambda d\tau)=0, \end{aligned} \tag{8}$$

$$\begin{aligned} &\frac{\partial u_2}{\partial\tau}+\frac{\partial^2 v_2}{\partial\xi^2}=0,\\ &\frac{\partial v_2}{\partial\tau}-\frac{\partial^2 u_2}{\partial\xi^2}+[2g_2\varphi_{02}^2u_2+2g_{12}\varphi_{01}\varphi_{02}u_1]\\ &\cdot\exp(-2\int\lambda d\tau)=0. \end{aligned} \tag{9}$$

Now, we consider that the variation of the perturbation obeys the expression $u_j=\cos(k\xi-\int\Lambda(\tau)d\tau)u_{0j}$, and $v_j=\sin(k\xi-\int\Lambda(\tau)d\tau)v_{0j}$, where $k\xi-\int\Lambda(\tau)d\tau$ is the modulation phase, k and Λ are, respectively, the wave number and the frequency of the modulation. By Eqs. (8) and (9), we can obtain the fourth order equation about eigenfrequency Λ,

$$\begin{aligned} &\Lambda^4-2k^2[k^2+(g_1\varphi_{01}^2+g_2\varphi_{02}^2)|\cos(2\sqrt{\Omega}t)|]\Lambda^2+k^4[(k^2\\ &+2g_1\varphi_{01}^2|\cos(2\sqrt{\Omega}t)|)(k^2+2g_2\varphi_{02}^2|\cos(2\sqrt{\Omega}t)|)\\ &-4g_{12}^2\varphi_{01}^2\varphi_{02}^2|\cos(2\sqrt{\Omega}t)|^2]=0. \end{aligned} \tag{10}$$

Then, we obtain the following explicitly time-dependent dispersion relation

$$\Lambda_{\pm}^2 = k^4\Big\{1+\frac{1}{k^2}[g_1\varphi_{01}^2+g_2\varphi_{02}^2 \pm\sqrt{4g_{12}^2\varphi_{01}^2\varphi_{02}^2+(g_1\varphi_{01}^2-g_2\varphi_{02}^2)^2}]|\cos(2\sqrt{\Omega}t)|\Big\}. \quad (11)$$

To discuss the MI more clearly, we write a more specific expression of dispersion relation as follows:

$$\Lambda_{\pm}^2 = \frac{1}{2}(B\pm\sqrt{B^2+4C}), \quad (12)$$

with

$$B = 2k^2[k^2+(g_1\varphi_{01}^2+g_2\varphi_{02}^2)|\cos(2\sqrt{\Omega}t)|],$$
$$C = -k^4[(k^2+2g_1\varphi_{01}^2|\cos(2\sqrt{\Omega}t)|)(k^2+2g_2\varphi_{02}^2 \cdot|\cos(2\sqrt{\Omega}t)|)-4g_{12}^2\varphi_{01}^2\varphi_{02}^2|\cos(2\sqrt{\Omega}t)|^2].$$

The expression given by Eq. (12) may be positive, negative, or complex, depending on the signs and magnitudes of the terms involved. The plane wave state is stable for all real k when $\Lambda^2 > 0$. When $\Lambda^2 < 0$, the instability sets in and the instability growth rate is defined as $\mathrm{Im}(\Lambda)$. In particular, when $C > 0$, the MI always sets in via the growth of the perturbations which are accounted for by $\Lambda_-^2 < 0$. A clear conclusion can be obtained from this discussion: when $C > 0$, the MI will always take place. To excite the MI, Λ must be complex with non-nil imaginary part, thus the right-hand side of Eq. (11) must be negative. We can obtain the local-instability growth rate $\mathrm{Im}(\Lambda)$. Considering $|\cos(2\sqrt{\Omega}t)| \le 1$, the condition for exciting the MI is simplified to

$$1+\frac{1}{k^2}\Big[g_1\varphi_{01}^2+g_2\varphi_{02}^2 \pm\sqrt{4g_{12}^2\varphi_{01}^2\varphi_{02}^2+(g_1\varphi_{01}^2-g_2\varphi_{02}^2)^2}\Big]<0, \quad (13)$$

and the local-instability growth rate is given by

$$\mathrm{Im}(\Lambda) = k^2\Big\{-1-\frac{1}{k^2}[g_1\varphi_{01}^2+g_2\varphi_{02}^2 \pm\sqrt{4g_{12}^2\varphi_{01}^2\varphi_{02}^2+(g_1\varphi_{01}^2-g_2\varphi_{02}^2)^2}]\Big\}^{1/2}. \quad (14)$$

Figures 1 and 2 show the instability growth rate of the system with $g_1 = g_2 = g$, $\varphi_{01} = \varphi_{02} = \varphi_0$. For the two-component BECs and for both the attractive and repulsive interactions, BECs are always susceptible to the MI. The unstable regions are gradually increasing with $|g_{12}|$ or $|g|$ (when $g < 0$). When g is fixed, the MI is symmetric with regard to g_{12}. However, when g_{12} is fixed, the MI is not symmetric with regard to g.

We can find from Eq. (11) that, for a trapped two-component BEC, a following trapping dependent modulational instability time scale exists

$$t_c = \frac{1}{2\sqrt{\Omega}}\arccos\{-k^2/[g_1\varphi_{01}^2+g_2\varphi_{02}^2 \pm\sqrt{4g_{12}^2\varphi_{01}^2\varphi_{02}^2+(g_1\varphi_{01}^2-g_2\varphi_{02}^2)^2}]\}. \quad (15)$$

That is, the MI will take place when $t < t_c$.

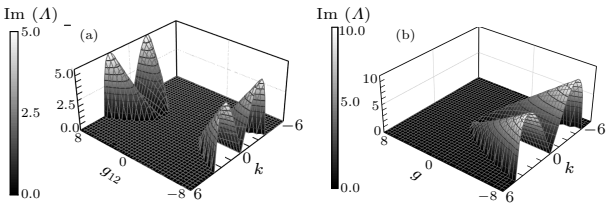

Fig. 1. (a) The MI gain with g_{12} for fixed $g = 3$. (b) The MI gain with g for fixed $g_{12} = 2$. Here $\varphi_{01} = \varphi_{02} = 1$.

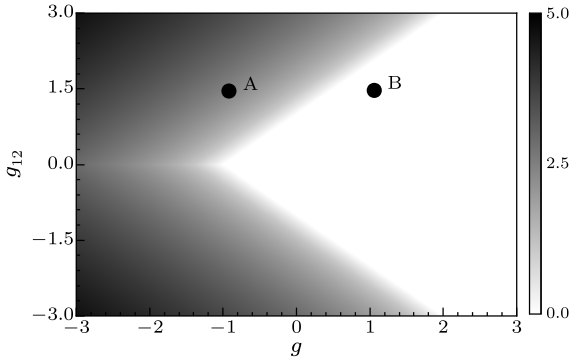

Fig. 2. $\mathrm{Im}(\Lambda)$ against g and g_{12} with $k = 1.5$ and $\varphi_{01} = \varphi_{02} = 1$.

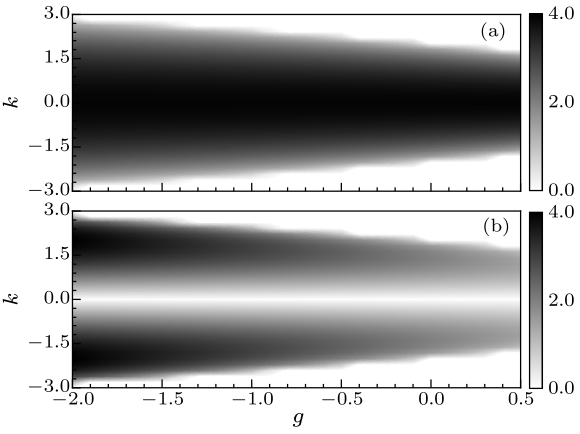

Fig. 3. (a) Behavior of t_c with g and k. (b) $\mathrm{Im}(\Lambda)$ with g and k. Here $\Omega = 0.04$, $g_{12} = 2$ and $\varphi_{01} = \varphi_{02} = 1$.

The timescale of MI is related to the trapping parameter, wave number, and atomic interactions, in the two-component BEC systems. This is clearly illustrated in Fig. 3, where t_c (Fig. 3(a)) and the corresponding instability growth rate of the system $\mathrm{Im}(\Lambda)$ (Fig. 3(b)) are shown. One can find that t_c and $\mathrm{Im}(\Lambda)$ are closely related. Note that, according to Eq. (13), for occurrence of MI, $0 < -k^2/[g_1\varphi_{01}^2+g_2\varphi_{02}^2 \pm \sqrt{4g_{12}^2\varphi_{01}^2\varphi_{02}^2+(g_1\varphi_{01}^2-g_2\varphi_{02}^2)^2}] < 1$ should be satisfied. Hence, Eq. (15) indicates that the time scale for exciting MI satisfies $t_c < \frac{1}{2\sqrt{\Omega}}\arccos(0) = t_s = \frac{\pi}{4\sqrt{\Omega}}$. This guarantees that our model of Eq. (5) is rational for the prediction of MI. This is also confirmed by the following numerical simulation of Eq. (1).

To confirm our theoretical analysis of MI, we use direct numerical simulation of Eq. (1). Initially we set $\psi_1(x,0) = \psi_2(x,0) = \psi_{\rm TF}[\varphi_{0j} + \varepsilon\cos(kx)]$, where $\psi_{\rm TF} = \sqrt{\max(0, 1 - V(x))}$ is the background wave function in the Thomas–Fermi approximation. We fix the wave amplitude to be $\varphi_{0j} = 1$, and the perturbation amplitude $\varepsilon = 0.01$. This value is sufficient small and will not cause significant variation of the results.

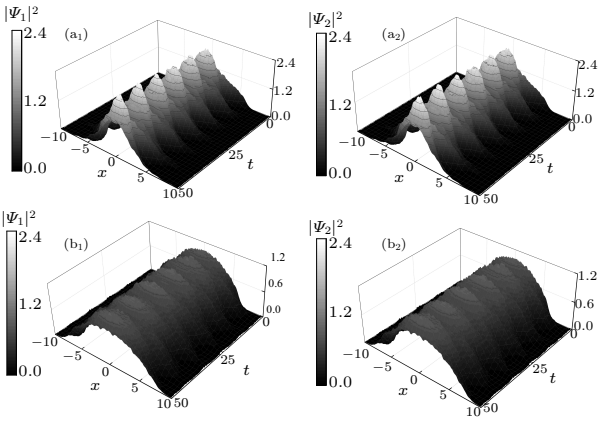

Fig. 4. The time evolution of the modulus square $|\psi_j|^2$ with the parameters as marked by A and B in Fig. 2: $g = -1$ (a$_1$, a$_2$), $g = 1$ (b$_1$, b$_2$). The other parameters are $\Omega = 0.04$, $k = g_{12} = 1.5$, and $\varphi_{01} = \varphi_{02} = 1$.

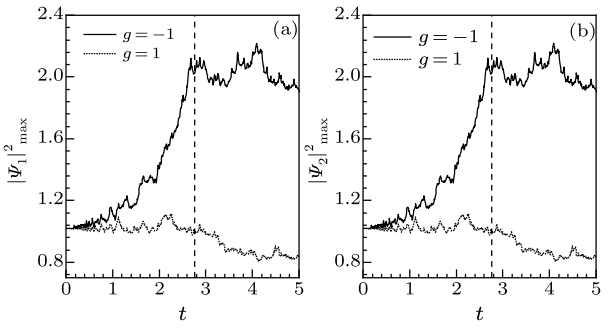

Fig. 5. The time evolution of the maximum modulus square $|\psi_j|^2$ with different atomic interactions g. The vertical-dashed line is $t_{\rm c}$ given by Eq. (15). Here $\Omega = 0.04$, $k = g_{12} = 1.5$ and $\varphi_{01} = \varphi_{02} = 1$.

For instance, Fig. 4 shows the time evolution of the modulus square $|\psi_j|^2$ with the parameters as marked in Fig. 2 by A and B; i.e., Figs. 4(a$_1$) and 4(a$_2$) for the modulational unstable case with $g_{12} = 1.5$, $g = -1$ and Figs. 4(b$_1$) and 4(b$_2$) for the modulational stable case with $g_{12} = 1.5$, $g = 1$. Figures 4(a$_1$) and 4(a$_2$) indicate that the density perturbation starts growing and finally reaches 2.4 time its initial value in both the components, and the perturbation density wave varies periodically. This means that the MI is excited. However, for the modulational stable case, as shown in Figs. 4(b$_1$) and 4(b$_2$) the density of both the components performs only small oscillations and indicates that the system almost maintains its initial state; i.e., the system is stable. The corresponding time evolution of the maximum modulus square $|\psi_j|^2_{\rm max}$ is also shown in Fig. 5. One can find that when $g = -1$ (see Figs. 4(a$_1$) and 4(a$_2$)), the MI takes place and $|\psi_j|^2$ increases with time as $t < t_{\rm c}$. The value of $t_{\rm c} \simeq 2.76$ predicted by Eq. (15) satisfies $t_{\rm c} < t_{\rm s} \simeq 3.93$ and is in good agreement with the numerical simulation. The numerical results shown in Figs. 4 and 5 confirm our theoretical predictions.

In summary, we have analytically investigated the modulational instability of trapped two-component BECs by means of the linear-stability analysis for small perturbations added to the plane wave states. Using a modified lens-type transformation and linear analysis, a new explicit time-dependent criterion for MI of trapped two-component BECs is obtained. The modulational properties are modified significantly by the external trapping and atomic interactions of two-component BECs. Furthermore, we validate our results by numerical simulation.

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