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Bound-State Soliton Solutions of the Nonlinear Schrödinger Equation and Their Asymmetric Decompositions *

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We study the asymmetric decompositions of bound-state (BS) soliton solutions to the nonlinear Schrödinger equation. Assuming that the BS solitons are split into multiple solitons with different displacements, we obtain more accurate decompositions compared to the symmetric decompositions. Through graphical techniques, the asymmetric decompositions are shown to overlap very well with the real trajectories of the BS soliton solutions.

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Soliton solutions to nonlinear partial differential equations have attracted significant attention since the finding of completely integrable partial differential equations. There is a wide range of literature on integrable nonlinear partial differential equations and their soliton solutions[1–4] (and references therein). Most of the literature addresses the behavior of solitons and their interactions in various integrable systems. Such knowledge is very valuable not only for the underlying integrable systems, but also for nearly-integrable systems that can be studied analytically by soliton perturbation theories. A remarkable characteristic of solitons is that the interactions between them are elastic. This means that, after each interaction, the solitons maintain the same velocity and amplitude evolving along the same orientation as before, except for phase shifts and displacements.[5]

The bound-state (BS) solitons of nonlinear partial differential equations are also extensively studied in literature. BS solitons are special solitons that have multiple components with the same velocity. In this Letter, we consider the asymmetric asymptotic behavior of BS solitons of the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{1}$$

The solitons of the NLS equation were first derived by Zakharov and Shabat[6] and have the following decomposition

$$u(x,t) \sim \sum_{j=1}^{N} u_j^{\pm}(x,t), \quad t \to \pm\infty, \tag{2}$$

where

$$u_j^{\pm}(x,t) = 2\beta_j \exp[-2i\alpha_j x + 4i(\alpha_j^2 - \beta_j^2)t - i(\psi_j^{\pm} + \pi/2)\text{sech}(2\beta_j x - 8\alpha_j\beta_j t - 2\delta_j^{\pm})],$$

$$e^{2(\delta_j^+ - \delta_j^-) + i(\psi_j^+ - \psi_j^-)} = \prod_{l=1}^{j-1}\left(\frac{k_j - k_l}{k_j - k_l^*}\right)^2 \prod_{m=j+1}^{N}\left(\frac{k_j - k_m^*}{k_j - k_m}\right)^2,$$

and $k_j = \alpha_j + i\beta_j$ $(j = 1, 2, 3, \ldots, N)$ are eigenvalues. When α_j and β_j $(j = 1, 2, 3, \ldots, N)$ go to the same values α and β respectively, the formulae of the BS solitons are derived. The BS soliton solution to the NLS equation was also derived in Ref. [6] by solving the associated Gelfand–Levitan–Marchenko equation. In addition, the decomposition for the second-order BS soliton was given in Ref. [6] and has the following symmetric form

$$u(x,t) \sim 2\beta e^{-2i\alpha x + 4i(\alpha^2 - \beta^2)t} \cdot \{\text{sech}[\theta + \ln|2^4\beta^2 t|] + \text{sech}[\theta - \ln|2^4\beta^2 t|]\}, \quad t \to \pm\infty. \tag{3}$$

The symmetric decompositions for arbitrary-order BS solitons were given in Refs. [7,8] (there are typos in the second-order BS soliton and its decomposition in Ref. [6], and the correct version is given in Ref. [7]). Although these decompositions illustrate the asymptotic behavior of BS solitons very well and describe the distances between components accurately, their trajectories do not overlap the real trajectories well. In this Letter, we analyze the asymmetric decompositions of BS solitons, which will result in very good approximations for the real trajectories.

The NLS equation is one of the most important equations in physics. It governs the evolution of the complex envelope of the weakly nonlinear one-dimensional wave packet in a generic physical system. Hence, this equation appears in many different fields, such as deep-water waves, optics, acoustics, and Bose–Einstein condensation. Over the past 50 years, this equation has received a great deal of attention.

The NLS equation is the compatible condition of the following Lax pair

$$\begin{aligned} \partial_x\psi(x,t,k) &= M\psi(x,t,k), \\ \partial_t\psi(x,t,k) &= N\psi(x,t,k), \end{aligned} \tag{4}$$

where M and N are 2×2 matrices defined by

$$\begin{aligned} M &= -ik\sigma_3 + U, \\ N &= -2ik^2\sigma_3 + 2kU - i(U^2 + U_x)\sigma_3, \end{aligned}$$

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with

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & u \\ -u^* & 0 \end{bmatrix},$$

and the superscript asterisk meaning complex conjugate. This means that $M_t - N_x + [M, N]$ leads to the NLS equation, where $[\cdot,\cdot]$ represents the commutator. To derive the BS solitons of the NLS equation, we could adopt the inverse scattering method,[9–11] the Darboux transformation,[12,13] or the Hirota method,[14] among others. In this study, we use the inverse scattering method to derive the BS solitons by developing the appropriate Riemann–Hilbert problem (RHP). Since the general procedure for constructing the RHP has been introduced many times in literature,[15,16] we do not introduce it in detail, and it is not the main point of this study.

Theorem 1: The RHP of the NLS equation is

$$M(x,t,k) = \begin{cases} \left[[\phi_-]_1(x,t,k) \quad \dfrac{[\phi_+]_2(x,t,k)}{a(k)} \right], \\ \qquad\qquad imk > 0, \\ \left[\dfrac{[\phi_+]_1(x,t,k)}{a^*(k^*)} \quad [\phi_-]_2(x,t,k) \right], \\ \qquad\qquad imk < 0, \end{cases} \tag{5}$$

and

$$\begin{aligned} M_+(x,t,k) = & M_-(x,t,k) e^{-i(kx+2k^2t)\sigma_3} \\ & \cdot J(k) e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathbb{R}, \end{aligned} \tag{6}$$

where $a(k)$ and $b(k)$ are scattering data, $r(k) = \frac{b(k)}{a(k)}$, $[\cdot]_j$ means the jth column of matrix $[\cdot]$

$$\begin{aligned} \phi(x,t,k) &= \psi(x,t,k)\, e^{i(kx+2k^2t)\sigma_3}, \\ M_\pm(x,t,k) &= \lim_{\varepsilon \to 0} M(x,t,k \pm i\varepsilon), \quad k \in \mathbb{R}, \end{aligned}$$

and the jump matrix $J(k)$ is

$$J(k) = \begin{bmatrix} 1 & r(k) \\ r^*(k) & 1 + |r(k)|^2 \end{bmatrix}.$$

In addition, the solution to the NLS equation can be constructed as

$$u(x,t) = \lim_{k \to \infty} 2ikM_{12}(x,t,k). \tag{7}$$

As shown in Refs. [6,7], the BS solitons are related to the higher-order poles of the reflection coefficient $r(k)$. When $r(k)$ has only one second-order pole k_0 on the upper half k-plane, it is reasonable to set

$$r(k) = r_0(k) + \frac{r_1}{k - k_0} + \frac{r_2}{(k - k_0)^2},$$

where $r_j (j = 1, 2)$ are complex parameters, and $r_0(k)$ is analytic on the upper half plane. Solving the RHP, we obtain the second-order BS soliton solution to the NLS equation as follows:

$$u(x,t) = \frac{|\hat{\Delta}|}{|\Delta|}, \tag{8}$$

where $\Delta = I + \Omega^* \Omega$ (I is the unit matrix)

$$\begin{aligned} \hat{\Delta} &= \begin{bmatrix} \eta_1 & \Delta_{12} \\ \eta_2 & \Delta_{22} \end{bmatrix}, \\ \Omega &= \begin{bmatrix} \Omega_{11} & \Omega_{21} \\ -\dfrac{r_2^* f_0^*}{k_0^* - k_0} & -\dfrac{r_2^* f_0^*}{(k_0^* - k_0)^2} \end{bmatrix}, \\ \eta_s &= \sum_{j=s}^{N} r_j f_{j-s}(x,t), \;\; f_l(x,t) \\ &= \lim_{k \to k_0} \frac{1}{l!} \frac{\partial^l}{\partial k^l} \exp[-2ik(x + 2\,kt)], \\ \Omega_{11} &= -\frac{r_1^* f_0^*}{k_0^* - k_0} - \frac{r_2^* f_1^*}{k_0^* - k_0} + \frac{r_2^* f_0^*}{(k_0^* - k_0)^2}, \\ \Omega_{21} &= -\frac{r_1^* f_0^*}{(k_0^* - k_0)^2} - \frac{r_2^* f_1^*}{(k_0^* - k_0)^2} + \frac{2r_2^* f_0^*}{(k_0^* - k_0)^3}. \end{aligned}$$

Let $r_1 = 1$, $r_2 = i$, and $k_0 = \alpha + i\beta$, then we obtain the explicit expression of the second-order BS soliton solution to the NLS equation

$$u(x,t) = \frac{u_N}{u_D} \times 2^{12}\beta^3 \times \exp\left[-2i\alpha x - 4i(\alpha^2 - \beta^2)t\right], \tag{9}$$

where

$$\begin{aligned} u_N = & \left[\frac{i}{64} - \frac{\beta^2 t}{16} - \left(\frac{i\alpha t}{16} + \frac{ix}{64} + \frac{i}{128}\right)\beta\right] \exp(\theta) \\ & + \beta^5 \left(i\alpha t + \frac{ix}{4} - \beta t + \frac{i}{8}\right) \exp(-\theta), \\ u_D = & 2^{12} \times \beta^4 \left[\frac{3}{128} + \beta^4 t^2 + \beta^2 \left(\alpha t + \frac{x}{4} + \frac{1}{8}\right)^2 \right. \\ & \left. - \left(\frac{\alpha t}{4} + \frac{x}{16} + \frac{1}{32}\right)\beta\right] + 256\beta^8 \exp(-2\theta) \\ & + \exp(2\theta), \end{aligned}$$

and $\theta = 2\beta(x + 4\alpha t)$. Let $\alpha = 0$ and $\beta = 0.5$, then we obtain the profile of the second-order BS soliton solution, and show it in Fig. 1, which consists of two solitons evolving on two curved trajectories.

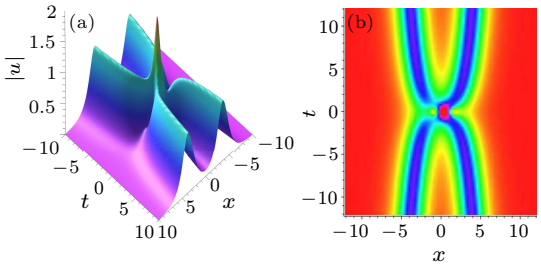

Fig. 1. The second-order bound-state soliton solution to the NLS equation with $\alpha = 0$ and $\beta = 0.5$.

Assuming the above BS soliton has the following asymmetric decomposition

$$\begin{aligned} u \sim & -2\beta\{\mathrm{sech}(\theta + \delta_1(t)) + \mathrm{sech}(\theta - \delta_2(t))\} \\ & \cdot \exp\left[-2i\alpha x - 4i(\alpha^2 - \beta^2)t\right], \;\; |t| \to \pm\infty, \end{aligned} \tag{10}$$

by comparing the denominators of $u(x,t)$ given by Eqs. (9) and (10), we have

$$e^{-2[\delta_1(t)-\delta_2(t)]} = 256\beta^8,\ e^{2\delta_2(t)} = 2^{12}\times\beta^8 t^2.$$

Solving the above equations leads to

$$\delta_1(t) = \ln|2^2 t|,\ \delta_2(t) = \ln|2^6\beta^4 t|. \tag{11}$$

Furthermore, the values of $\delta_1(t)$ and $\delta_2(t)$ can be verified directly through the numerator of $u(x,t)$. Actually, from the numerators of $u(x,t)$ given by Eqs. (9) and (10), it follows

$$-4\beta e^{\delta_2(t)} = -2^8\beta^5 t,\quad -4\beta e^{2\delta_2(t)-\delta_1(t)} = -2^{12}\beta^9 t, \tag{12}$$

which also yields Eq. (11). Thus the asymmetric decomposition of the second-order BS soliton is

$$\begin{aligned} u \sim & -2\beta\{\mathrm{sech}[\theta+\ln|2^2 t|]+\mathrm{sech}[\theta-\ln|2^6\beta^4 t|]\} \\ & \cdot\exp\left[-2i\alpha x-4i(\alpha^2-\beta^2)t\right],\quad |t|\to\pm\infty. \end{aligned} \tag{13}$$

This means that the second-order BS soliton is split into two solitons that have different displacements. The distance between these two solitons keeps the same value $\ln|2^8\beta^4 t|$ for both the symmetric and the asymmetric decompositions. In Fig. 2, we show the evolutions of the BS soliton and its decomposition. The green lines in the background represent the evolution of the second-order BS soliton, and the blue and red lines represent its decomposition. Obviously, the lines overlap very well, which verifies the fidelity of the asymmetric decomposition.

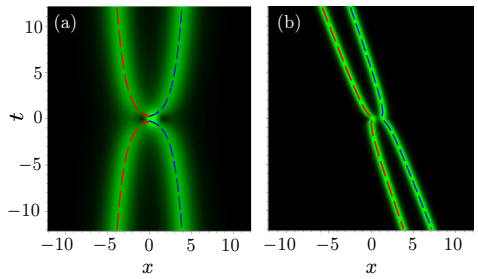

Fig. 2. The decomposition of second-order bound-state soliton: (a) $\alpha = 0$, $\beta = 0.5$, and (b) $\alpha = 0.1$, $\beta = 2$.

When $r(k)$ has only one third-order pole $k_0 = \alpha + i\beta$ on the upper half k-plane, it is reasonable to set

$$r(k) = r_0(k) + \frac{r_1}{k-k_0} + \frac{r_2}{(k-k_0)^2} + \frac{r_3}{(k-k_0)^3},$$

where $r_j (j = 1,2,3)$ are complex parameters, and $r_0(k)$ is analytic on the upper half plane. Solving the RHP, we obtain the third-order BS soliton solution to the NLS equation with $r_1 = 1$, $r_2 = i$, and $r_3 = 1$

$$u = \frac{u_N}{u_D}\times 2^{24}\times i\beta^4\exp[-2i\alpha x-4i(\alpha^2-\beta^2)t], \tag{14}$$

where

$$\begin{aligned} u_N =& 2^{-19}\times\xi_1\exp(5\theta)+\xi_2\beta^6\exp(3\theta) \\ & +\beta^{14}\xi_3\exp(\theta), \\ u_D =& \exp(6\theta)+2^{20}\times\beta^6\xi_4\exp(4\theta)+2^{26} \\ & \times\beta^{12}\xi_5\exp(2\theta)+2^{18}\beta^{18}, \end{aligned}$$

$$\xi_0 = \alpha^2t^2+\frac{\alpha t}{2}\Big(x-\frac{1}{2}\Big)+\frac{x^2}{16}-\frac{x}{16}-\frac{1}{32},$$

$$\begin{aligned} \xi_1 =& 128\beta^4t^2+256i\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)\beta^3 t \\ & -[128\alpha^2t^2+(112i+64\alpha x-32\alpha)t+8x^2 \\ & -8x+8]\beta^2+12(8\alpha t+2x-1)\beta-15, \end{aligned}$$

$$\begin{aligned} \xi_2 =& 2\beta^8t^4+4\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)^2\beta^6t^2-\frac{3\beta^5 t}{8}(4t \\ & +i)\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)+\{2\alpha^4t^4-i\alpha^2t^3-(1 \\ & -2x)\alpha^3t^3+\frac{1}{256}\Big[44+192\Big(x-\frac{1}{2}\Big)^2\alpha^2 \\ & +(-128x+64)i\alpha\Big]t^2+\frac{1}{256}\Big[32\Big(x-\frac{1}{2}\Big)^3\alpha \\ & -16ix^2+16ix+14i\Big]t+\frac{x^4}{128}-\frac{x^3}{64} \\ & +\frac{3x^2}{256}-\frac{x}{256}-\frac{1}{256}\}\beta^4+\frac{15}{32}\Big[-\frac{16}{5}\alpha^2t^2 \\ & +it+\Big(-\frac{8}{5}x+\frac{4}{5}\Big)\alpha t-\frac{x^2}{5}+\frac{x}{5} \\ & -\frac{1}{10}\times\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)\Big]\beta^3+\Big[\frac{31}{64}\alpha^2t^2 \\ & +\Big(-\frac{35i}{512}+\frac{31\alpha x}{128}-\frac{31\alpha}{256}\Big)t+\frac{31x^2}{1024} \\ & -\frac{31x}{1024}+\frac{29}{2048}\Big]\beta^2+\Big(-\frac{9\alpha t}{128}-\frac{9x}{512} \\ & +\frac{9}{1024}\Big)\beta+\frac{21}{8192}, \end{aligned}$$

$$\begin{aligned} \xi_3 =& \beta^2t^2+\frac{i\beta t}{4}-2i\alpha\beta t^2-\frac{i\beta xt}{2}-\frac{\alpha xt}{2}+\frac{x}{16}+\frac{1}{32} \\ & -\alpha^2t^2+\frac{\alpha t}{4}-\frac{x^2}{16}-\frac{it}{8}, \end{aligned}$$

$$\begin{aligned} \xi_4 =& \beta^8t^4+2\xi_0\beta^6t^2-\frac{3t^2}{2}\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)\beta^5 \\ & +\Big[\alpha^4t^4+\Big(x-\frac{1}{2}\Big)\alpha^3t^3+\frac{21}{64}t^2+\Big(\frac{3}{16}+\frac{3}{8}x^2 \\ & -\frac{3}{8}x\Big)\alpha^2t^2+\frac{\alpha t}{16}(x^2-x+1)\Big(x-\frac{1}{2}\Big) \\ & +\frac{1}{256}(x^2-x+1)^2\Big]\beta^4-\Big(\alpha t+\frac{x}{4}-\frac{1}{8}\Big)\Big(\xi_0 \\ & +\frac{3}{32}\Big)\beta^3+\Big(\frac{3\alpha^2t^2}{8}+\frac{3\alpha xt}{16} \\ & -\frac{3\alpha t}{32}+\frac{21}{2048}+\frac{3x^2}{128}-\frac{3x}{128}\Big)\beta^2-\Big(\frac{15x}{1024} \\ & +\frac{15\alpha t}{256}-\frac{15}{2048}\Big)\beta+\frac{57}{16384}, \end{aligned}$$

$$\xi_5 = \beta^8 t^4 + 2\Big(\xi_0 + \frac{3}{32}\Big)\beta^6 t^2 + \xi_0^2\beta^4 - \frac{\xi_0}{2}(\alpha t + \frac{x}{4} - \frac{1}{8})\beta^3 + \frac{9}{64}\Big(\alpha t + \frac{x}{4} - \frac{1}{4}\Big) \times \Big(\alpha t + \frac{x}{4}\Big)\beta^2 - \Big(\frac{3x}{512} + \frac{3\alpha t}{128} - \frac{3\beta}{1024}\Big) + \frac{15}{8192}.$$

Taking a similar method to that above, the third-order BS soliton has the following asymmetric decomposition: $t \to \pm\infty$

$$u \sim 2i\beta\{\mathrm{sech}[\theta + \ln(16|\beta|t^2)] + \mathrm{sech}[\theta - 3\ln|2\beta|] + \mathrm{sech}[\theta - \ln(2^{10}|\beta|^7 t^2)]\} \times \exp[-2i\alpha x - 4i(\alpha^2 - \beta^2)t]. \quad (15)$$

We display the evolutions of the third-order BS soliton and its decompositions in Fig. 3. It is clearly shown that the asymmetric decompositions overlap the real trajectories very well.

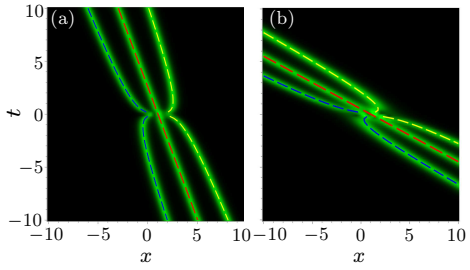

Fig. 3. The decomposition of third-order bound-state soliton: (a) $\alpha = 0.1$, $\beta = 2$, and (b) $\alpha = 0.5$, $\beta = 1$.

Symmetric decompositions have already provided accurate estimates for the asymptotic properties of the BS solitons,[17,18] such as the distances between components and the long-time behavior. Here the asymmetric decompositions accurately illustrate the displacements of individual components.

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