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Dynamics of Quantum Fisher Information in Homodyne-Mediated Feedback Control *

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We investigate the quantum Fisher information (QFI) dynamics of a dissipative two-level system in homodyne-mediated quantum feedback control. The analytical results demonstrate that the maximum values and stable values of the QFI can be greatly enhanced via feedback control. The quantum feedback plays a more evident role in the improvement of classical Fisher information. The classical part can reach a high stable value, while the quantum part eventually decays to zero whatever the feedback parameter is.

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Parameter estimation is a fundamental problem in gravitational-wave detectors,[1] frequency spectroscopy,[2] interferometry,[3] and atomic clocks.[4] Quantum parameter estimation has been extensively investigated in recent decades.[5−11] It aims to enhance the resolution and measurement precision of an unknown physical parameter using the properties of quantum mechanics. Nevertheless, actual quantum systems are inevitably coupled with their surrounding environment.[12−16] Quantum states which are very sensitive to a change of the parameters are usually also sensitive to noise disturbance. This will have a negative impact on the quantum parameter estimation scheme. Hence, how to overcome the influence of environmental noise on the precision of parameter estimation is one of the core issues in the field of quantum metrology.

To enhance parameter-estimation precision, various methods such as decoherence-free subspace,[7] dynamical decoupling,[8] quantum error correction,[9] and reservoir engineering[10] have been used. Quantum feedback control is regarded as a very potential method to deal with the problem of decoherence.[17−22] Quantum feedback based on photo-detection measurement[23] was first used to enhance the parameter precision of optimal quantum estimation of a dissipative qubit. Inspired by Ref. [23], Zhang *et al.* studied the steady quantum Fisher information (QFI) of two driven and collectively damped qubits with homodyne-mediated feedback.[24] The same research group also studied the parameter precision of detection efficiency.[25] This work aims to further study the applications of quantum feedback control in parameter estimation of quantum decoherence. We investigate the QFI dynamics of a dissipative two-level system by homodyne-mediated quantum feedback. To study the effect of quantum feedback on the QFI, we separate the QFI into two parts. The classical and quantum parts in QFI are respectively studied. The results show that the appropriate choice of feedback parameters evidently increases the QFI. The quantum feedback plays a more important role in the dynamics evolution of the classical part than the quantum part.

QFI is a central quantity in quantum metrology. The definition of the QFI with respect to the parameter θ is $F_\theta := \mathrm{Tr}(\rho_\theta L_\theta^2)$,[26] where the symmetry logarithmic derivative L_θ is determined by $\partial_\theta \rho_\theta = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta)$. Utilizing the spectrum decomposition $\rho_\theta = \sum_{i=1}^{M} P_i |\Psi_i\rangle\langle\Psi_i|$, the QFI can be separated into two parts,[6,27]

$$F_\theta = F_{\mathrm{C}}(\rho) + F_{\mathrm{Q}}(\rho), \tag{1}$$

where $F_{\mathrm{C}}(\rho)$ indicates the classical Fisher information

$$F_{\mathrm{C}}(\rho) = \sum_{i=1}^{M} \frac{(\partial_\theta P_i)^2}{P_i}, \tag{2}$$

and $F_{\mathrm{Q}}(\rho)$ is regarded as the quantum contribution

$$F_{\mathrm{Q}}(\rho) = \sum_{i=1}^{M} P_i F_{\mathrm{Q}}(|\Psi_i\rangle) - \sum_{i\neq j}^{M} \frac{8P_iP_j}{P_i+P_j} |\langle\Psi_i|\partial_\theta\Psi_j\rangle|^2, \tag{3}$$

with $F_{\mathrm{Q}}(|\Psi_i\rangle) = 4(\langle\partial_\theta\Psi_i|\partial_\theta\Psi_i\rangle - |\langle\Psi_i|\partial_\theta\Psi_i\rangle|^2)$.

In this work, we consider a two-level atom resonantly coupled to a heavily damped single-mode cavity with the coupling constant g and decay rate κ, and driven by a classical field with Rabi frequency Ω. In the large damping limit of the cavity, the cavity modes can be adiabatically eliminated by neglecting the terms greater than the second order,[17,18] then the effective damping rate of the qubit $\Gamma = g^2/\kappa$ can be

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obtained. Furthermore, in the limit $\Gamma \gg \gamma$, the effect of the spontaneous emission is ignored. The master equation prompting the atomic evolution is given by ($\hbar = 1$)[19]

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[c]\rho, \tag{4}$$

where the driving Hamiltonian of the system is $H = \Omega(\sigma_- + \sigma_+)$, and the Lindblad operator is $c = \sqrt{\Gamma}\sigma_-$. The operators $\sigma_- = |g\rangle\langle e|$ and $\sigma_+ = |e\rangle\langle g|$ are the lowering and raising operators with the two levels of the atom $|e\rangle$ and $|g\rangle$, respectively. The Liouvillian superoperator $\mathcal{D}$ for irreversible evolution is defined by $\mathcal{D}[c]\rho \equiv c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)$.

In the theory of Markovian feedback, we introduce a control Hamiltonian $H_{\rm fb} = I(t)F$ depending on the signal $I(t)$ from the homodyne detection of the cavity output. The feedback Hamiltonian is constantly applied to the atom according to the continuous homodyne detection photocurrent $I(t)$ from the detector. The effective feedback master equation of the system can be written as[20,21]

$$\dot{\rho} = -i\Big[H + \frac{1}{2}(c^\dagger F + Fc), \rho\Big] + \mathcal{D}[c - iF]\rho. \tag{5}$$

By comparison, we choose the simple and effective feedback Hamiltonian $F = \lambda\sigma_y$, where $\lambda \in \mathbb{R}$ is a controllable feedback parameter, and σ_y is the usual Pauli matrix.

In the following we discuss in detail the QFI dynamics corresponding to the general superposition initial state $|\psi_0\rangle = \cos\alpha|e\rangle + \sin\alpha|g\rangle$. The evolved density matrix of the qubit can be exactly solved, and the corresponding matrix elements in the basis $\{|e\rangle, |g\rangle\}$ are

$$\begin{aligned}\rho_{11}(t) &= \frac{e^{-\xi t}(\Gamma + 2\sqrt{\Gamma}\lambda + 2e^{\xi t}\lambda^2 + \xi\cos(2\alpha))}{2\xi},\\ \rho_{12}(t) &= \frac{1}{2}e^{-\frac{1}{2}t(\sqrt{\Gamma}+2\lambda)^2}\sin(2\alpha),\end{aligned} \tag{6}$$

with $\xi = \Gamma + 2\sqrt{\Gamma}\lambda + 2\lambda^2$. Based on the normalizing condition and the coherence between the basis states, other matrix elements are $\rho_{22}(t) = 1 - \rho_{11}(t)$ and $\rho_{21}(t) = \rho_{12}^*(t)$. We first concentrate on $\alpha = 0, \pi/2$ and $\pi/4$.

For the initial state $|e\rangle$, the specific expressions of the evolved density matrix can be obtained by Eq. (6), taking $\alpha = 0$. According to Eqs. (2) and (3), the analytical expressions of the classical and quantum parts of the QFI are obtained as

$$\begin{aligned}F_{\rm C}(\rho) &= \frac{[(e^{\xi t} - 1)\lambda^2 + \xi t(\sqrt{\Gamma} + \lambda)^2]^2}{\Gamma\xi^2(e^{\xi t} - 1)[e^{\xi t}\lambda^2 + (\sqrt{\Gamma} + \lambda)^2]},\\ F_{\rm Q}(\rho) &= 0.\end{aligned} \tag{7}$$

Similarly, for the initial state $|g\rangle$, $F_{\rm Q}(\rho) = 0$. The classical Fisher information is

$$F_{\rm C}(\rho) = \frac{\lambda^2(\sqrt{\Gamma} + \lambda)^2(1 - e^{\xi t} + \xi t)^2}{\Gamma\xi^2(e^{\xi t} - 1)[\lambda^2 + e^{\xi t}(\sqrt{\Gamma} + \lambda)^2]}. \tag{8}$$

The above analytical solutions show that the QFI only depends on the classical part $F_{\rm C}(\rho)$ for the initial states $|e\rangle$ and $|g\rangle$. Furthermore, the feedback control only plays a role in the evolution of the classical part $F_{\rm C}(\rho)$, and the quantum part $F_{\rm Q}(\rho)$ is always zero no matter how high the feedback strength λ is. Figure 1 illustrates the time evolution of $F_{\rm C}(\rho)$ for different feedback strengths λ with the initial states $|e\rangle$ and $|g\rangle$. In the absence of feedback, the QFI increases from zero to a maximum value and rapidly decays to zero for the initial state $|e\rangle$, and the QFI is always zero for the initial state $|g\rangle$. The maximum values and stable values of the QFI are greatly enhanced by introducing homodyne-mediated quantum feedback, and the QFI increases to a steady value monotonously for the initial state $|g\rangle$. However, the stable value of the QFI does not enhance with the increase of the feedback strength λ and an optimal value should exist. We will discuss the QFI of steady states in more detail in the following.

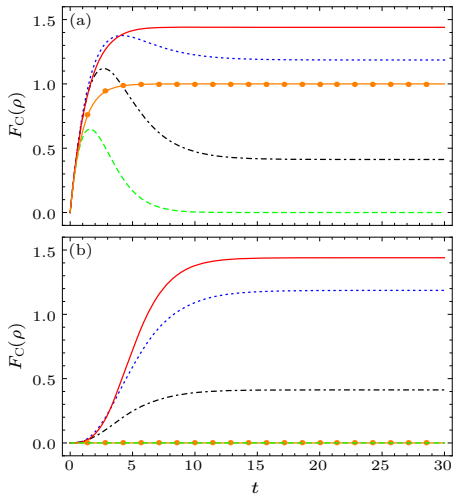


Fig. 1. The time evolution of $F_{\rm C}(\rho)$ for different feedback strengths λ corresponding to the initial states (a) $|e\rangle$ and (b) $|g\rangle$. Here $\lambda = 0$ (green dashed curve), $\lambda = -0.35$ (black dot-dashed curve), $\lambda = -0.55$ (blue dotted curve), $\lambda = -0.75$ (red solid curve), and $\lambda = -1.0$ (orange curve with circle dot). The other parameters are $\Gamma = 1$ and $\Omega = 0$.

For the equal-weighted superposition initial state $(|e\rangle + |g\rangle)/\sqrt{2}$, we also calculate the time evolution of the classical part $F_{\rm C}(\rho)$, the quantum part $F_{\rm Q}(\rho)$, and the QFI $F_\Gamma(\rho)$, respectively. However, their analytical expressions are relatively cumbersome so they are not given in the work. The QFI is jointly contributed by the classical part $F_{\rm C}(\rho)$ and the quantum part $F_{\rm Q}(\rho)$. In Fig. 2, we plot the dynamic evolution of $F_{\rm C}(\rho)$, $F_{\rm Q}(\rho)$, and $F_\Gamma(\rho)$ for different feedback strengths λ, respectively. The QFI $F_\Gamma(\rho)$ is evidently enhanced by choosing the appropriate quantum feedback parameter, and the maximum instantaneous value of QFI in time evolution is markedly larger than the ones for the initial states $|e\rangle$ and $|g\rangle$ in Fig. 1, but the maximum stable value of QFI is not improved. Furthermore,

we find that quantum feedback plays a more important role in the evolution of the classical part $F_{\rm C}(\rho)$ than the quantum part $F_{\rm Q}(\rho)$ by comparing Fig. 2(a) against Fig. 2(b). Although the instantaneous values of $F_{\rm Q}(\rho)$ can be increased for different feedback strengths λ, all the values of $F_{\rm Q}(\rho)$ eventually decay to zero. Nevertheless, the classical part $F_{\rm C}(\rho)$ can reach a high stable value for an appropriate feedback strength λ. Therefore, the quantum feedback actually enhances the stable values of the classical Fisher information.

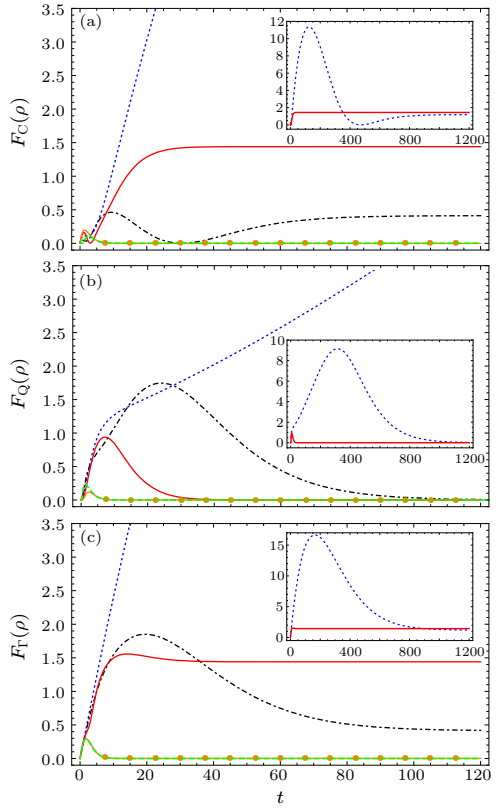


Fig. 2. The time evolution of (a) $F_{\rm C}(\rho)$, (b) $F_{\rm Q}(\rho)$, and (c) $F_{\Gamma}(\rho)$ under different feedback strengths λ for the initial state $(|e\rangle + |g\rangle)/\sqrt{2}$. The other parameters and corresponding values of λ are the same as those in Fig. 1.

Without loss of generality, we study the dynamic behavior of the QFI for a general superposition initial state $|\psi_0\rangle = \cos\alpha|e\rangle + \sin\alpha|g\rangle$. The analytic expressions of $F_{\rm C}(\rho)$, $F_{\rm Q}(\rho)$, and $F_{\Gamma}(\rho)$ can be acquired. The expressions are very complex, and we only demonstrate their time evolutions in Fig. 3. The effect of quantum feedback on the improvement of $F_{\rm C}(\rho)$ is still very evident. Here $F_{\rm C}(\rho)$ can reach high stable values for different initial superposition states, but $F_{\rm Q}(\rho)$ eventually decays to zero. Furthermore, we find that the maximum value of the QFI appears at $\alpha = \pi/4$ and $3\pi/4$ corresponding to the initial states $(|g\rangle \pm |e\rangle)/\sqrt{2}$ with $\Gamma = 1$ and $\lambda = -0.75$.

We have discussed the QFI dynamics when the qubit is in a pure state initially. Next we investigate the QFI for a mixed state $\rho_0 = p|e\rangle\langle e| + (1-p)|g\rangle\langle g|(p \in (0,1))$. The evolved density matrix elements are as follows:

$$\rho_{11}(t) = pe^{-\xi t} + \frac{(1-e^{-\xi t})\lambda^2}{\xi}, \quad \rho_{12}(t) = 0. \quad (9)$$

The quantum part is also $F_{\rm Q}(\rho) = 0$. The classical Fisher information is

$$F_{\rm C}(\rho) = \{(\sqrt{\Gamma}+\lambda)^2[p\xi^2 t - \lambda^2(1-e^{\xi t}+\xi t)]^2\} \cdot \{\Gamma\xi^2[\lambda^2 + e^{\xi t}(\sqrt{\Gamma}+\lambda)^2 - p\xi][(e^{\xi t}-1)\lambda^2 + p\xi]\}^{-1}. \quad (10)$$

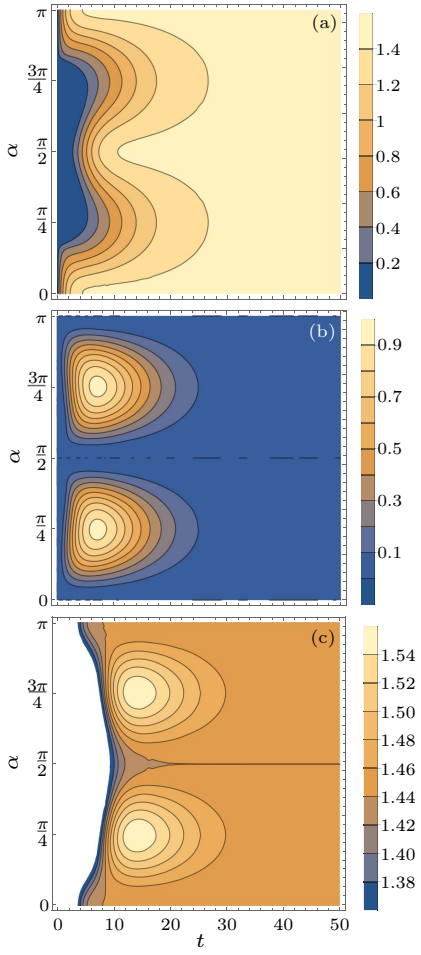


Fig. 3. The values of (a) $F_{\rm C}(\rho)$, (b) $F_{\rm Q}(\rho)$, and (c) $F_{\Gamma}(\rho)$ with respect to α and t for the superposition initial state $|\psi_0\rangle = \cos\alpha|e\rangle + \sin\alpha|g\rangle$ with $\Gamma = 1$, $\Omega = 0$, and $\lambda = -0.75$.

The dynamic evolutions of $F_{\rm C}(\rho)$ with and without feedback control are displayed in Fig. 4. In the absence of the feedback (i.e., $\lambda = 0$), $F_{\rm C}(\rho) = \frac{pt^2}{e^{t\Gamma}-p}$, which has large exponential decay in the long-time limit, and the stable value of $F_{\rm C}(\rho)$ is 0. The value of $F_{\rm C}(\rho)$ increases with the probability in the excited state p, i.e., the optimal initial state is $|e\rangle$ without the feedback. When the feedback parameter is chosen as $\lambda = -0.75$, the stable value of $F_{\rm C}(\rho)$ is 1.44 in the long-time limit. We can easily find that the stable value of $F_{\rm C}(\rho)$ is evidently improved in the presence

of the feedback for arbitrary initial states. As a short summary, the QFI can be enhanced by the feedback and the improvement of the classical part is more evident whether the initial state is pure or mixed.

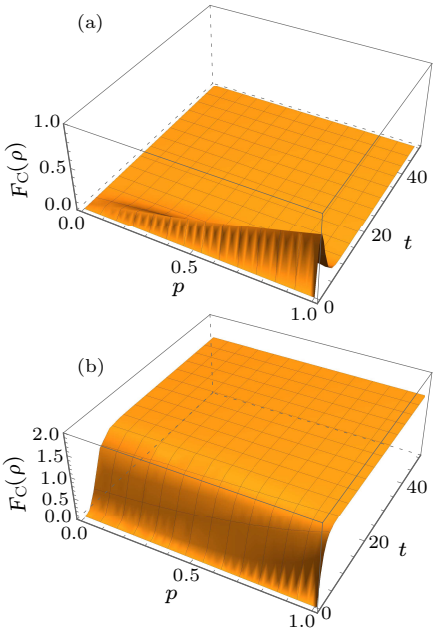


Fig. 4. The classical Fisher information $F_C(\rho)$ as functions of the excited-state probability p and time t (a) without feedback $\lambda = 0$ and (b) with feedback $\lambda = -0.75$. The other parameters are $\Gamma = 1$ and $\Omega = 0$.

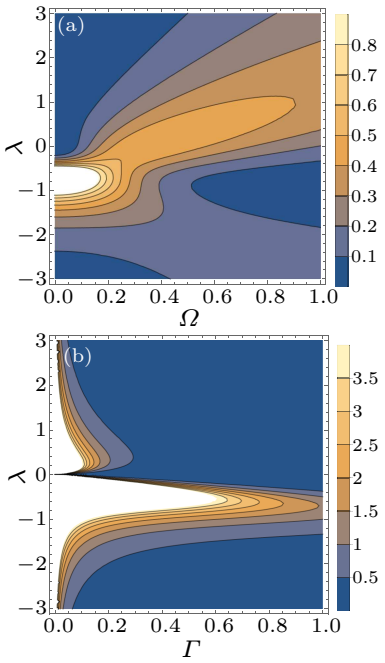


Fig. 5. (a) The steady-state QFI F_∞ as functions of the Rabi frequency Ω and the feedback control strength λ with the damping rate $\Gamma = 1$, and (b) F_∞ as functions of λ and Γ with $\Omega = 0$.

The above results show that the system can reach the dynamic balance for the different initial states. Now we study QFI of the steady state because it is very important in realistic quantum information processing. The stable solution of QFI for arbitrary initial states with respect to the damping parameter Γ is

$$F_\infty = \frac{\Gamma^2\lambda^2 + 16\Omega^2(\sqrt{\Gamma}+\lambda)^2}{\Gamma(\Gamma^2 + 2\Gamma^{3/2}\lambda + 2\Gamma\lambda^2 + 8\Omega^2)^2}. \tag{11}$$

From Fig. 5(a), it is easy to find that the maximum value of QFI appears when the Rabi frequency of the classical field is $\Omega = 0$. The maximum value is approximately equal to 1.46 with $\lambda \approx -0.71$. Figure 5(b) indicates that the stable value of the QFI decreases with the increase of the damping rate Γ, which implies that the greater the decoherence parameter is, the lower the estimation precision becomes. The stable value of the QFI is zero in the absence of the feedback, while the estimation precision can be significantly strengthened by tuning the optimal feedback strength λ. The larger stable values can be obtained in the range of $\lambda < 0$.

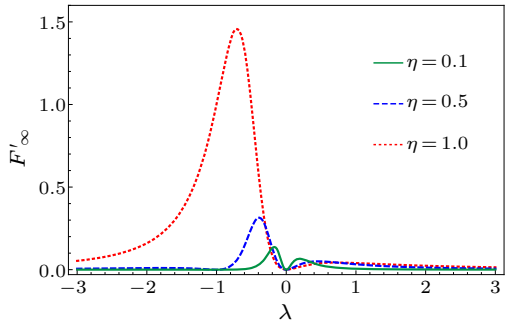


Fig. 6. Variation of the steady-state QFI F'_∞ with respect to the feedback parameter λ for different detection efficiencies η with $\Gamma = 1$ and $\Omega = 0$.

Until now, all of the analyses are based on the perfect detection process. However, in practical feedback schemes, the detection efficiency is a very important factor as feedback is conditioned on the measurement result. To take this factor into account, the corresponding master equation (5) should be modified as[28]

$$\dot{\rho} = -i[H + \frac{1}{2}(c^\dagger F + Fc), \rho] + \mathcal{D}[c - iF]\rho + \frac{1-\eta}{\eta}\mathcal{D}[F]\rho, \tag{12}$$

where η represents the efficiency of the detector. As a result, the analytic expression of steady-state QFI with respect to Γ is

$$F'_\infty = \frac{\eta^2\lambda^2(\sqrt{\Gamma}+\lambda)^2}{\Gamma(\Gamma\eta + 2\sqrt{\Gamma}\eta\lambda + \lambda^2)(\Gamma\eta + 2\sqrt{\Gamma}\eta\lambda + 2\lambda^2)^2}. \tag{13}$$

We display the effect of detection efficiencies on the QFI of the steady state in Fig. 6. It is obvious that the imperfect detection results in the decreasing steady-state QFI. This is reasonable because less efficient detection will lead to the weakening of effective control. However, we can find the optimal value of λ corresponding to the maximum value of the steady-state QFI for different detection efficiencies η.

In summary, we have investigated the QFI dynamics for a dissipative qubit system by quantum feedback

based on homodyne measurement. To study the effect of quantum feedback on the dynamical evolution of QFI, the classical Fisher information and the quantum contribution are discussed. The analytical results show that homodyne-mediated quantum feedback evidently increases the maximum values and stable values of the QFI. The quantum feedback plays a more important role in the dynamic evolution of the classical part, whether the initial state is pure or mixed. The classical part can reach a high stable value for an appropriate feedback strength. Nevertheless, the stable values of the quantum part are independent of the feedback parameter values and always remain at zero. Therefore, quantum feedback actually enhances the stable values of the classical Fisher information.

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