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Multiple Soliton Solutions of Alice–Bob Boussinesq Equations *

Hui Li(李辉)[1], S. Y. Lou(楼森岳)[1,2]**

[1]School of Physical Science and Technology, Ningbo University, Ningbo 315211
[2]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062

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Three Alice–Bob Boussinesq (ABB) nonlocal systems with shifted parity ($\hat{P}_{\rm s}$), delayed time reversal ($\hat{T}_{\rm d}$) and $\hat{P}_{\rm s}\hat{T}_{\rm d}$ nonlocalities are investigated. The multi-soliton solutions of these models are systematically found from the $\hat{P}_{\rm s}$, $\hat{T}_{\rm d}$ and $\hat{P}_{\rm s}\hat{T}_{\rm d}$ symmetry reductions of a coupled local Boussinesq system. The result shows that for ABB equations with $\hat{P}_{\rm s}$ and/or $\hat{T}_{\rm d}$ nonlocality, an odd number of solitons is prohibited. The solitons of the $\hat{P}_{\rm s}$ nonlocal ABB and $\hat{T}_{\rm d}$ nonlocal ABB equations must be paired, while any number of solitons is allowed for the $\hat{P}_{\rm s}\hat{T}_{\rm d}$ nonlocal ABB system. t-breathers, x-breathers and rogue waves exist for all three types of nonlocal ABB system. In particular, different from classical local cases, the first-order rogue wave can have not only four leaves but also five and six leaves.

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In quantum mechanics, physical quantities are described by Hermitian operators such that the physical observable quantities possess real eigenvalues. In 1998, Bender and Boettcher found that a non-Hermitian Hamiltonian with parity and charge conjugate (PC) symmetry or equivalently parity and time reversal (PT) symmetry may also possess real eigenvalues if the potential of the Hamiltonian is time-independent.[1] This phenomenon has attracted a great deal of attention from scientists because it can be realized in optical systems with the help of gain or lost effects.[2] In 2013, a PT symmetric (or more exactly PC symmetric because nonlinear potentials are time-dependent) nonlinear nonlocal Schrödinger system was proposed by Ablowitz and Musslimani.[3] After that, the study of nonlocal systems (named Alice–Bob (AB) systems) became one of the hot topics in nonlinear science.[4–19] In this Letter, using the obtained idea[4,5] we investigate a special Alice–Bob Boussinesq (ABB) system

$$\begin{aligned}&A_{\rm tt}-\Big[A_{xx}+\delta A+\frac{9}{4}A^2+\frac{3}{2}AB-\frac{3}{4}B^2\Big]_{xx}=0,\\&B=\hat{f}A\equiv A^{\hat{f}},\ \hat{f}\in\mathcal{G}\equiv\{1,\hat{P}_{\rm s},\hat{T}_{\rm d},\hat{P}_{\rm s}\hat{T}_{\rm d}\},\\&A^{\hat{P}_{\rm s}}=A(-x+x_0,t),\ A^{\hat{T}_{\rm d}}=A(x,-t+t_0),\end{aligned}\tag{1}$$

with arbitrary constants x_0 and t_0.

For $\hat{f}=1$, Eq. (1) is just the usual local Boussinesq equation which is widely used in physics, especially in fluid dynamics. For $\hat{f}\neq 1$, Eq. (1) expresses three types of nonlocal ABB systems with three different nonlocalities, the parity nonlocal ABB (PNABB) system ($B=A^{\hat{P}_{\rm s}}$), the time-reversal nonlocal ABB (TNABB) system ($B=A^{\hat{T}_{\rm d}}$) and the parity and time-reversal nonlocal ABB (PTNABB) ($B=A^{\hat{P}_{\rm s}\hat{T}_{\rm d}}$) system.

The ABB system in Eq. (1) can be derived by applying the consistent correlated bang approach to the usual Bussinesq equation[4,6]

$$u_{\rm tt}-(3u^2+\delta u+u_{xx})_{xx}=0.\tag{2}$$

It is known that if $\hat{g}$ is a second-order operator with $\hat{g}^2=1$, then an arbitrary function A can be separated into an invariant part and an antisymmetric part in the following way

$$A=u+v,\ u=\frac{A+A^{\hat{g}}}{2},\ v=\frac{A-A^{\hat{g}}}{2}.\tag{3}$$

Substituting Eq. (3) into Eq. (1) and separating the symmetric and antisymmetric parts, we have

$$u_{\rm tt}-(u_{xx}+\delta u+3u^2)_{xx}=0,\tag{4}$$
$$v_{\rm tt}-(v_{xx}+\delta v+6uv)_{xx}=0,\tag{5}$$

under the symmetric and antisymmetric conditions

$$A=u+v,\ \hat{f}u=u,\ \hat{f}v=-v,\ \hat{f}\in\{\hat{P}_{\rm s},\hat{T}_{\rm d},\hat{P}_{\rm s}\hat{T}_{\rm d}\}.\tag{6}$$

In other words, the nonlocal system (1) is just a (discrete) symmetry reduction of the enlarged system (4) and (5). The integrable system (4) and (5) is just one of the simplest integrable couplings, named as the Z_2-Boussinesq equation. Thus solving the ABB in Eq. (1) is equivalent to solving the integrable coupling (4) and (5) with the parity and time reversal conditions (6). Because of the integrability of the Z_n-Boussinesq system for arbitrary n,[20] it is straightforward to write the Lax pair of the nonlocal ABB in Eq. (1) as

$$\psi_{xxx}=\begin{pmatrix}M&0\\Z&M\end{pmatrix}\psi,\tag{7}$$

$$\psi_t=\sqrt{-3}\psi_{xx}+\frac{\sqrt{-3}}{2}\begin{pmatrix}A+A^{\hat{f}}&0\\A-A^{\hat{f}}&A+A^{\hat{f}}\end{pmatrix}\psi,\tag{8}$$

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**Corresponding author. Email: lousenyue@nbu.edu.cn

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with $M \equiv -\frac{1}{4}[3(A+A^{\hat{f}})+\delta]\partial_x+\frac{1}{2}\int(A+A^{\hat{f}})_t\mathrm{dx}+\lambda_1-\frac{3}{8}(A+A^{\hat{f}})_x$, $Z \equiv -\frac{3(A-A^{\hat{f}})}{4}\partial_x+\frac{1}{2}\int(A-A^{\hat{f}})_t\mathrm{dx}+\lambda_2-\frac{3}{8}(A-A^{\hat{f}})_x$ and λ_1 and λ_2 being arbitrary constants.

For the Boussinesq Eq. (4), it is well known that the multiple soliton solution possesses the form[4,21]

$$u = 2(\ln F)_{xx}, \tag{9}$$

$$F = \sum_{\nu} \exp\Big(\sum_{j=1}^{N}\nu_j\xi_j + \sum_{i\le j}^{N}\nu_j\nu_i\theta_{ij}\Big), \tag{10}$$

where the summation of ν should be carried out for all permutations of $\nu_i = 0, 1$, $i = 1, 2, \ldots, N$ and

$$\xi_j = k_j x - \omega_j t + \xi_{0j}, \quad \omega_i^2 = k_i^4 + \delta k_i^2,$$
$$\exp(\theta_{ij}) = \frac{k_i k_j(2k_i^2+2k_j^2-3k_ik_j+\delta)-\omega_i\omega_j}{k_ik_j(2k_i^2+2k_j^2+3k_ik_j+\delta)-\omega_i\omega_j}, \tag{11}$$

with arbitrary constants k_i and ξ_{i0} for all i.

From the expressions (9) and (10) it is quite difficult to find its $\hat{f}(\hat{f} \in \{\hat{P}_{\mathrm{s}}, \hat{T}_{\mathrm{d}}, \hat{P}_{\mathrm{s}}\hat{T}_{\mathrm{d}}\})$ invariant form. Fortunately, from the obtained results[4] we know that if we rewrite Eq. (11) as

$$\xi_j = \eta_j - \frac{1}{2}\sum_{i=1}^{j-1}\theta_{ij} - \frac{1}{2}\sum_{i=j+1}^{N}\theta_{ij}, \tag{12}$$

$$\eta_j = k_j\Big(x - \frac{1}{2}x_{0j}\Big) - \omega_j\Big(t - \frac{1}{2}t_{0j}\Big), \tag{13}$$

with arbitrary constants x_{0j} and t_{0j}, then the N-soliton solution of the Boussinesq Eq. (2) can be rewritten as[4]

$$u = 2(\ln F_N)_{xx} = 2\Big[\ln\sum_{\nu}K_\nu\cosh\Big(\frac{1}{2}\sum_{j=1}^{N}\nu_j\eta_j\Big)\Big]_{xx}, \tag{14}$$

where the summation of $\nu = \{\nu_1, \nu_2, \ldots, \nu_N\}$ should be carried out for all non-dual permutations of $\nu_i = 1, -1, i = 1, 2, \ldots, N$ (ν and ν' are dual if $\nu = -\nu'$), and

$$K_\nu = \prod_{i>j}\sqrt{k_ik_j(2k_i^2+2k_j^2-3\nu_i\nu_jk_ik_j+\delta)-\omega_i\omega_j}$$
$$\equiv \prod_{i>j}a_{ij}^{\nu_i\nu_j}, a_{ij}^{\nu_i\nu_j} \equiv a_{ij}^{\pm}\text{if } \nu_i\nu_j = \pm 1. \tag{15}$$

From the expressions (13) and (14), it is straightforward to see that for $\hat{f} = \hat{P}_{\mathrm{s}}\hat{T}_{\mathrm{d}}$, the $\hat{f}$-invariant condition $\hat{f}u = u$ requires

$$x_{0j} = x_0, \quad t_{0j} = t_0, \quad j = 1, 2, \ldots, N, \tag{16}$$

with arbitrary positive integer N. From the constraint condition (16), we know that for the PTNABB system (Eq. (1) with $\hat{f} = \hat{P}_{\mathrm{s}}\hat{T}_{\mathrm{d}}$), arbitrary numbers of solitons are allowed. However, all the soliton interaction positions are fixed at $\{x = \frac{x_0}{2}, t = \frac{t_0}{2}\}$ because of the loss of the x and t translation invariants.

Similar to the analyses for another ABB system[8] and the ABKP system,[9] there may be some constraints on the multi-soliton solution (14) for the other two nonlocal ABB systems, the PNABB system (Eq. (1) with $\hat{f} = \hat{P}_{\mathrm{s}}$) and the TNABB system (Eq. (1) with $\hat{f} = \hat{T}_{\mathrm{d}}$).

For $N = 1$, Eq. (14) possesses the form

$$u = 2(\ln F_1)_{xx}, \ F_1 = \cosh\Big(\frac{\eta_1}{2}\Big),$$
$$\eta_1 = k_1\Big(x - \frac{x_{01}}{2}\Big) \pm \sqrt{k_1^4+\delta k_1^2}\Big(t - \frac{t_{01}}{2}\Big). \tag{17}$$

It is clear that it is impossible for the single soliton solution (17) to select the arbitrary constants k_1, x_{01} and t_{01} such that the $\hat{f}$ invariant condition $\hat{f}u = u$ is satisfied for $\hat{f} = \hat{P}_{\mathrm{s}}$ or $\hat{f} = \hat{T}_{\mathrm{d}}$. In fact, for any nontrivial traveling wave solution $u = u(kx+\omega t+\theta_0)$ is not $\hat{P}_{\mathrm{s}}$ or $\hat{T}_{\mathrm{d}}$-invariant. Furthermore, one can obtain that for any odd $N = 2n-1$, the solution (14) is not $\hat{f}$-invariant for $\hat{f} \in \{\hat{P}_{\mathrm{s}}, \hat{T}_{\mathrm{d}}\}$. In other words, an odd number of solitons for the PNABB and TNABB systems is prohibited.

For $N = 2$, solution (14) becomes

$$u = 2(\ln F_2)_{xx},$$
$$F_2 = a_{12}^{-}\cosh\Big(\frac{\eta_1+\eta_2}{2}\Big) + a_{12}^{+}\cosh\Big(\frac{\eta_1-\eta_2}{2}\Big),$$
$$a_{12}^{\pm} = \sqrt{k_1k_2(2k_1^2+2k_2^2\pm 3k_1k_2+\delta)-\omega_1\omega_2}. \tag{18}$$

From expression (18), it is obvious that if we select

$$\eta_2 = \pm\eta_1^{\hat{f}}, \tag{19}$$

then the field u will be naturally $\hat{f}$-invariant.

Fortunately, the condition (19) can be readily satisfied by fixing the parameters k_2, ω_2, x_{0i} and t_{0i}, that is, for $\hat{f} = \hat{P}_{\mathrm{s}}$,

$$x_{0i} = x_0, \quad t_{02} = t_{01}, \quad k_2 = \mp k_1, \quad \omega_2 = \pm\omega_1; \tag{20}$$

and for $\hat{f} = \hat{T}_{\mathrm{d}}$,

$$t_{0i} = t_0, \quad x_{02} = x_{01}, \quad k_2 = \pm k_1, \quad \omega_2 = \mp\omega_1. \tag{21}$$

Thus we have two soliton solutions

$$F_2 = \sqrt{4k_1^2+\delta}\cosh\Big[\omega_1\Big(t - \frac{t_{01}}{2}\Big)\Big]$$
$$+ \sqrt{k_1^2+\delta}\cosh\Big[k_1\Big(x - \frac{x_0}{2}\Big)\Big], \ \hat{f} = \hat{P}_{\mathrm{s}}, \tag{22}$$

with arbitrary constant t_{01} for the PNABB system and

$$F_2 = \sqrt{4k_1^2+\delta}\cosh\Big[\omega_1\Big(t - \frac{t_0}{2}\Big)\Big]$$
$$+ \sqrt{k_1^2+\delta}\cosh\Big[k_1\Big(x - \frac{x_{01}}{2}\Big)\Big], \quad \hat{f} = \hat{P}_{\mathrm{s}}, \tag{23}$$

with arbitrary constant x_{01} for the TNABB system.

For $N=4$, F_4 in Eq. (14) becomes

$$\begin{aligned}F_4 =& K_{\{2,4\}}\cosh(\xi_1^+ - \xi_2^+) + K_{\{2,3\}}\cosh(\xi_1^- - \xi_2^-)\\ &+ K_{\{3\}}\cosh(\xi_1^- + \xi_2^+) + K_{\{2\}}\cosh(\xi_1^+ - \xi_2^-)\\ &+ K_{\{1\}}\cosh(-\xi_1^- + \xi_2^+) + K_{\{3,4\}}\cosh(\xi_1^- + \xi_2^-)\\ &+ K_{\{\}}\cosh(\xi_1^+ + \xi_2^+) + K_{\{4\}}\cosh(\xi_1^+ + \xi_2^-),\end{aligned} \tag{24}$$

$$\xi_1^\pm = \frac{1}{2}(\eta_1 \pm \eta_3), \quad \xi_2^\pm = \frac{1}{2}(\eta_2 \pm \eta_4), \tag{25}$$

where the notation $K_{\{i_1,\ldots,i_j\}}$ is a simplified one for $K_{\{\nu\}}$ with the permutation $\{\nu\} \equiv \{\nu_i = -1, i = i_1,\ldots,i_j; \nu_i = 1, i \neq i_1,\ldots,i_j\}$. For instance, for $n=4$, $K_{\{\nu\}}$ in Eq. (24) possesses the forms $K_{\{\}} = K_{\{1,1,1,1\}}$, $K_{\{4\}} = K_{\{1,1,1,-1\}}$, $K_{\{2,4\}} = K_{\{1,-1,1,-1\}}$ and so on.

From Eq. (25) we know that if we select

$$\eta_3 = \eta_1^{\hat{f}}, \quad \eta_4 = \eta_2^{\hat{f}}, \tag{26}$$

then $\xi_i^\pm$ will be all symmetric or antisymmetric with respect to the operator $\hat{f}$.

To realize that the pair condition (26) is quite trivial as in the $N=2$ case, we fix the constants as

$$\begin{aligned}&k_3 = -k_1, \quad k_4 = -k_2, \quad \omega_3 = \omega_1, \quad \omega_4 = \omega_2,\\ &x_{01} = x_{02} = x_{03} = x_{04} = x_0, \quad t_{03} = t_{01}, \quad t_{04} = t_{02},\end{aligned} \tag{27}$$

for the PNABB system and

$$\begin{aligned}&k_3 = k_1, \quad k_4 = k_2, \quad \omega_3 = -\omega_1, \quad \omega_4 = -\omega_2,\\ &t_{01} = t_{02} = t_{03} = t_{04} = t_0, \quad x_{03} = x_{01}, \quad x_{04} = x_{02},\end{aligned} \tag{28}$$

for the TNABB system.

For the parameter selections (27) and/or (28), we have

$$a_{14}^\pm = a_{23}^\pm, \quad a_{34}^\pm = a_{12}^\pm. \tag{29}$$

After considering the relations (29), one can directly check

$$K_{\{4\}} = K_{\{2\}}, \quad K_{\{3\}} = K_{\{1\}}. \tag{30}$$

Using the relation (30) or equivalently (29), Eq. (24) becomes

$$\begin{aligned}F_4 =& K_{\{\}}\cosh(\xi_1^+ + \xi_2^+) + K_{\{34\}}\cosh(\xi_1^- + \xi_2^-)\\ &+ K_{\{24\}}\cosh(\xi_1^+ - \xi_2^+) + K_{\{23\}}\cosh(\xi_1^- - \xi_2^-)\\ &+ K_{\{4\}}[\cosh(\xi_1^+ + \xi_2^-) + \cosh(\xi_1^+ - \xi_2^-)]\\ &+ K_{\{3\}}[\cosh(\xi_2^+ + \xi_1^-) + \cosh(\xi_2^+ - \xi_1^-)].\end{aligned} \tag{31}$$

Now we are ready to write down the invariant conditions for the $N=2n$ soliton solutions (14) for the shifted parity nonlocal and delayed time reversal nonlocal ABB Eq. (1).

Taking the constant pair conditions

$$k_{n+i} = -k_i, \omega_{n+i} = \omega_i, \; x_{0i} = x_0, \; i = 1,2,\ldots,n, \tag{32}$$

with arbitrary constants k_i and $t_{0i} = t_{0(n+i)}$ for the PNABB system and

$$k_{n+i} = k_i, \quad \omega_{n+i} = -\omega_i, \quad t_{0i} = t_0, \quad i = 1,2,\ldots,n, \tag{33}$$

with arbitrary constants k_i and $x_{0i} = x_{0(n+i)}$ for the TNABB equation, one can directly verify that the following relations

$$\eta_{n+i} = \eta_i^{\hat{f}}, \tag{34}$$

$$a_{i(n+j)}^{\nu_i\nu_{n+j}} = a_{j(n+i)}^{\nu_j\nu_{n+i}}, \quad a_{(n+i)(n+j)}^{\nu_{n+i}\nu_{n+j}} = a_{ij}^{\nu_i\nu_j} \tag{35}$$

are correct for both $\hat{f} = \hat{P}_{\rm s}$ and $\hat{f} = \hat{T}_{\rm d}$. Thus thanks to the paired conditions (32) and/or (33), the $N=2n$ solution (14) becomes an $\hat{f}$-invariant solution.

To solve $\hat{f}$-antisymmetric v Eq. (5), we can apply the symmetry knowledge on the usual Boussinesq Eq. (4). It is known that there are infinitely many symmetries for the integrable Boussinesq Eq. (4). Two of the simplest symmetry solutions are u_t and u_x related to the space-time translation invariance. Thus we have a special $\hat{f}$-antisymmetric v-solution

$$v = \begin{cases} au_x + bu_t, & \hat{f} = \hat{P}_{\rm s}\hat{T}_{\rm d},\\ au_x, & \hat{f} = \hat{P}_{\rm s},\\ bu_t, & \hat{f} = \hat{T}_{\rm d},\end{cases} \tag{36}$$

with arbitrary constants a and b.

Finally, the multiple soliton solutions for ABB can be written as

$$A = \begin{cases} (1 + a\partial_x + b\partial_t)u, & \hat{f} = \hat{P}_{\rm s}\hat{T}_{\rm d},\\ (1 + a\partial_x)u, & \hat{f} = \hat{P}_{\rm s},\\ (1 + b\partial_t)u, & \hat{f} = \hat{T}_{\rm d},\end{cases} \tag{37}$$

with u given by (14) with (13) and (15) under the conditions (16), (32) and (33) for $\hat{f} = \hat{P}_{\rm s}\hat{T}_{\rm d}$, $\hat{f} = \hat{P}_{\rm s}$ and $\hat{f} = \hat{T}_{\rm d}$, respectively.

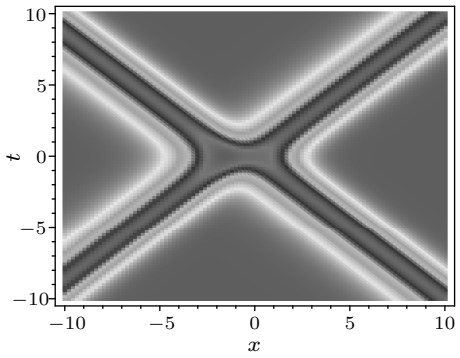

Fig. 1. Two-soliton interaction of the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$.

Figure 1 is a density plot of the two-soliton interaction solution described by Eq. (37) for the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$ under the parameter selections ($t_{0j} = x_{0j} = x_0 = t_0 = \delta = 0$ for Figs. 1–4)

$$N = 2, \quad k_1 = -k_2 = a = 1. \tag{38}$$

Figure 2 is a density plot of the t-breather described by Eq. (37) for the PNABB (1) with $\hat{f} = \hat{P}_{\rm s}$ under the parameter selections

$$N = 2, \quad k_1 = -k_2 = 0.5, \quad d = -2, \quad a = 1. \tag{39}$$

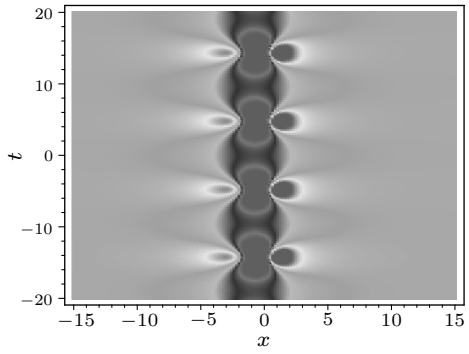

Fig. 2. The t-breather of the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$.

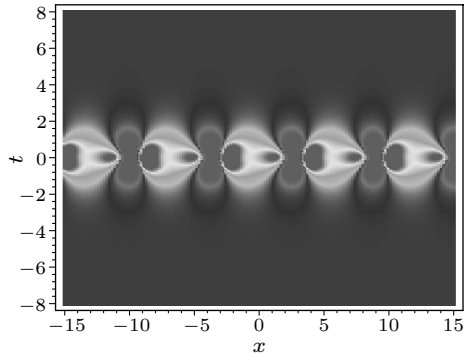

Fig. 3. The x-breather of the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$.

Figure 3 is a density plot of the x-breather described by Eq. (37) for the PNABB (1) with $\hat{f} = \hat{P}_{\rm s}$ under the parameter selections

$$N = 2, \quad k_1 = -k_2 = \sqrt{-1}, \quad d = -2, \quad a = 1. \tag{40}$$

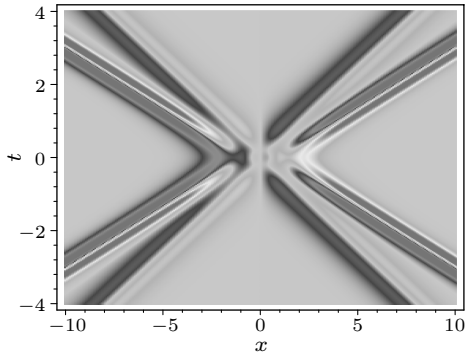

Fig. 4. Four-soliton interaction solution of the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$.

Figure 4 is a density plot of the four-soliton interaction solution described by Eq. (37) for the PNABB Eq. (1) with $\hat{f} = \hat{P}_{\rm s}$ under the parameter selections

$$N = 4, \quad k_1 = -k_3 = 3, \quad k_2 = -k_4 = -d = 2, \quad a = 0.1. \tag{41}$$

From the t-breather (Fig. 2) and x-breather (Fig. 3), we know that there are rogue wave solutions which correspond to the limit cases of the breathers with infinite periods. It is quite simple to verify that the single rogue wave solution of the nonlocal ABB Eq. (1) possesses the form (37) with

$$u = 2\partial_x^2 \ln\left[-\frac{3}{\delta} + x'^2 - \delta t'^2\right], \quad x' = x - x_1, \quad t' = t - t_1 \tag{42}$$

where the constants x_1 and t_1 should be fixed as

$$x_1 = \frac{x_0}{2}, \ t_1 = \frac{t_0}{2}, \ \text{for } \hat{f} = \hat{P}_{\rm s}\hat{T}_{\rm d},$$
$$x_1 = \frac{x_0}{2}, \quad \text{for } \hat{f} = \hat{P}_{\rm s},$$
$$t_1 = \frac{t_0}{2}, \quad \text{for } \hat{f} = \hat{T}_{\rm d}.$$

From Eq. (42) we know that the algebraic solution (37) with (42) exists only for $\delta \neq 0$ and is analytical for $\delta < 0$. Figure 5 displays the structures of the rogue wave for the PNABB equation. Figure 5(a) shows the structure of the four-leaf rogue wave described by (37) with (42) and the parameter selections $\delta = -3$ and $a = x_0 = t_1 = 0$. Figure 5(b) shows the structure of the six-leaf rogue wave under the parameter selections $x_0 = t_1 = 0$, $\delta = -3$ and $a = 30$. In fact, the first-order rogue waves of the nonlocal ABB systems can be changed from four leaves to five leaves and six leaves as the parameter a increases. However, in local systems, first-order rogue waves can only have four leaves.

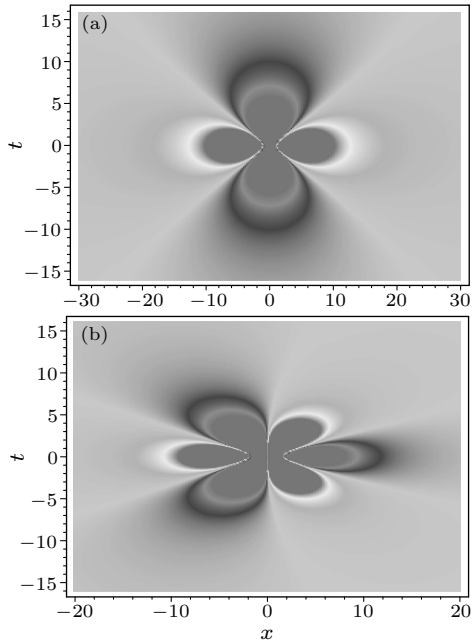

Fig. 5. Rogue wave solution of the PNABB system (1) with $\hat{f} = \hat{P}_{\rm s}$. (a) Four-leaf rogue wave and (b) six-leaf rogue wave.

In fact, multiple breather solutions can be read off from the multiple soliton solution expression (37) by appropriately selecting parameters, and the multiple higher order rogue waves can be obtained from the

limit cases of the multiple breathers. The final form of the multiple rogue waves of the nonlocal ABB also possesses the form (37) with

$$u = 2(\ln G_n)_{xx}, \quad G_n = \sum_{m=0}^{n(n+1)/2} \sum_{j=0}^{m} a_{jm} x'^{2j} t_1^{2m-2j},$$

with suitable constants a_{jm}. For instance, for the second rogue wave (for $\delta < 0$), we have

$$\begin{aligned} G_2 =& \delta^3(\delta t'^2 - x'^2)^3 + \delta^2(17\delta t'^2 - 5x'^2)(\delta t'^2 - 5x'^2) \\ &+ 25\delta(5x'^2 + 19\delta t'^2) + 2\alpha\delta x'(3\delta^2 t'^2 + \delta x'^2 + 1) \\ &+ 2t'\delta\beta(\delta^2 t'^2 + 3\delta x'^2 - 5) - \alpha^2\delta + \beta^2 + 1875, \end{aligned}$$

where x' and t' are defined the same as in Eq. (42), and α and β are arbitrary constants.

In summary, three special ABB systems with different nonlocalities $\hat{f} = \hat{P}_{\rm s}\hat{T}_{\rm d}$, $\hat{P}_{\rm s}$ and $\hat{T}_{\rm d}$ are investigated. Some types of multi-soliton solutions including paired solitons, x- and t-breathers, and rogue waves are obtained. The soliton structures of the ABB systems are quite different from those of the usual Boussinesq equation. In particular, the odd numbers of solitons are prohibited for the PNABB and TNABB system. Quite different from the local cases, first-order rogue waves of the ABB systems may have four leaves, five leaves or six leaves depending on the different selections of the antisymmetric parameter a.

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