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Emergent Quantum Dynamics of Vortex-Line under Linear Local Induction Approximation *

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Using the linear local induction approximation, we investigate the self-induced motion of a vortex-line that corresponds to the motion of a particle in quantum mechanics. Provided Kelvin waves, the effective Schrödinger equation, physical quantity operators, and the corresponding path-integral formula can be obtained. In particular, the effective Planck constant defined by parameters of vortex-line motion shows the mathematical relation between the two fields. The vortexline–particle mapping may help in understanding particle motion in quantum mechanics.

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Lord Kelvin studied the properties of vortex-lines 150 years ago, which consisted of fluid rotating around a centerline. He predicted that the vortex-line could carry linear waves (Kelvin waves),[1] which were confirmed by experiment in ^{4}He superfluid[2] 50 years ago. In subsequent years, single vortex-lines have attracted much attention from researchers. In 1967, Batchelor[3] presented a formulation of a non-local three-dimensional Biot–Sarvart equation to approximate the local expression of motion, which became known as the local induction approximation (LIA). In 1972, Hasimoto[4] developed a transformation that mapped the LIA equation onto the nonlinear Schrödinger equation (NLS), and identified the soliton-wave solutions along the vortex-line. From experiments in 1998, Donnelly and Barenghi[5] reported on the properties of vortex-lines in ^{4}He superfluid at low temperatures. Recently, quantum turbulence based on LIA and NLS has attracted considerable attention among researchers. The energy transferred between Kelvin waves of different wave-numbers through nonlinear coupling is believed to be the mechanism underlying superfluid turbulence.[6] The relevant work on vortex-line motion is more concentrating on soliton-wave solutions of NLS[7][8] and quantum turbulence.[9,10]

From another perspective, quantum mechanics (QM) is a fundamental theory of modern physics that has successfully explained most experimental results and had wide applications, while the Schrödinger equation is fundamental in QM theory.[11] During the last 100 years, scientists have discovered several models of motion (approximately) described by the Schrödinger equation, such as the propagation of acoustic waves in plasma,[12] the motion of light pulses in nonlinear optical fibers,[12] and the recently discovered evolution of self-gravitational-induced waves in disks surrounding a massive body.[13] Here we point out that under the linear LIA (LLIA), the self-induced motion of a vortex-line is characterized by an effective linear Schrödinger equation, together with more mathematical connection that makes it possible to visualize the QM wave function intuitively.

In this Letter, we theoretically investigate the correspondence between the self-induced motion of the vortex-line and the particle motion in QM under LLIA. First, we present a brief derivation of the effective Schrödinger equation from the Biot–Savart equation applying the LLIA, the redefined vortex-line wave function being different from that of Hasimoto,[4] and theoretically discussed about the condition under which the linearity approximation is valid. Then, the momentum differential operator is naturally deduced from the original definition of momentum in the fluid, and so does the angular momentum. The commutation relation of coordinate and momentum operators of Kelvin wave are easily to be inferred. Specifically, the results can also be described by the effective Planck constant which is defined by the parameters of properties of vortex-line. After considering the Hamiltonian, the effective Schrödinger equation and the corresponding path-integral formula based on LLIA are subsequently obtained.

From the original Biot–Savart equation, we derive briefly the expression of the self-induced motion of a vortex-line that satisfies an effective linear Schrödinger equation under the LLIA.

The velocity $\boldsymbol{v}$ at certain point in fluid field associated with a vortex-line is described by the Biot-Savart equation

$$\boldsymbol{v} = \frac{\Gamma}{4\pi}\int_{L'}\frac{d\boldsymbol{l}'\times\boldsymbol{R}}{|\boldsymbol{R}|^3}, \tag{1}$$

where Γ denotes the vorticity of vortex-line, L' the path of integration, $d\boldsymbol{l}'$ the element of integration, and $\boldsymbol{R}$ the position vector from the source point to the field point. The Helmholtz vorticity theorem states that Γ remains the same value along the vortex-line. These parameters are shown in Fig. 1.

Assume that the vortex-line is a single-valued function of z' in the coordinate system $x'y'z'$. The self-induced motion, as suggested by the name, indicates that apart from the whole field's effect and the point itself, each element on the vortex-line has an inductive effect on a certain point (x_0, y_0, z_0) of the line, which will drive the changes in the shape of the vortex-line. It will be easier for us in the following deriva-

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tion if we make a coordinate transformation. We take (x_0, y_0, z_0) as the reference point, and the new coordinate system xyz and the new parameters in the following derivation are shown in Fig. 2. After setting $\boldsymbol{R} = (x_0, y_0, z_0) - (x', y', z') = (x, y, z)$, the velocity becomes

$$\boldsymbol{v} = \frac{\Gamma}{4\pi}\int_{L'\neq z_0}\frac{d\boldsymbol{l}'\times\boldsymbol{R}}{|\boldsymbol{R}|^3} = \frac{\Gamma}{4\pi}\int_{z\neq 0}\boldsymbol{f}(z)dz, \tag{2}$$

$$\boldsymbol{f}(z) = \frac{(\frac{y}{z}-\frac{\partial y}{\partial z})\boldsymbol{i} + (\frac{\partial x}{\partial z}-\frac{x}{z})\boldsymbol{j} + (\frac{x}{z}\frac{\partial y}{\partial z}-\frac{y}{z}\frac{\partial x}{\partial z})\boldsymbol{k}}{z|z|(1+\frac{x^2+y^2}{z^2})^{\frac{3}{2}}}. \tag{3}$$

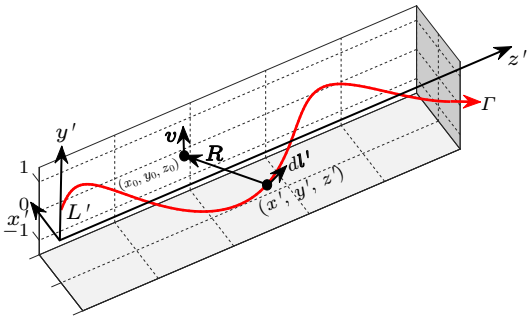

Fig. 1. The schematic diagram of parameters of vortex-line with the coordinate system $x'y'z'$, where the red line represents the vortex-line whose vorticity is Γ, and the two black points stand for the field point and the source point separately.

Details of the calculations are given in Supplementary A. After expanding $\boldsymbol{f}(z)$ in a Taylor series to second order at $z = 0$, we obtain the polarity of the integral in the finite interval $[-l, 0^-)\cup(0^+, l]$, as LIA does.[4] Here we take the ith component as an example,

$$\int_{-l\neq 0}^{l} f_i dz = \lim_{\sigma\to 0^+}\ln\Big(\frac{l}{\sigma}\Big)\frac{\partial^2 y}{\partial z^2}\Big|_{z=0}\Big[1+\Big(\frac{\partial y}{\partial z}\Big|_{z=0}\Big)^2 + \Big(\frac{\partial x}{\partial z}\Big|_{z=0}\Big)^2\Big]^{-\frac{3}{2}}, \tag{4}$$

where l is any positive finite number, as shown in Fig. 2. To avoid the integral diverging, we need the second derivative $\partial^2 y/\partial z^2|_{z=0} = 0$, and the value of the integral value in Eq. (4) vanishes. The same result holds for f_j. Because z_0 is arbitrary, the second-order derivative of the vortex-line must be zero to make the integral finite. If the nonzero higher-order terms are considered, it would inevitably lead to a finite second-order derivative at a certain point and a divergent integral. Therefore, we need to consider the radius of the vortex-line core, which in practice exists and alters the interval of integration. Following Batchelor in regard to the application of the LIA,[3] the polar integration can be dealt with by considering the effect of the radius on the denominator of Eq. (3). The expression for velocity eventually reduces to

$$\boldsymbol{v} = \frac{\Gamma\ln\varepsilon}{4\pi}\Big[-\boldsymbol{i}\frac{\partial^2 y}{\partial z^2} + \boldsymbol{j}\frac{\partial^2 x}{\partial z^2} + \boldsymbol{k}\Big(\frac{\partial^2 y}{\partial z^2}\frac{\partial x}{\partial z} - \frac{\partial^2 x}{\partial z^2}\frac{\partial y}{\partial z}\Big)\Big]\cdot\Big[1+\Big(\frac{\partial x}{\partial z}\Big)^2+\Big(\frac{\partial y}{\partial z}\Big)^2\Big]^{-\frac{3}{2}}, \tag{5}$$

where $\ln\varepsilon$ is a parameter actually related to the vortex-line radius and the length of the curvature radius, while the numerical change of this parameter $\ln\varepsilon$ can be ignored during the experiment process. Donnelly's experimental results yielded a value for $\ln\varepsilon$ in ^{4}He superfluid.[5,14]

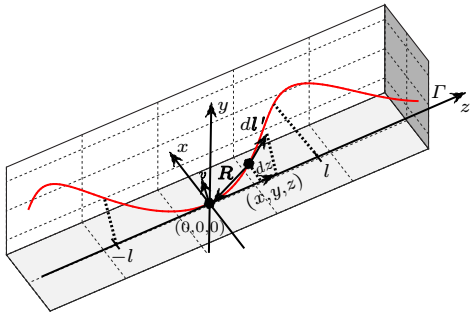

Fig. 2. The schematic diagram of new parameters with the coordinate system xyz. We have selected the field point which belongs to the vortex-line as the reference point. The self-induced motion means that each element on the vortex-line has an inductive effect on the reference point, which will make the shape of vortex-line change. The parameter l is an arbitrary parameter, while the interval of integral $[-l, 0^-)\cup(0^+, l]$ is based on the local induction approximation to solve the polarity of the integration.

Equation (5) shows that the self-induced velocity is only related to the first and second-order derivatives of the vortex-line, which is consistent with the conclusion that the velocity is only related to the curvature of the vortex-line. Note that we do not need to pay attention to the specific meaning of x, y here because of

$$\frac{d^2x_0}{dz_0^2} = \frac{d^2x}{dz^2},\quad \frac{d^2y_0}{dz_0^2} = \frac{d^2y}{dz^2}; \tag{6}$$

and the first-order derivative only involves the square term here. Therefore, in the following derivation, we do not distinguish between the position of the field point (x_0, y_0, z_0) and the position vector (x, y, z). After resetting the position of the field point to (x, y, z), the velocity can be expressed as $\boldsymbol{v} = \frac{d}{dt}(x, y, z)$.

Then, by introducing a complex number $\psi = x + iy$, the equations of motion are expressed as

$$\begin{aligned}\frac{d\psi}{dt} &= i\frac{\Gamma\ln\varepsilon}{4\pi}\frac{\partial^2\psi}{\partial z^2}\Big[1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}\Big]^{-\frac{3}{2}},\\ \frac{dz}{dt} &= -i\frac{\Gamma\ln\varepsilon}{8\pi}\Big(\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z} - \frac{\partial^2\psi^*}{\partial z^2}\frac{\partial\psi}{\partial z}\Big)\Big[1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}\Big]^{-\frac{3}{2}}.\end{aligned} \tag{7}$$

Given the relations for the derivatives, a functional expression is finally derived,

$$i\frac{\partial\psi}{\partial t} = -\frac{\Gamma\ln\varepsilon}{4\pi}\left(\frac{\psi'}{\sqrt{1+\psi^{*\prime}\psi'}}\right)'. \tag{8}$$

If we introduce the linear approximation, we can derive an effective linear Schrödinger equation for the vortex-line as follows:

$$i\frac{\partial\psi}{\partial t} = -\frac{\Gamma\ln\varepsilon}{4\pi}\frac{\partial^2\psi}{\partial z^2}. \tag{9}$$

Next, we need a sufficient condition for the validity of the linear approximation, i.e., LLIA. Consider

the Kelvin wave $\psi = ae^{i(kz-\omega t)}$, where a denotes the radius (amplitude), k the wave number, and ω the angular frequency of the Kelvin wave. Since the Kelvin wave is the eigen-solution of Eqs. (8) and (9), the dispersion relations are

$$\omega_{\rm n} = \frac{\Gamma \ln \varepsilon}{4\pi} \frac{k^2}{\sqrt{1+a^2k^2}}, \quad \omega_{\rm l} = \frac{\Gamma \ln \varepsilon}{4\pi} k^2, \tag{10}$$

where $\omega_{\rm n}(k)$ establishes the dispersion relation of the nonlinear equation (8) under LIA and $\omega_{\rm l}(k)$ that for the linear equation (9) under LLIA. Clearly, if $a^2k^2 \ll 1$, the dispersion relations will be nearly the same, then the LLIA will be valid. To make this condition more explicit, we introduce a characteristic evolution time T_0, which can be the time from experiment beginning to evolution end. At the beginning of the revolution process ($t = 0$), the phases of the two Kelvin waves are $\varphi_{\rm n} = 0$ and $\varphi_{\rm l} = 0$. After time T_0, the phase difference is $\pi/2$, that is,

$$\varphi_{\rm l} - \varphi_{\rm n} = (\omega_{\rm l} - \omega_{\rm n})T_0 \sim \frac{\pi}{2}. \tag{11}$$

We then find the LLIA condition to be

$$a < \frac{2\pi\sqrt{k^2T_0\Gamma \ln \varepsilon - \pi^2}}{k(k^2T_0\Gamma \ln \varepsilon - 2\pi^2)}, \tag{12}$$

which means that the effect of nonlinearity may be ignored if the radius of the Kelvin wave does not go beyond the constant value determined by the algebraic expression on the right-hand side of Eq. (12) during the revolution time of T_0, then the linear approximation, LLIA specifically, is reasonable.

Indeed, in the ^{4}He superfluid experiment, the radius of the Kelvin wave is about 10^{-2} to 10^{-4} cm,[15] the vorticity is $\Gamma \approx 9.97 \times 10^{-8}\,{\rm m}^2/{\rm s}$,[16] the wavenumber is about $5000\,{\rm m}^{-1}$,[16] and $\ln \varepsilon \approx 0.8$.[5,14] Hence the characteristic evolution time is about

$$T_0 = \frac{2\pi^2}{k^2\Gamma \ln \varepsilon \left(1 - \frac{1}{\sqrt{1+a^2k^2}}\right)} \approx 10 \sim 100\,{\rm s}.$$

This implies that the evolution time may last at least for 10 seconds without breaking the LLIA in the ^{4}He superfluid experiment.

Moreover, in the limitation $a \to 0$, the LLIA is well defined mathematically. To express the physical meaning of Eq. (12) more intuitively, some numerical simulations based on Eq. (12) are shown in Supplementary C.

We have obtained the 'effective' Schrödinger equation based on the Biot–Savart equation under the LLIA. Next, we derive expressions for the operators representing momentum, angular momentum, and energy (i.e., the Hamiltonian) from their original definitions in a fluid. It is clearly shown that the operators are similar to those in QM, especially when we take use the effective Planck constant defined by the parameters of vortex-line, the effective Schrödinger equation and operators do share the same form as QM.

We change the coordinate system so that the central axis of the vortex-line lies along the z-axis, and consider instances for which ψ is relatively small,

$$\int_L x dz = 0, \quad \int_L y dz = 0, \quad \Rightarrow \int_L \psi dz = 0.$$

The definition of the momentum of the vortex is[17]

$$\boldsymbol{p} = \frac{1}{2}\int_V \rho \boldsymbol{r} \times \boldsymbol{\omega} dV, \tag{13}$$

where ρ is the density of the fluid. The origin $o = (0,0,0)$ is the reference point for $\boldsymbol{r}$. For the vortex-line, the projection of the momentum along the z-axis is

$$p_z = \frac{\Gamma\rho}{2}\int_L (x dy - y dx) = -i\frac{\Gamma\rho}{2}\int_L \psi^* \frac{\partial \psi}{\partial z} dz, \tag{14}$$

the details are given in Supplementary B. We introduce a 'normalized' wave function $\psi_{\rm n}$,

$$\psi_{\rm n} = \sqrt{\frac{\pi}{V}}\psi, \tag{15}$$

where $V = \int_L \pi\psi^*\psi dz$. Then, p_z is written in the form

$$p_z = \int_L \psi_{\rm n}^* \Big(- i\frac{\Gamma\rho V}{2\pi}\frac{\partial}{\partial z}\Big)\psi_{\rm n} dz. \tag{16}$$

Finally, the operator representing momentum is obtained as follows:

$$\hat{p} = -i\frac{\Gamma\rho V}{2\pi}\frac{\partial}{\partial z}. \tag{17}$$

The momentum of the Kelvin wave is then

$$p = \frac{\Gamma\rho}{2}a^2Lk = \frac{\Gamma\rho V}{2\pi}k, \tag{18}$$

where L is the length of the interval of integration along z, $V = \pi a^2 L$ corresponds to our definition. It can be called the effective 'de Broglie relation'. In addition, the commutation relation is obtained from Eq. (17),

$$[\hat{z}, \hat{p}] = i\frac{\Gamma\rho V}{2\pi}. \tag{19}$$

The definition for the angular momentum of a vortex is[17]

$$\boldsymbol{M} = -\frac{1}{2}\int_V \rho r^2 \boldsymbol{\omega} dV. \tag{20}$$

By setting $(0,0,0)$ as the reference point, the projected angular momentum along the z-axis of the vortex-line is

$$M_z = -\frac{\Gamma\rho}{2}\int_L (x^2 + y^2 + z^2) dz. \tag{21}$$

For the length of the vortex-line, the integral is infinite. Hence we must subtract the trivial straight vortex-line to extract the additional (effective) angular momentum of the curved vortex-line, $L_z = M_z - M_{0z}$. Finally, we obtain

$$L_z = -\frac{\Gamma\rho}{2}\int_L \psi^*\psi dz = -\frac{\Gamma\rho V}{2\pi}, \tag{22}$$

for which the effective angular momentum of vortex-line is proportional to V, the 'volume' of the vortex-line. As indicated in Fig. 3, when rotating the vortex-line (red line) around the z-axis, the Kelvin wave

forms a tube (blue lines), and the definition of V is just the volume inside the tube. As Γ and ρ are considered as constants and V can be proved to be constant because of the conservation of angular momentum, we define a new parameter

$$\hbar_{\rm eff} = \frac{\Gamma\rho V}{2\pi}. \tag{23}$$

Given expressions (17)–(19) and (22), we refer to this parameter as the effective Planck constant.

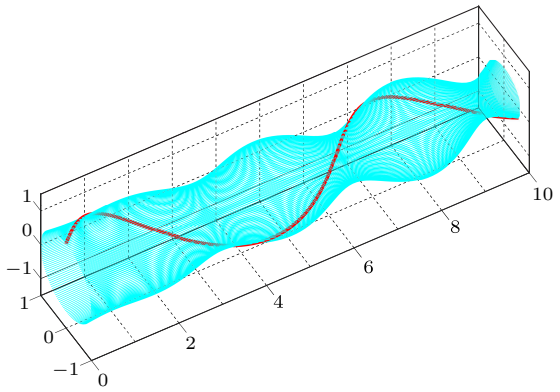

Fig. 3. The illustration of Kelvin wave of vortex-line. The red line denotes the vortex-line, and the blue one is the tube generated by vortex-line rotating around z-axis. The volume of vortex-line is actually the volume of the blue tube.

The commutation relation can be expressed as

$$[\hat{z}, \hat{p}] = i\hbar_{\rm eff}, \tag{24}$$

and the momentum relation and angular momentum can have the form

$$p = \hbar_{\rm eff} k, \quad L_z = -\hbar_{\rm eff}. \tag{25}$$

In the field of superfluids, according to the quantization condition for a quantized vortex-line (QV), we have[18]

$$\Gamma = \frac{h}{m}, \tag{26}$$

where h is Planck's constant and m the mass of one atom, which is the fundamental unit of the fluid. The total mass inside the vortex-line volume is

$$M = \rho V = nm, \tag{27}$$

where n is the number of atoms. Hence, the effective Planck constant is

$$\hbar_{\rm eff} = n\hbar. \tag{28}$$

These results are consistent with those in Ref. [19].

The definition of the Hamiltonian (kinetic energy actually) of the vortex-line is[17]

$$H = \frac{1}{8\pi}\iint \rho \frac{\boldsymbol{\omega}\cdot\boldsymbol{\omega}'}{R} dV dV'. \tag{29}$$

For a vortex-line under LIA, the energy is obtained as

$$H \overset{\rm LIA}{\approx} \frac{\rho\Gamma^2}{8\pi}\int_L 2\ln\varepsilon \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dz, \tag{30}$$

which is consistent with the results of Ref. [6]. The detailed calculation is given in Appendix B.

To simplify the expression, we set the energy of the trivial solution of a straight vortex-line to be zero and focus on the change in energy,

$$H = \frac{\rho\Gamma^2 \ln\varepsilon}{8\pi}\int \frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z} dz, \tag{31}$$

where we consider the condition for LLIA. Using the definition of ψ and performing an integration by parts, we obtain

$$H = -\int \psi_{\rm n}^* \frac{\rho V\Gamma^2 \ln\varepsilon}{8\pi^2}\frac{\partial^2}{\partial z^2}\psi_{\rm n} dz,$$

where $\psi_{\rm n}$ has been defined as the 'normalized' wave function as in Eq. (16). Hence, we arrive at the Hamiltonian operator

$$\hat{H} = -\frac{\rho V\Gamma^2 \ln\varepsilon}{8\pi^2}\frac{\partial^2}{\partial z^2}. \tag{32}$$

If we multiply both sides of Eq. (9) by the coefficient $\frac{\Gamma\rho V}{2\pi}$, then

$$i\frac{\Gamma\rho V}{2\pi}\frac{\partial\psi}{\partial t} = -\frac{\rho V\Gamma^2\ln\varepsilon}{8\pi^2}\frac{\partial^2\psi}{\partial z^2}, \tag{33}$$

which takes the form

$$i\hbar_{\rm eff}\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{34}$$

There are obvious similarities between the Biot–Savart equation for vortex-lines under LLIA and the Schrödinger equation in QM describing a free particle.

The shape of a vortex-line curve is called a state, which can be expressed as a linear superposition of different Kelvin waves. As discussed above, when the radius of the Kelvin wave vanishes in the mathematical limit, which means the vortex-line curves very little as if its volume only contains very few atoms, all the Kelvin waves will satisfy the linear Schrödinger equation. This is the completeness relation of Hilbert space. We borrow the Dirac symbol $\psi \to |\psi\rangle$ to express the states as

$$\psi(z) = \langle z|\psi\rangle, \tag{35}$$

and the completeness relation is denoted by $\int |z\rangle\langle z| dz = 1$, Using Eq. (18), we have the Fourier transformation

$$\langle z|p\rangle = \frac{1}{\sqrt{2\pi\hbar_{\rm eff}}} e^{ipz/\hbar_{\rm eff}}, \tag{36}$$

which will be used later.

Defining the operator $\hat{Q}$ and propagator $U(t', t)$ from time t to t' as

$$\hat{Q} = \frac{\pi\ln\varepsilon}{\Gamma\rho^2 V^2}\hat{p}^2, \quad |\psi(t')\rangle = U(t', t)|\psi(t)\rangle, \tag{37}$$

respectively, and considering $\hat{Q}$ to be time-independent, the propagator is written as $U(t', t) = e^{-i\hat{Q}(t'-t)}$.

In the coordinate representation, we make use of Eqs. (35) and (36) to obtain

$$\psi(z',t') = \int dz U(z',t';z,t)\psi(z,t), \tag{38}$$

where $U(z',t';z,t) = \langle z'|U(t',t)|z\rangle$ is a 'matrix element' of the propagator, which is here abbreviated to U_z.

We then divide $t'-t$ into N equal parts, $t'-t = N\Delta t$, and write $t_k = t + k\Delta t$, $k = 0,1,2,\cdots$. U_z may be expressed as a multiplication of propagators from time t_k to t_{k+1}. The kth matrix element is $U_{\rm zk} = \left\langle z_{k+1}|e^{-i\hat{Q}\Delta t}|z_k\right\rangle$. Applying a Fourier transformation, we obtain, in the limits $N\to\infty$ and $\Delta t\to 0$, an expression for the matrix element,

$$U_{\rm zk} = \int dp_k \frac{1}{2\pi\hbar_{\rm eff}} e^{i\frac{\Delta t}{\hbar_{\rm eff}}\left(p_k\dot{z}_k - \frac{\ln\varepsilon}{2\rho V}p_k^2\right)}.$$

The Fourier transformation we obtain in the limits $N\to\infty$ and $\Delta t\to 0$ is applied,

$$U_{\rm zk} = \int dp_k \frac{1}{2\pi\hbar_{\rm eff}} e^{i\frac{\Delta t}{\hbar_{\rm eff}}\left(p_k\dot{z}_k - \frac{\ln\varepsilon}{2\rho V}p_k^2\right)}.$$

We then have

$$U_z = \int \mathscr{D}_\Omega \exp\left[i\int_t^{t'} dt \frac{1}{\hbar_{\rm eff}}\left(p\dot{z} - \frac{\ln\varepsilon}{2\rho V}p^2\right)\right], \tag{39}$$

where $\int\mathscr{D}_\Omega = \lim_{N\to\infty}\int\frac{dp_N\ldots dp_1 dz_{N-1}\ldots dz_1}{(2\pi\hbar_{\rm eff})^N}$. If we define $H' = \frac{\ln\varepsilon}{2\rho V}p^2$ to be the Hamiltonian of a 'free particle', the system satisfies the Feynman path integral. The effective mass of a free particle becomes

$$m_{\rm eff} = \frac{\rho V}{\ln\varepsilon} \overset{\rm QV}{=} n\frac{m}{\ln\varepsilon}. \tag{40}$$

Indeed, the Hamiltonian of the vortex-line Eq. (32) itself simplifies to $\hat{H} = \frac{\hat{p}^2}{2m_{\rm eff}}$.

Finally, we describe the movement of the wave by evaluating the Feynman path integral of the particle. The Feynman propagator is expressed as

$$U_z = \int \mathscr{D}_\Omega \exp\left[i\int_t^{t'} dt\frac{\mathscr{L}}{\hbar_{\rm eff}}\right] = \int \mathscr{D}_\Omega e^{iS/\hbar_{\rm eff}}, \tag{41}$$

where $\mathscr{L}$ is the Lagrangian of the system, and S is the action. The derivation of path integral shows the correspondence between the vortex-line motion and a particle's motion in quantum mechanics.

In this study, the motion of vortex-line is mapped mathematically onto the movement of a particle in quantum mechanics (QM) under linear local induction approximation (LLIA). On the basis of the conservation of the vortex-line volume, we define an effective Planck constant, which bridges classical mechanics with QM, including the momentum and angular momentum operators, the commutation relation, as well as the Hamiltonian form of the effective Schrödinger equation that can all be described by the effective Planck constant. The path integral shows the corresponding between the vortex-line motion and the free particle's movement. Beyond the present derivations, we would like to share our considerations about the correspondence of a classical vortex-line motion and quantum mechanics, as listed in Table 1.

Table 1. Mathematical Correspondence with QM.

	Vortexline motion	Quantum mechanics
Schrödinger equation	$i\hbar_{\rm eff}\frac{\partial\psi}{\partial t} = \hat{H}\psi$	$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
Commutation relation	$[\hat{z},\hat{p}] = i\hbar_{\rm eff}$	$[\hat{z},\hat{p}] = i\hbar$

Finally, we show our current view about why the two systems share the same mathematical form. The two systems have the same symplectic structure, which is used to describe the Hamiltonian mechanics. Since both equations of motion can be transformed into the following form

$$i\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*}, \tag{42}$$

where H is Hamiltonian of the system, ψ is the wave function and ψ^* is the conjugate wave function, it is obvious that the well-defined wave function is necessary. In addition, the symplectic volume is conserved under the symplectic transformation, which in principle makes it possible to define the effect Planck constant to describe the system uniformly.

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Supplemental Material: Emergent Quantum Dynamics of Vortex-Line under Linear Local Induction Approximation

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A. Schrödinger Equation

The detailed calculations of Eq.(2) to Eq.(3) in our manuscript is given by

$$\vec{v} = \frac{\Gamma}{4\pi}\int_{L'\neq z_0}\frac{\mathrm{d}\vec{l'}\times\vec{R}}{|\vec{R}|^3} \tag{A.1}$$

$$= \frac{\Gamma}{4\pi}\int_{z\neq 0}\frac{(y\mathrm{d}z - z\mathrm{d}y)\vec{i} + (z\mathrm{d}x - x\mathrm{d}z)\vec{j} + (x\mathrm{d}y - y\mathrm{d}x)\vec{k}}{\sqrt{x^2+y^2+z^2}^3} \tag{A.2}$$

$$= \frac{\Gamma}{4\pi}\int_{z\neq 0}\frac{(y - z\frac{\partial y}{\partial z})\vec{i} + (z\frac{\partial x}{\partial z} - x)\vec{j} + (x\frac{\partial y}{\partial z} - y\frac{\partial x}{\partial z})\vec{k}}{\sqrt{x^2+y^2+z^2}^3}\mathrm{d}z \tag{A.3}$$

$$= \frac{\Gamma}{4\pi}\int_{z\neq 0}\frac{(y - z\frac{\partial y}{\partial z})\vec{i} + (z\frac{\partial x}{\partial z} - x)\vec{j} + (x\frac{\partial y}{\partial z} - y\frac{\partial x}{\partial z})\vec{k}}{z^2|z|}(1+\frac{x^2+y^2}{z^2})^{-\frac{3}{2}}\mathrm{d}z \tag{A.4}$$

$$= \frac{\Gamma}{4\pi}\int_{z\neq 0}\frac{(\frac{y}{z} - \frac{\partial y}{\partial z})\vec{i} + (\frac{\partial x}{\partial z} - \frac{x}{z})\vec{j} + (\frac{x}{z}\frac{\partial y}{\partial z} - \frac{y}{z}\frac{\partial x}{\partial z})\vec{k}}{z|z|}(1+\frac{x^2+y^2}{z^2})^{-\frac{3}{2}}\mathrm{d}z. \tag{A.5}$$

From Eq.(3) to Eq.(4): Here we use $\vec{i}$ component as an example to demonstrate that the integral polarity is inevitable. The Taylor expansion of numerator is

$$\begin{cases} x(z) = x(0) + \frac{\partial x}{\partial z}|_{z=0}(z-0) + \frac{1}{2}\frac{\partial^2 x}{\partial z^2}|_{z=0}(z-0)^2 + \frac{1}{6}\frac{\partial^3 x}{\partial z^3}|_{z=0}(z-0)^3 + o(z^4), \\ y(z) = y(0) + \frac{\partial y}{\partial z}|_{z=0}(z-0) + \frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}(z-0)^2 + \frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}(z-0)^3 + o(z^4), \end{cases} \tag{A.6}$$

$$\rightarrow \begin{cases} \frac{x}{z} = \frac{\partial x}{\partial z}|_{z=0} + \frac{1}{2}\frac{\partial^2 x}{\partial z^2}|_{z=0}z + \frac{1}{6}\frac{\partial^3 x}{\partial z^3}|_{z=0}z^2 + o(z^3), \\ \frac{y}{z} = \frac{\partial y}{\partial z}|_{z=0} + \frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}z + \frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}z^2 + o(z^3), \end{cases} \tag{A.7}$$

it is necessary to consider the definition of x, y, z, when $z \to 0$, $x, y \to 0$.

The ith component of $\vec{f}(z)$ can be expressed as

$$f_i(z) = [\frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}z + \frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}z^2 + o(z^3)]\frac{1}{z|z|}[1 + (\frac{\partial y}{\partial z}|_{z=0})^2 + (\frac{\partial x}{\partial z}|_{z=0})^2 + o(z^3)]^{-\frac{3}{2}} \tag{A.8}$$

$$= \begin{cases} [\frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}\frac{1}{z} + \frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}][1 + (\frac{\partial y}{\partial z}|_{z=0})^2 + (\frac{\partial x}{\partial z}|_{z=0})^2]^{-\frac{3}{2}}, z > 0, \\ -[\frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}\frac{1}{z} + \frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}][1 + (\frac{\partial y}{\partial z}|_{z=0})^2 + (\frac{\partial x}{\partial z}|_{z=0})^2]^{-\frac{3}{2}}, z < 0. \end{cases} \tag{A.9}$$

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So the integral is

$$\int_{-l\neq 0}^{l} f_i \mathrm{d}z = \lim_{\sigma\to 0^+}\int_{-l}^{-\sigma} -[\frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}\frac{1}{z}+\frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}][1+(\frac{\partial y}{\partial z}|_{z=0})^2+(\frac{\partial x}{\partial z}|_{z=0})^2]^{-\frac{3}{2}}\mathrm{d}z$$
$$+\int_{\sigma}^{l}[\frac{1}{2}\frac{\partial^2 y}{\partial z^2}|_{z=0}\frac{1}{z}+\frac{1}{6}\frac{\partial^3 y}{\partial z^3}|_{z=0}][1+(\frac{\partial y}{\partial z}|_{z=0})^2+(\frac{\partial x}{\partial z}|_{z=0})^2]^{-\frac{3}{2}}\mathrm{d}z \tag{A.10}$$
$$=\lim_{\sigma\to 0^+}[\ln(\frac{l}{\sigma})\frac{\partial^2 y}{\partial z^2}|_{z=0}+0][1+(\frac{\partial y}{\partial z}|_{z=0})^2+(\frac{\partial x}{\partial z}|_{z=0})^2]^{-\frac{3}{2}}, \tag{A.11}$$

where l can be any finite number.

We next show how to derive Eq.(11). The equations of motion are

$$\begin{cases}\dot{\psi}=\dfrac{\mathrm{d}\psi}{\mathrm{d}t}=\mathrm{i}\dfrac{\Gamma\ln\varepsilon}{4\pi}\dfrac{\partial^2\psi}{\partial z^2}/\sqrt{1+\dfrac{\partial\psi}{\partial z}\dfrac{\partial\psi^*}{\partial z}}^3,\\ \dot{z}=\dfrac{\mathrm{d}z}{\mathrm{d}t}=-\mathrm{i}\dfrac{\Gamma\ln\varepsilon}{8\pi}(\dfrac{\partial^2\psi}{\partial z^2}\dfrac{\partial\psi^*}{\partial z}-\dfrac{\partial^2\psi^*}{\partial z^2}\dfrac{\partial\psi}{\partial z})/\sqrt{1+\dfrac{\partial\psi}{\partial z}\dfrac{\partial\psi^*}{\partial z}}^3.\end{cases}$$

Considering the relation of derivative symbols, the equation is

$$\frac{\partial\psi}{\partial t}=\frac{\mathrm{d}\psi}{\mathrm{d}t}-\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial\psi}{\partial z} \tag{A.12}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}[\frac{\partial^2\psi}{\partial z^2}+\frac{1}{2}(\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z}-\frac{\partial^2\psi^*}{\partial z^2}\frac{\partial\psi}{\partial z})\frac{\partial\psi}{\partial z}]/\sqrt{1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}}^3 \tag{A.13}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}[(\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z})-\frac{1}{2}(\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z}+\frac{\partial^2\psi^*}{\partial z^2}\frac{\partial\psi}{\partial z})\frac{\partial\psi}{\partial z}]/\sqrt{1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}}^3 \tag{A.14}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}[\frac{\partial^2\psi}{\partial z^2}(1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z})-\frac{1}{2}(\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z}+\frac{\partial^2\psi^*}{\partial z^2}\frac{\partial\psi}{\partial z})\frac{\partial\psi}{\partial z}]/\sqrt{1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}}^3 \tag{A.15}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}[\frac{\partial^2\psi}{\partial z^2}\sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}}-\frac{\frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi^*}{\partial z}+\frac{\partial^2\psi^*}{\partial z^2}\frac{\partial\psi}{\partial z}}{2\sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}}}\frac{\partial\psi}{\partial z}]/\sqrt{1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}}^2 \tag{A.16}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}[(\frac{\partial}{\partial z}\frac{\partial\psi}{\partial z})\sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}}-(\frac{\partial}{\partial z}\sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}})\frac{\partial\psi}{\partial z}]/\sqrt{1+\frac{\partial\psi}{\partial z}\frac{\partial\psi^*}{\partial z}}^2 \tag{A.17}$$
$$=\frac{\mathrm{i}\Gamma\ln\varepsilon}{4\pi}\frac{\partial}{\partial z}\frac{\frac{\partial\psi}{\partial z}}{\sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}}}. \tag{A.18}$$

By rewriting the equation in another form, we derive Eq.(12)

$$\mathrm{i}\frac{\partial\psi}{\partial t}=-\frac{\Gamma\ln\varepsilon}{4\pi}(\frac{\psi'}{\sqrt{1+\psi^{*\prime}\psi'}})'. \tag{A.19}$$

B. Operators

B.1. Momentum

We show the detailed calculations of momentum operator. The momentum is defined by

$$\vec{p}=\frac{1}{2}\int_V \rho\vec{r}\times\vec{\omega}\mathrm{d}V. \tag{B.1}$$

For vortex-line, we have

$$\vec{p} = \frac{\Gamma\rho}{2}\int_L \vec{r}\times \mathrm{d}\vec{l} = \frac{\Gamma\rho}{2}\int_L (y\mathrm{d}z - z\mathrm{d}y)\vec{i} + (z\mathrm{d}x - x\mathrm{d}z)\vec{j} + (x\mathrm{d}y - y\mathrm{d}x)\vec{k},$$

where we've already set the reference point of $\vec{r}$ at $o = (0,0,0)$. The momentum of z-axis is

$$p_z = \frac{\Gamma\rho}{2}\int_L (x\mathrm{d}y - y\mathrm{d}x) = \frac{\Gamma\rho}{2}\int_L \frac{\psi+\psi^*}{2}\frac{\partial(\psi-\psi^*)}{2\mathrm{i}\partial z} - \frac{\psi-\psi^*}{2\mathrm{i}}\frac{\partial(\psi+\psi^*)}{2\partial z}\mathrm{d}z \tag{B.2}$$

$$= -\mathrm{i}\frac{\Gamma\rho}{4}\left(\int_L \psi^*\frac{\partial\psi}{\partial z}\mathrm{d}z - \int_L \psi\frac{\partial\psi^*}{\partial z}\mathrm{d}z\right) \tag{B.3}$$

$$= -\mathrm{i}\frac{\Gamma\rho}{4}\left(\int_L \psi^*\frac{\partial\psi}{\partial z}\mathrm{d}z - \psi\psi^*|_{-\infty}^{+\infty} + \int_L \psi^*\frac{\partial\psi}{\partial z}\mathrm{d}z\right) \tag{B.4}$$

$$= -\mathrm{i}\frac{\Gamma\rho}{2}\int_L \psi^*\frac{\partial\psi}{\partial z}\mathrm{d}z + \mathrm{i}\frac{\Gamma\rho}{4}\psi\psi^*|_{-\infty}^{+\infty}. \tag{B.5}$$

When ψ is a kind of 'bound state', there is $\psi\psi^*|_{-\infty}^{+\infty} = 0$.

B.2. Angular Momentum

The augular momentum is defined by

$$\vec{M} = -\frac{1}{2}\int_V \rho r^2\vec{\omega}\mathrm{d}V. \tag{B.6}$$

As to vortexline, we have

$$\vec{M} = -\frac{\Gamma\rho}{2}\int_L r^2\mathrm{d}\vec{l} = -\frac{\Gamma\rho}{2}\int_L (x^2+y^2+z^2)(\vec{i}\mathrm{d}x + \vec{j}\mathrm{d}y + \vec{k}\mathrm{d}z),$$

where $(0,0,0)$ is the reference point. And the z-axis component is

$$M_z = -\frac{\Gamma\rho}{2}\int_L (x^2+y^2+z^2)\mathrm{d}z. \tag{B.7}$$

The projected angular momentum is given by

$$L_z = M_z - M_{0z} \tag{B.8}$$

$$= -\frac{\Gamma\rho}{2}\int_L (x^2+y^2+z^2-z^2)\mathrm{d}z \tag{B.9}$$

$$= -\frac{\Gamma\rho}{2}\int_L (x^2+y^2)\mathrm{d}z \tag{B.10}$$

$$= -\frac{\Gamma\rho}{2}\int_L \psi^*\psi\mathrm{d}z = -\frac{\Gamma\rho V}{2\pi}, \tag{B.11}$$

where we have assumed that Γ is along the positive direction of z-axis.

B.3. Hamiltonian

Finally, we calculate the Hamiltonian,

$$H = \frac{1}{8\pi}\iint \rho\frac{\vec{\omega}\cdot\vec{\omega}'}{R}\mathrm{d}V\mathrm{d}V'. \tag{B.12}$$

As to vortexline, the expression is

$$H = \frac{\rho\Gamma^2}{8\pi}\iint \frac{\mathrm{d}\vec{l}\cdot\mathrm{d}\vec{l'}}{R} \tag{B.13}$$

$$= \frac{\rho\Gamma^2}{8\pi}\int_L \mathrm{d}z \int_{L'\neq z} \mathrm{d}z' \frac{\frac{\partial x}{\partial z}\frac{\partial x'}{\partial z} + \frac{\partial y}{\partial z}\frac{\partial y'}{\partial z} + 1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{B.14}$$

$$\overset{\text{LIA}}{\approx} \frac{\rho\Gamma^2}{8\pi}\int_L \mathrm{d}z \int_{z-l\neq z0}^{z+l} \mathrm{d}z' \frac{\frac{\partial x}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial z} + 1}{\sqrt{[(\frac{\partial x}{\partial z})^2+(\frac{\partial y}{\partial z})^2+1][(z-z')^2+\sigma^2]}}. \tag{B.15}$$

we utilize a method under LIA for processing velocity integral polarity to derive integral value near $z' \to z$. The main contribution of the integral exists near the area of polarity. Besides, we further approximate coordinate differences to linear items of Taylor Expansion, $\frac{x-x'}{z-z'} \to \frac{\partial x}{\partial z}$, $\frac{y-y'}{z-z'} \to \frac{\partial y}{\partial z}$, and the first order derivatives are equal in the integral interval ($\frac{\partial x'}{\partial z} \to \frac{\partial x}{\partial z}$, $\frac{\partial y'}{\partial z} \to \frac{\partial y}{\partial z}$). So, the expression goes to

$$H \overset{\text{LIA}}{\approx} \frac{\rho\Gamma^2}{8\pi}\int_L 2\ln\varepsilon\sqrt{(\frac{\partial x}{\partial z})^2+(\frac{\partial y}{\partial z})^2+1}\mathrm{d}z \tag{B.16}$$

$$= \frac{\rho\Gamma^2\ln\varepsilon}{4\pi}\int_L \sqrt{1+\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}}\mathrm{d}z \tag{B.17}$$

$$\overset{\text{LLIA}}{\approx} \frac{\rho\Gamma^2\ln\varepsilon}{4\pi}\int_L (1+\frac{1}{2}\frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z})\mathrm{d}z. \tag{B.18}$$

Finally, we have

$$H = \frac{\rho\Gamma^2\ln\varepsilon}{8\pi}\int \frac{\partial\psi^*}{\partial z}\frac{\partial\psi}{\partial z}\mathrm{d}z \tag{B.19}$$

$$= \frac{\rho\Gamma^2\ln\varepsilon}{8\pi}(\frac{\partial\psi}{\partial z}\psi^*|_{-\infty}^{\infty} - \int \psi^*\frac{\partial^2\psi}{\partial z^2}\mathrm{d}z) \tag{B.20}$$

$$= -\int \psi^*\frac{\rho\Gamma^2\ln\varepsilon}{8\pi}\frac{\partial^2}{\partial z^2}\psi\mathrm{d}z \tag{B.21}$$

$$= -\int \psi_{\mathrm{n}}^*\frac{\rho V\Gamma^2\ln\varepsilon}{8\pi^2}\frac{\partial^2}{\partial z^2}\psi_{\mathrm{n}}\mathrm{d}z \tag{B.22}$$

C. Numerical Simulations

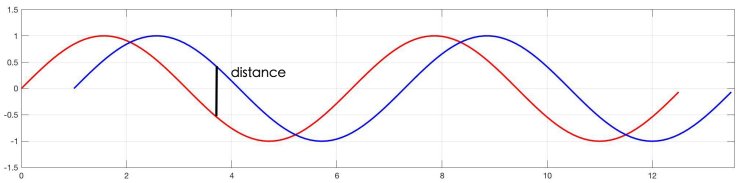


图 1: Distances Between Two Evolution Results: the red line stands for the evolution result at a certain time of Eq.(12) under LIA, the blue line is the result of Eq.(13) under LLIA, and the distance is the maximum vertical coordinate difference of the Kelvin waves at a certain time

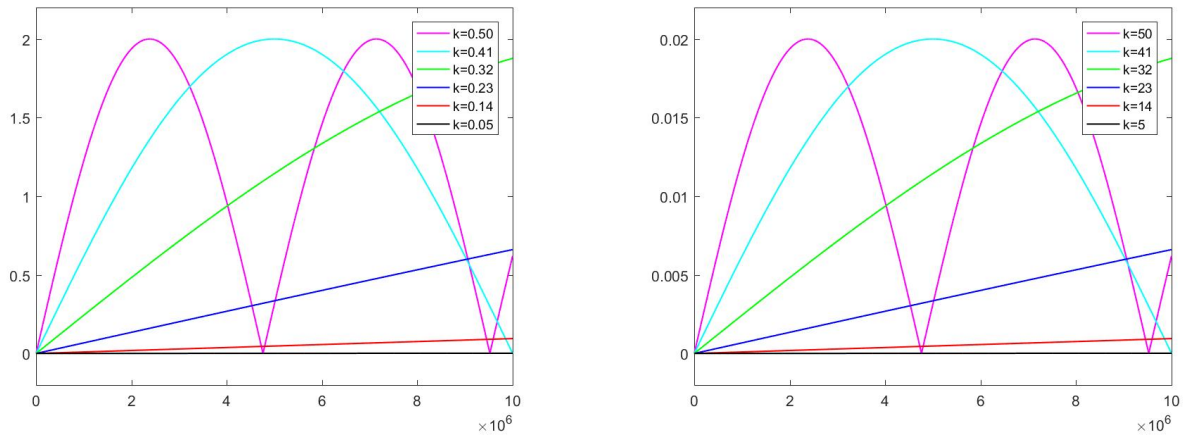
FIG. 1 shown here is the sketch map of our simulation, the abscissa axis of FIG. 1 is the z-axis of the Kelvin wave, while the vertical axis in stand of the x-axis. We substitute a series of Kelvin waves with certain

radius and wave numbers into the two equations - the nonlinear equation under LIA and the linear equation under LLIA, separately. For clearly discussion, we assume the red line in FIG. 1 follows the nonlinear equation, while the blue line obeys the linear equation. At first, the phases of Kelvin waves are set to be zero. After a time of evolution, the phases difference of two Kelvin waves gradually appears. It's easy to know that the blue line moves 'faster' than the red line, the phase difference leads to the appearance of 'distance' - the maximum vertical coordinate difference of the Kelvin waves at a certain time, which is shown in FIG. 1. We record the variation tendency of the distance during the evolution, and do the records of different wave numbers with different radius.

Here, FIG. 2 shows the results of the distance variation tendency. During the simulation, we set a characteristic time in simulation to ignore the influence of the parameters of the Kelvin wave,

$$t' = \frac{\Gamma \ln \epsilon}{2\pi} t,$$

while the dimension conforms to the International System of Units.



(a)a=1.00 (b)a=0.01

图 2: Distance Time Varying Figures: periodic boundary condition was used in simulation, the number of Kelvin wave cycles is 10, the number of space gridding is 5000, the time evolution iterations is 10^7, and $t' = \frac{\Gamma \ln \epsilon}{2\pi} t$ is characteristic time in simulation. Time step was controlled separately: (a) $a = 1.00$, time step is 10^{-4} and the length of time evolution is 1000; (b) $a = 0.01$, time step is 10^{-8} and the length of time evolution is 0.1. The space step depends on the wave number.

The abscissa axis of FIG. 2 is the number of characteristic time step of evolution, while the vertical axis in stand of the distance which is shown in FIG. 1. The FIG. 2(a) shows the distance variation of six Kelvin waves with radius a all equal to 1 but six different wave numbers ranging from $k = 0.05$ to $k = 0.50$, the time step is set to be 10^{-4} so the total evolution time T_0 is 1000 in the simulation program. It's the same that the FIG. 2(b) shows the results of Kelvin waves $a = 0.01$ and k is from $k = 5$ to $k = 50$, the time step is 10^{-8} and the total evolution time T_0 is 0.1. We can clearly see that during the evolution time, Kelvin wave with $a = 1$ and $k = 0.5$ - which we call it Kelvin wave A - has experienced two cycles of distance variation, while the phase different of Kelvin wave with $a = 1$ and $k = 0.05$ - which we call it Kelvin wave B - seems to stay zero. From the result we can say, during the evolution time, the linear approximation of Kelvin wave B is acceptable, and the linear approximation of Kelvin wave A is not good. The same results of Kelvin waves with $a = 0.01$. When compared with the two subfigures, it's not difficult to realize that as the radius going smaller, the larger wave number

of Kelvin wave can fit the LLIA. If we assume that the radius goes to zero mathematically, which means the vortex-line curve very little, Kelvin wave with all wave numbers can actually fit the linear approximation, then the deeply correspondence between the two fields is easier to be talked about.