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Methods for Derivation of Density Matrix of Arbitrary Multi-Mode Gaussian States from Its Phase Space Representation *

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We present a method for derivation of the density matrix of an arbitrary multi-mode continuous variable Gaussian entangled state from its phase space representation. An explicit computer algorithm is given to reconstruct the density matrix from Gaussian covariance matrix and quadrature average values. As an example, we apply our method to the derivation of three-mode symmetric continuous variable entangled state. Our method can be used to analyze the entanglement and correlation in continuous variable quantum network with multi-mode quantum entanglement states.

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Phase space method is an efficient tool to describe entanglement and dynamics evolution of continuous variable (CV) Gaussian entangled state.[1,2] However, the drawback of phase space is that it applies only to the Gaussian state and Gaussian channels. In practice, non-Gaussian state and non-Gaussian processing are more general. Actually, non-Gaussian operations are the necessary building blocks for continuous variable entanglement distillation,[3–5] error correction[6] and many other information processing protocols.[7–9] After non-Gaussian operation, the states in the output are non-Gaussian and the density matrix in the Fock state basis could provide all the information of the non-Gaussian state.

The most straightforward way to obtain the density matrix is to represent every interaction and every state with photon number basis. Thus, the number of the density matrix entries scales exponentially with the optical mode number N, which imposes a great limit to the optical modes that we can process and analyze in realistic CV quantum networks. An alternative method is to write the non-Gaussian state as a linear combination of a few Gaussian states in phase space. Then, by deriving the density matrix of each Gaussian state one by one and again using the linear combination, one can finally obtain the density matrix of the non-Gaussian state.[10–12] This method has been successfully applied to describe the quantum states generated in continuous variable entanglement distillation,[13] photon catalysis[14] and noiseless linear amplification.[15]

Deriving the density matrix from the phase space is non-trivial and requires large amounts of calculation. However, research in this area is rare and the already-known methods can be divided into two categories. First, the Hermite polynomials method: for a Gaussian state, the Husimi Q function of the quantum state has a Gaussian form and the density matrix entries in the Hilbert space can be expressed by terms of multivariate Hermite polynomials.[16] Second, the characteristic functions method: the density matrix entries can be expressed by the multi-variant differential of the characteristic function of the Gaussian state.[17,18] In Ref. [17] the density matrix elements in the Hilbert Space are obtained assuming that the Gaussian entangled state resides in two-mode Hilbert space and has zero-displacement (i.e., quadrature average values are zero). However, their density matrix is not normalized and later in Ref. [12] we improve Chen's idea and develop an auto-normalized state transfer algorithm for zero-displaced two-mode Gaussian entanglement.

Up till now, most of the studies in this area have been restricted in two-mode entangled states, which imposes a general limit on the number of modes that we can deal with in the continuous variable information processing. In this Letter, we present an algorithm to derive the density matrix of an arbitrary multi-mode Gaussian entangled state from its phase space representation. This result is of great value in the study of the exact photon number population in the Fock state space and may find widespread applications in the study of the entanglement dynamics of multi-mode CV entanglement.[19,20]

For an N-mode CV quantum system, each mode is an infinite dimensional subsystem and can be conveniently expressed by field quadrature position operator $\hat{x}_k = (\hat{a}_k + \hat{a}_k^\dagger)/\sqrt{2}$ and momentum operator $\hat{p}_k = (\hat{a}_k - \hat{a}_k^\dagger)/(i\sqrt{2})$, where $k = 1, 2, \ldots N$ de-

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notes the index of mode, $\hat{a}_k$ and $\hat{a}_k^\dagger$ denote the photon annihilation and creation operators for the kth mode. For any density matrix ρ in the Hilbert space, there is a corresponding characteristic function defined in phase space $\chi(\zeta) = \mathrm{Tr}[\exp[i\hat{R}\zeta^{\mathrm{T}}\rho]]$, with $\hat{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_N, \hat{p}_N)$ and $\zeta \in \mathbb{R}^{2N}$ being a $2N$-dimensional real row vector. The characteristic function of a Gaussian state can be completely described by the covariance matrix and quadrature average values, $\chi(\zeta) \equiv \chi_G(\zeta, V, \overline{\boldsymbol{R}}) = \exp[-\frac{1}{2}\zeta V\zeta^{\mathrm{T}} + i\overline{\boldsymbol{R}}\zeta^{\mathrm{T}}]$, with V being the $2N \times 2N$ covariance matrix (CM)

$$V_{lm} = \frac{1}{2}\mathrm{Tr}[(\hat{R}_l\hat{R}_m + \hat{R}_m\hat{R}_l)\rho] - \mathrm{Tr}[\hat{R}_l\rho]\mathrm{Tr}[\hat{R}_m\rho], \quad (1)$$

with $l, m = 1, 2, 3, \ldots, 2N$ and $\overline{\boldsymbol{R}}$ being the average values of quadrature operators $\hat{R}$. To derive the density matrix for characteristic function χ, we note that the relation follows

$$\rho = \int \frac{d\boldsymbol{\mu}}{\pi^N}\,\mathrm{Tr}[\rho D(\boldsymbol{\mu})]D(-\boldsymbol{\mu}), \quad (2)$$

with $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{C}^N$, $D(\boldsymbol{\mu}) = \exp[\boldsymbol{\mu}\boldsymbol{a}^\dagger - \boldsymbol{\mu}^*\boldsymbol{a}] = \exp[(\boldsymbol{\mu}, \boldsymbol{\mu}^*)(\boldsymbol{a}^\dagger, -\boldsymbol{a})^{\mathrm{T}}]$ being the N-mode displacement operator.

The matrix elements of ρ can be conveniently obtained through[12]

$$\begin{aligned}&\langle k_1, k_2, \ldots, k_N|D(-\boldsymbol{\mu})|m_1, m_2, \ldots, m_N\rangle \\ &= \frac{\exp[-\frac{|\boldsymbol{\mu}|^2}{2}]}{\sqrt{\prod_{i=1}^N k_i!\prod_{i=1}^N m_i!}}\prod_{i=1}^N \partial_{t_i}^{k_i}\prod_{i=1}^N \partial_{t_i'}^{m_i} \\ &\times \exp[\boldsymbol{t}\boldsymbol{t}'^{\mathrm{T}} - \boldsymbol{t}\boldsymbol{\mu}^{\mathrm{T}} + \boldsymbol{t}'\boldsymbol{\mu}^{*\mathrm{T}}]\big|_{\boldsymbol{t}=\boldsymbol{t}'=0}, \end{aligned} \quad (3)$$

where $\boldsymbol{t} = (t_1, t_2, \ldots, t_N)$ and $\boldsymbol{t}' = (t_1', t_2', \ldots, t_N')$ are N-dimensional real vectors.

It should be noted that $(\boldsymbol{a}^\dagger, -\boldsymbol{a})^{\mathrm{T}} = iL_N(\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_N, \hat{p}_N)^{\mathrm{T}}$,

$$L_N = \left(\begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes I_N\right)P,$$

$$P_{kl} = \begin{cases} \delta_{2k-1,l} & k \le N, \\ \delta_{2(k-N),l} & k > N, \end{cases} \quad (4)$$

with I_N the identity matrix in N dimensions.

We obtain (after integrating over $\boldsymbol{\mu}$) the following theorem.

Theorem 1:

$$\begin{aligned}&\langle k_1, k_2, \ldots, k_N|\rho|m_1, m_2, \ldots, m_N\rangle \\ &= \frac{\prod_{i=1}^N \partial_{t_i}^{k_i}\prod_{i=1}^N \partial_{t_i'}^{m_i} F\big|_{\boldsymbol{t}=\boldsymbol{t}'=0}}{\sqrt{\prod_{i=1}^N k_i!\prod_{i=1}^N m_i!}}, \end{aligned} \quad (5)$$

with $F = \exp[\frac{1}{2}(\boldsymbol{t}, \boldsymbol{t}')R(\boldsymbol{t}, \boldsymbol{t}')^{\mathrm{T}}]\Big/\sqrt{\det(V + I_{2N}/2)}$, $R = \sigma_x \otimes I_N + (\sigma_z \otimes I_N)L_N^*(V + I_{2N}/2)^{-1}L_N^\dagger(\sigma_z \otimes I_N)$, and σ_x, σ_z the usual Pauli matrices.

Then, one can check that the state ρ as given by Eq. (5) is now automatically normalized; i.e., $\mathrm{Tr}[\rho] = 1$. In the following, we further simplify this theorem and finally present a computer algorithm that enables us to perform a numerical calculation. We emphasize that the theorem presented here is quite general and versatile, making it very useful for further applications, especially when entanglement has to be quantitatively evaluated with density matrix in the Fock state basis.

In what follows, we give a detailed derivation of the density matrix from a Gaussian-state covariance matrix with $N = 3$. For simplicity, we use the notation $t_{N+i} = t_i'(i = 1, 2, \ldots N)$. For a given three-mode Gaussian state with CM V and zero displacements, the density matrix is given by

$$\begin{aligned}&\langle k_1, k_2, k_3|\rho|m_1, m_2, m_3\rangle \\ &= \frac{\partial_{t_1}^{k_1}\partial_{t_2}^{k_2}\partial_{t_3}^{k_3}\partial_{t_4}^{m_1}\partial_{t_5}^{m_2}\partial_{t_6}^{m_3}\exp[\tau]\big|_{t_1=t_2=t_3=t_4=t_5=t_6=0}}{\sqrt{k_1!k_2!k_3!m_1!m_2!m_3!\det(V + I_6/2)}}, \end{aligned} \quad (6)$$

with $R = \sigma_x \otimes I_3 + \sigma_z \otimes I_3 L_3^*(V + I_6/2)^{-1}L_3^\dagger\sigma_z \otimes I_3$, $\tau = [\frac{1}{2}(t_1, t_2, t_3, t_4, t_5, t_6)R(t_1, t_2, t_3, t_4, t_5, t_6)^{\mathrm{T}}] = \sum_{i=1}^6\sum_{j=i}^6 c_{ij}t_it_j$. In this equation, c_{ij} can be expressed by a 6×6 upper-triangular matrix

$$c_{ij} = \begin{cases} R_{ii}/2 & i = j, \\ (R_{ij} + R_{ji})/2 & i < j, \\ 0 & i > j. \end{cases} \quad (7)$$

The exponential can be calculated as

$$\exp[\tau] = \sum_{n=0}^\infty \frac{1}{n!}(\sum_{i=1}^6\sum_{j=i}^6 c_{ij}t_it_j)^n \quad (8)$$

$$\begin{aligned}&= \sum_{n=0}^\infty \sum_{\hat{\boldsymbol{n}}\in\Omega(n)} \frac{1}{\prod_{i=1}^6\prod_{j=i}^6 \hat{\boldsymbol{n}}_{ij}!}\prod_{i=1}^6\prod_{j=i}^6 (c_{ij}t_it_j)^{\hat{\boldsymbol{n}}_{ij}} \\ &= \sum_{n=0}^\infty \sum_{\hat{\boldsymbol{n}}\in\Omega(n)} \frac{\prod_{i=1}^6\prod_{j=i}^6 (c_{ij})^{\hat{\boldsymbol{n}}_{ij}}}{\prod_{i=1}^6\prod_{j=i}^6 \hat{\boldsymbol{n}}_{ij}!} \\ &\times t_1^{2\hat{\boldsymbol{n}}_{11}+\hat{\boldsymbol{n}}_{12}+\hat{\boldsymbol{n}}_{13}+\hat{\boldsymbol{n}}_{14}+\hat{\boldsymbol{n}}_{15}+\hat{\boldsymbol{n}}_{16}} \\ &\times t_2^{2\hat{\boldsymbol{n}}_{22}+\hat{\boldsymbol{n}}_{12}+\hat{\boldsymbol{n}}_{23}+\hat{\boldsymbol{n}}_{24}+\hat{\boldsymbol{n}}_{25}+\hat{\boldsymbol{n}}_{26}} \\ &\times t_3^{2\hat{\boldsymbol{n}}_{33}+\hat{\boldsymbol{n}}_{13}+\hat{\boldsymbol{n}}_{23}+\hat{\boldsymbol{n}}_{34}+\hat{\boldsymbol{n}}_{35}+\hat{\boldsymbol{n}}_{36}} \\ &\times t_4^{2\hat{\boldsymbol{n}}_{44}+\hat{\boldsymbol{n}}_{14}+\hat{\boldsymbol{n}}_{24}+\hat{\boldsymbol{n}}_{34}+\hat{\boldsymbol{n}}_{45}+\hat{\boldsymbol{n}}_{46}} \\ &\times t_5^{2\hat{\boldsymbol{n}}_{55}+\hat{\boldsymbol{n}}_{15}+\hat{\boldsymbol{n}}_{25}+\hat{\boldsymbol{n}}_{35}+\hat{\boldsymbol{n}}_{45}+\hat{\boldsymbol{n}}_{56}} \\ &\times t_6^{2\hat{\boldsymbol{n}}_{66}+\hat{\boldsymbol{n}}_{16}+\hat{\boldsymbol{n}}_{26}+\hat{\boldsymbol{n}}_{36}+\hat{\boldsymbol{n}}_{46}+\hat{\boldsymbol{n}}_{56}}, \end{aligned} \quad (9)$$

where $\hat{\boldsymbol{n}}$ is again a 6×6 upper-triangular matrix, and $\hat{\boldsymbol{n}}_{\boldsymbol{ij}}$ is the corresponding matrix element for the ith

row and jth column

$$\hat{\boldsymbol{n}} = \begin{pmatrix} \hat{\boldsymbol{n}}_{11}\hat{\boldsymbol{n}}_{12} & \ldots, \hat{\boldsymbol{n}}_{15} & \hat{\boldsymbol{n}}_{16} \\ 0\hat{\boldsymbol{n}}_{22} & \hat{\boldsymbol{n}}_{23}\ldots, & \hat{\boldsymbol{n}}_{26} \\ \vdots\vdots & \vdots\vdots & \vdots \\ 00 & 0\hat{\boldsymbol{n}}_{55} & \hat{\boldsymbol{n}}_{56} \\ 00 & 00 & \hat{\boldsymbol{n}}_{66} \end{pmatrix}, \tag{10}$$

with $\Omega(n)$ being the set of all possible matrices $\hat{\boldsymbol{n}}$ such that

$$\Omega(n) = \{\hat{\boldsymbol{n}} | \sum_{i=1}^{6}\sum_{j=i}^{6} \hat{\boldsymbol{n}}_{ij} = n\}. \tag{11}$$

Under the condition $t_i = 0$ in Eq. (6), the density matrix can be explicitly written as

$$\begin{aligned} &\langle k_1, k_2, k_3|\rho|m_1, m_2, m_3\rangle \\ &= \frac{1}{\sqrt{\det(V + I_6/2)}} \sum_{n=0}^{\infty} \sum_{\hat{\boldsymbol{n}}\in\Omega(\boldsymbol{n})} \\ &\quad \cdot \prod_{i=1}^{6}\prod_{j=i}^{6} (c_{ij})^{\hat{\boldsymbol{n}}_{ij}} f(k_1, k_2, k_3, \hat{\boldsymbol{n}}), \end{aligned} \tag{12}$$

with

$$\begin{aligned} &f(k_1, k_2, k_3, \hat{\boldsymbol{n}}) \\ &= \sqrt{\binom{k_1}{\hat{\boldsymbol{n}}_{11}, \hat{\boldsymbol{n}}_{11}, \hat{\boldsymbol{n}}_{12}, \hat{\boldsymbol{n}}_{13}, \hat{\boldsymbol{n}}_{14}, \hat{\boldsymbol{n}}_{15}, \hat{\boldsymbol{n}}_{16}}} \\ &\quad \times \sqrt{\binom{k_2}{\hat{\boldsymbol{n}}_{22}, \hat{\boldsymbol{n}}_{22}, \hat{\boldsymbol{n}}_{12}, \hat{\boldsymbol{n}}_{23}, \hat{\boldsymbol{n}}_{24}, \hat{\boldsymbol{n}}_{25}, \hat{\boldsymbol{n}}_{26}}} \\ &\quad \times \sqrt{\binom{k_3}{\hat{\boldsymbol{n}}_{33}, \hat{\boldsymbol{n}}_{33}, \hat{\boldsymbol{n}}_{13}, \hat{\boldsymbol{n}}_{23}, \hat{\boldsymbol{n}}_{34}, \hat{\boldsymbol{n}}_{35}, \hat{\boldsymbol{n}}_{36}}} \\ &\quad \times \sqrt{\binom{m_1}{\hat{\boldsymbol{n}}_{44}, \hat{\boldsymbol{n}}_{44}, \hat{\boldsymbol{n}}_{14}, \hat{\boldsymbol{n}}_{24}, \hat{\boldsymbol{n}}_{34}, \hat{\boldsymbol{n}}_{45}, \hat{\boldsymbol{n}}_{46}}} \\ &\quad \times \sqrt{\binom{m_2}{\hat{\boldsymbol{n}}_{55}, \hat{\boldsymbol{n}}_{55}, \hat{\boldsymbol{n}}_{15}, \hat{\boldsymbol{n}}_{25}, \hat{\boldsymbol{n}}_{35}, \hat{\boldsymbol{n}}_{45}, \hat{\boldsymbol{n}}_{56}}} \\ &\quad \times \sqrt{\binom{m_3}{\hat{\boldsymbol{n}}_{66}, \hat{\boldsymbol{n}}_{66}, \hat{\boldsymbol{n}}_{16}, \hat{\boldsymbol{n}}_{26}, \hat{\boldsymbol{n}}_{36}, \hat{\boldsymbol{n}}_{46}, \hat{\boldsymbol{n}}_{56}}}, \end{aligned} \tag{13}$$

and $\binom{k}{n_1,n_2,n_3,n_4,n_5,n_6,n_7} = k!\delta_{k,\sum_{i=1}^{7} n_i}/(\prod_{i=1}^{7} n_i!)$ being the multinomial coefficient.

Moreover, from the definition of multinomials, we know

$$2n = 2\sum_{i=1}^{6}\sum_{j=i}^{6} n_{ij} = \sum_{i=1}^{3} k_i + \sum_{i=1}^{3} m_i, \tag{14}$$

$$2\hat{\boldsymbol{n}}_{11} + \hat{\boldsymbol{n}}_{12} + \hat{\boldsymbol{n}}_{13} + \hat{\boldsymbol{n}}_{14} + \hat{\boldsymbol{n}}_{15} + \hat{\boldsymbol{n}}_{16} = k_1, \tag{15}$$

$$2\hat{\boldsymbol{n}}_{22} + \hat{\boldsymbol{n}}_{12} + \hat{\boldsymbol{n}}_{23} + \hat{\boldsymbol{n}}_{24} + \hat{\boldsymbol{n}}_{25} + \hat{\boldsymbol{n}}_{26} = k_2, \tag{16}$$

$$2\hat{\boldsymbol{n}}_{33} + \hat{\boldsymbol{n}}_{13} + \hat{\boldsymbol{n}}_{23} + \hat{\boldsymbol{n}}_{34} + \hat{\boldsymbol{n}}_{35} + \hat{\boldsymbol{n}}_{36} = k_3, \tag{17}$$

$$2\hat{\boldsymbol{n}}_{44} + \hat{\boldsymbol{n}}_{14} + \hat{\boldsymbol{n}}_{24} + \hat{\boldsymbol{n}}_{34} + \hat{\boldsymbol{n}}_{45} + \hat{\boldsymbol{n}}_{46} = m_1, \tag{18}$$

$$2\hat{\boldsymbol{n}}_{55} + \hat{\boldsymbol{n}}_{15} + \hat{\boldsymbol{n}}_{25} + \hat{\boldsymbol{n}}_{35} + \hat{\boldsymbol{n}}_{45} + \hat{\boldsymbol{n}}_{56} = m_2, \tag{19}$$

$$2\hat{\boldsymbol{n}}_{66} + \hat{\boldsymbol{n}}_{16} + \hat{\boldsymbol{n}}_{26} + \hat{\boldsymbol{n}}_{36} + \hat{\boldsymbol{n}}_{46} + \hat{\boldsymbol{n}}_{56} = m_3. \tag{20}$$

Finally, the matrix elements $\langle k_1k_2k_3|\rho|m_1m_2m_3\rangle$ are given by

$$\begin{aligned} &\langle k_1, k_2, k_3|\rho|m_1, m_2, m_3\rangle \\ &= \sum_{\hat{\boldsymbol{n}}\in\Omega(\sum_{i=1}^{3}\frac{k_i+m_i}{2})} \prod_{i=1}^{6}\prod_{j=i}^{6} (c_{ij})^{\hat{\boldsymbol{n}}_{ij}} \frac{f(k_1, k_2, k_3, \hat{\boldsymbol{n}})}{\sqrt{\det(V+I_6/2)}}. \end{aligned} \tag{21}$$

To save computer time and storage, every mode is truncated at $\overline{D}$; i.e., only the contributions of the terms $|0\rangle, |1\rangle, \ldots, |D-1\rangle$ are finally kept. The value of $\overline{D}$ is chosen to be as large as possible such that our state transfer is performed with very high precision, corresponding to an error less than 5×10^{-5}.

A possible computer algorithm that realizes this state transfer looks as follows:

Algorithm START

(1) *Set* $\overline{D} = 6$.

(2) Generate all the possible $(\hat{\boldsymbol{n}}_{11}, \hat{\boldsymbol{n}}_{12}, \ldots, \hat{\boldsymbol{n}}_{66})$ satisfying Eqs. (14)–(20).

(3) *FOR* each $(k_1, k_2, k_3, m_1, m_2, m_3)$ such that $0 \le k_1, k_2, k_3, m_1, m_2, m_3 \le \overline{D} - 1$

IF $k_1 + k_2 + k_3 + m_1 + m_2 + m_3$ is even

(i) Generate the set $\Omega(\sum_{i=1}^{3}\frac{k_i+m_i}{2})$, i.e., all possible $\hat{\boldsymbol{n}}$ such that

$$\sum_{i=1}^{6}\sum_{j=i}^{6} \hat{\boldsymbol{n}}_{ij} = \sum_{i=1}^{3} \frac{k_i + m_i}{2}. \tag{22}$$

(ii) Calculate Eq. (21).

ELSE

$$\langle k_1k_2k_3|\rho|m_1m_2m_3\rangle = 0. \tag{23}$$

END IF

END FOR

Algorithm END

With the set Ω and the definition of c_{ij} in Eq. (7), we can calculate the density matrix entry $\langle k_1, k_2, k_3|\rho|m_1, m_2, m_3\rangle$. Furthermore, by varying the values of $(k_1, k_2, k_3, m_1, m_2, m_3)$, the total density matrix ρ can be numerically obtained.

To check the correctness of our algorithm, here we use the genuinely multipartite entangled 3-mode symmetric Gaussian states as an example. These states

can be experimentally prepared using a sequence of $N-1$ phase-free beam splitters and N squeezed input states,[21–23] with the corresponding CM can be given by

$$V_N = |\mathcal{I}\rangle\langle\mathcal{I}| \otimes \boldsymbol{\epsilon} + (\boldsymbol{\alpha} - \boldsymbol{\epsilon}) \otimes I_N, \tag{24}$$

in which $|\mathcal{I}\rangle$ is the unnormalized N-dimensional real vector $|\mathcal{I}\rangle = (1, 1, 1 \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^N$ and $\boldsymbol{\alpha} = \mathrm{diag}(a, b), \boldsymbol{\epsilon} = \mathrm{diag}(c, d)$ are 2×2 diagonal matrices, with (Eq. (32) in Ref. [22] multiplied by a factor 2 due to the different definitions of CV)

$$\begin{aligned} a &= \frac{1}{2N}[e^{2r_1} + (N-1)e^{-2r_2}], \\ c &= \frac{1}{2N}(e^{2r_1} - e^{-2r_2}), \\ b &= \frac{1}{2N}[e^{-2r_1} + (N-1)e^{2r_2}], \\ d &= \frac{1}{2N}(e^{-2r_1} - e^{2r_2}) \end{aligned} \tag{25}$$

and r_1 being the squeezing parameter of one squeezed input and r_2 being the squeezing parameter of the other two equally squeezed inputs. For simplicity, we consider the so-called unbiased states for which $r_1 = r_1(r_2) = \frac{1}{2}\ln[(N-1)\sinh(2r_2)] + \frac{1}{2}\ln[\sqrt{1 + [(N-1)\sinh(2r_2)]^{-2}} + 1]$.

With our computer algorithm, we are able to calculate the density matrix ρ for arbitrary three-mode Gaussian state. Assuming that the density matrix obtained from our algorithm is ρ_{algo}. One good way to evaluate how well our algorithm works is to calculate the covariance matrix V_{algo} of our output matrix ρ_{algo}. Then, we can use the maximal norm $||A||_\infty = \max\{a_{ij}\}$ to measure the error during our state derivation

$$V_{\mathrm{err}} = ||V_{\mathrm{alog}} - V_N||_\infty. \tag{26}$$

We can use the logarithmic negativity[24] to quantify the entanglement, which is more important in practical quantum communications.[25,26]

The logarithmic negativity of a multipartite state ρ is defined as[24]

$$E_N(\rho) = \log_2(1 + 2\mathcal{N}(\rho)), \tag{27}$$

in which $\mathcal{N}(\rho)$ is given by the absolute value of the sum of negative eigenvalues of the partially transposed (e.g., with respect to the first mode) density operator ρ^{Γ_1}. For Gaussian entangled states, we can evaluate the logarithmic negativity from the CM V_N, which follows

$$E_{\mathrm{th}} = \max[0, -\log_2\widetilde{\nu}], \tag{28}$$

with ν being the minimal positive symplectic eigenvalues of TV_NT for $T = (\ 1, 00, -1\ ) \oplus I_4$.[27]

Because this algorithm also gives us a method for calculating the density matrix of ρ in the Fock state space, we can numerically find E_N by a straightforward diagonalization of the partially transposed density matrix. The comparison between the theoretic and numerical values $|E_N - E_{\mathrm{th}}|$ gives us another figure of merit to evaluate the performance of our algorithm.

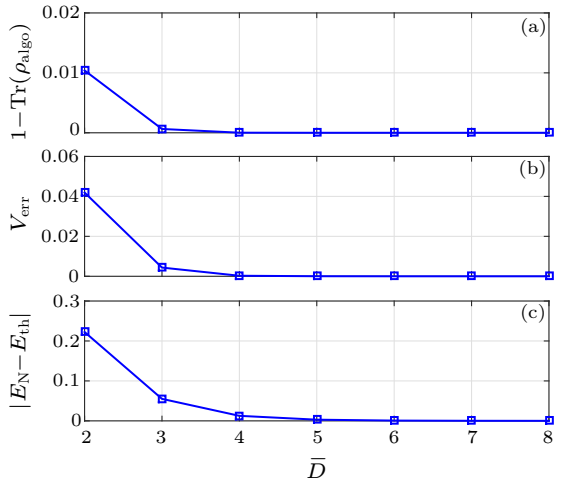

Fig. 1. Numerical check of correctness of our numerical algorithm for $r_2 = 0.2$, $N = 3$. (a) Error of trace of density matrix: $1 - \mathrm{Tr}[\rho_{\mathrm{algo}}]$. (b) Error of covariance matrix. (c) Error of logarithmic negativity.

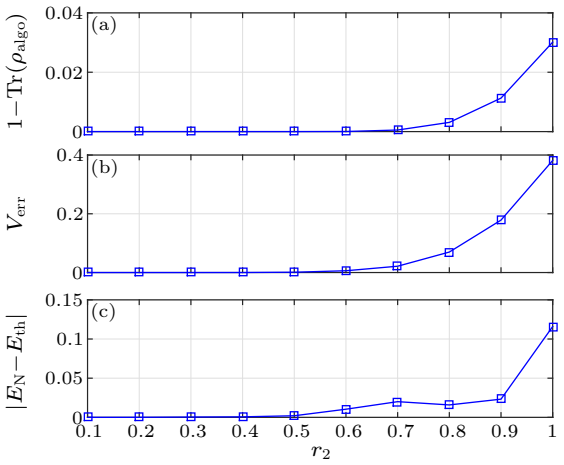

Fig. 2. Performance of our numerical algorithm for increasing squeezing r. Other parameters are chosen as $N = 3$, $\overline{D} = 8$. (a) Error of trace of density matrix. (b) Error of covariance matrix. (c) Error of logarithmic negativity.

In Fig. 1, we take $r_2 = 0.2$ as an example and give a plot of the performance of our numerical algorithm for different values of truncation number $\overline{D}$. As shown in Fig. 1, the figure of merit of the truncated state ρ_{algo} quickly converges for a larger $\overline{D}$. When $\overline{D} = 6$, the error due to our truncation is almost negligible. In particular, Fig. 1(a) shows the error of the trace of our derived state ρ_{algo}, which once again proves the auto-normalization of our algorithm.

In Fig. 2, we fix $\overline{D} = 8$ and evaluate the perfor-

mance of state transferring, for variations of squeezing r. The error increases with r and when $r = 0.6$ the average photon number in each modes of symmetric state is about 0.6342. Although this is much smaller than $\overline{D} = 8$, we can see that this is still strong enough to influence the entanglement E_N.

Finally, we give a remark on the storage complexity and computational complexity. In fact, for $\overline{D} = 3, 4, 5, 6, 7, 8$, there are $785, 11874, 1.3 \times 10^5, 1.1 \times 10^6, 7.18 \times 10^6, 4.0 \times 10^7$ groups of possible $(\hat{\boldsymbol{n}}_{11}, \hat{\boldsymbol{n}}_{12}, \ldots, \hat{\boldsymbol{n}}_{66})$. Especially for $\overline{D} = 8$, it costs 806 Megabyte storage and 590 min to complete the numerical evaluation with a single-core 2.0 Ghz-CPU-equipped personal computer (PC). Large $\overline{D}$ requires more storage and more computation, which is necessary for multi-mode entanglement state with larger average photon number. Fortunately, we note that the computational complexity in the large squeezing scenario can be further improved thanks to the fact that our algorithm deals with the density matrix entries one by one, which can be certainly updated with a parallel computation technique and run with the multi-core environments.

In conclusion, we have provided an auto-normalized computer method for evaluating arbitrary multi-mode continuous variable Gaussian entangled stats. Finally, we apply our method into the symmetric three-mode Gaussian entangled state. The performance of our algorithm is numerically evaluated, in terms of errors of trace of density matrix, covariance matrix, and logarithmic negativity. We envisage that our state transfer method will be widely used in characterizing CV entanglement dynamics evolution and multipartite nonlocality test[28] in infinite-dimensional quantum systems.

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