CHIN. PHYS. LETT. Vol. 36, No. 12 (2019) 120501

# Soliton Molecules, Asymmetric Solitons and Hybrid Solutions for (2+1)-Dimensional Fifth-Order KdV Equation *

Zhao Zhang(张钊)[1], Shu-Xin Yang(杨淑心)[1,2], Biao Li(李彪)[1**]

[1]*School of Mathematics and Statistics, Ningbo University, Ningbo 315211*
[2]*School of Foundation Studies, Zhejiang Pharmaceutical College, Ningbo 315199*

(Received 30 September 2019)

*Soliton molecules were first discovered in optical systems and are currently a hot topic of research. We obtain soliton molecules of the (2+1)-dimensional fifth-order KdV system under a new resonance condition called velocity resonance in theory. On the basis of soliton molecules, asymmetric solitons can be obtained by selecting appropriate parameters. Based on the N-soliton solution, we obtain hybrid solutions consisting of soliton molecules, lump waves and breather waves by partial velocity resonance and partial long wave limits. Soliton molecules, and some types of special soliton resonance solutions, are stable under the meaning that the interactions among soliton molecules are elastic. Both soliton molecules and asymmetric solitons obtained may be observed in fluid systems because the fifth-order KdV equation describes the ion-acoustic waves in plasmas, shallow water waves in channels and oceans.*

*PACS: 05.45.Yv, 02.30.Ik, 47.20.Ky, 52.35.Mw* DOI: 10.1088/0256-307X/36/12/120501

Nonlinear partial differential equations have many important applications in physics, physiology, biology, ecology and other fields.[1–6] Korteweg-de Vries (KdV)-type equations, with the first-order nonlinear and dispersive terms being retained and in balance, have been used as approximate models governing weakly nonlinear long waves from different dynamical contexts.[7,8] In this work, we consider the (2 + 1)-dimensional fifth-order KdV equation[9]

$$36u_t + u_{5x} + 15u_x u_{xx} + 15uu_{3x} + 45u^2u_x - 5u_{xxy} - 15uu_y - 15u_x\int u_y dx - 5\int u_{yy}dx = 0, \quad (1)$$

which describes the ion-acoustic waves in plasmas, shallow water waves in channels and oceans, and pulse waves in large arteries.[10] Here $u = u(x, y, t)$ denotes a scalar function of the space variables $x, y$, and time variable $t$.

From Ref. [9], the $N$-soliton solutions of Eq. (1) have the form of

$$u(x, y, t) = 2(\ln f)_{xx}, \quad (2)$$

where

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{j<s}^{N} \mu_j\mu_s A_{js} + \sum_{j=1}^{N} \mu_j\eta_j\Big), \quad (3)$$

with

$$\eta_j = k_j x + w_j t + p_j y + \phi_j,$$
$$k_j^6 - 5k_j^3 p_j - 5{p_j}^2 + 36k_j w_j = 0, \quad (4)$$

and

$$e^{A_{js}} = -\{(k_j - k_s)^6 - 5(k_j - k_s)^3(p_j - p_s) + 36(w_j - w_s)(k_j - k_s) - 5(p_j - p_s)^2\} \cdot \{(k_j + k_s)^6 - 5(k_j + k_s)^3(p_j + p_s) + 36(w_j + w_s)(k_j + k_s) - 5(p_j + p_s)^2\}^{-1}. \quad (5)$$

In addition to the $N$-soliton solutions, Eq. (2) includes many kinds of resonant excitations.[11,12] Breather solutions can be obtained by resonance condition $k_i = \overline{k_j}, p_i = \overline{p_j}$, and we can obtain rational solutions by further resonant conditions $k_s \mapsto 0, p_s \mapsto 0$.

Recently, soliton molecules, which are bound states of solitons, have become popular research objects. In 2012, soliton molecules were numerically predicted in Bose–Einstein condensates.[13] In 2017, researchers resolved the evolution of femtosecond soliton molecules in the cavity of a few-cycle mode-locked laser by means of an emerging time-stretch technique.[14] In 2018, Liu *et al.* experimentally observed the real-time dynamics of the entire buildup process of stable soliton molecules for the first time.[15] Just this month, Lou[16] proposed a velocity resonance mechanism to theoretically obtain soliton molecules of (1+1)-dimensional systems.

In this Letter, we introduce a new possibility—the velocity resonant mechanism—to form soliton molecules for the (2+1)-dimensional fifth-order KdV system. On the basis of soliton molecules, we obtain asymmetric solitons that can be observed in real systems, especially in the ocean via selections on the solution parameters. Furthermore, we study the interaction between soliton molecules, lump waves and breather waves by partial velocity resonance and partial long wave limits.

*Supported by the National Natural Science Foundation of China under Grant Nos 11775121, 11805106 and 11435005, and K.C. Wong Magna Fund in Ningbo University.
**Corresponding author. Email: libiao@nbu.edu.cn
© 2019 Chinese Physical Society and IOP Publishing Ltd

To obtain soliton molecules from Eq. (2), we introduce a new resonance condition ($k_i \neq \pm k_j, p_i \neq \pm p_j$), velocity resonance:

$$\frac{k_i}{k_j} = \frac{p_i}{p_j} = \frac{w_i}{w_j}. \tag{6}$$

Combined with Eq. (4), we can obtain the following nonsingular analytical resonant excitations:

$$k_i = \sqrt{\frac{-{k_j}^3 + 5p_j}{k_j}},\ p_i = \frac{p_j}{k_j}\sqrt{\frac{-{k_j}^3 + 5p_j}{k_j}}, \tag{7}$$

or

$$k_i = -\sqrt{\frac{-{k_j}^3 + 5p_j}{k_j}},\ p_i = -\frac{p_j}{k_j}\sqrt{\frac{-{k_j}^3 + 5p_j}{k_j}}. \tag{8}$$

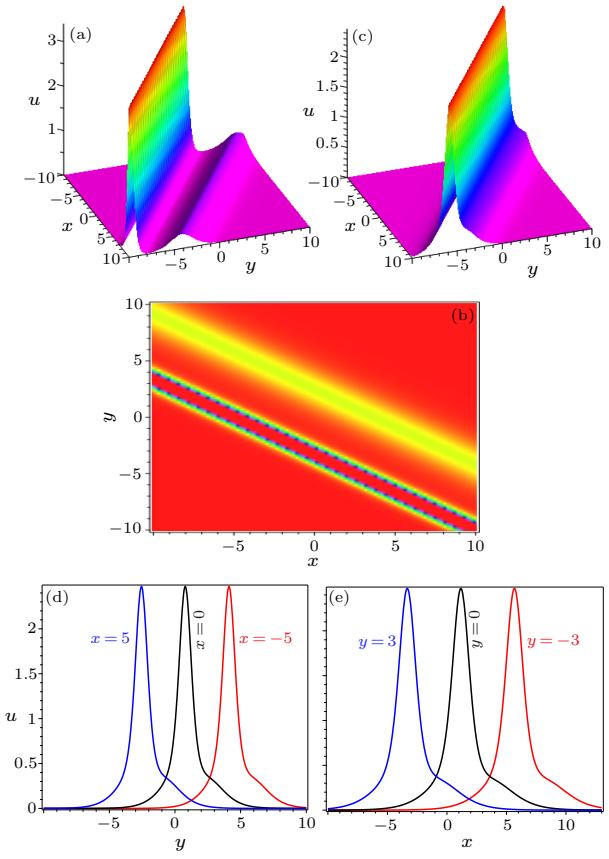


**Fig. 1.** The solutions $u$ to Eq. (1) at $t = 0$: (a) soliton molecule structure for the (2+1)-dimensional fifth-order KdV system descried by Eq. (2) with parameter selections of Eq. (9); (b) the density plot of soliton molecule with parameter selections of Eq. (9); (c) asymmetric soliton for the (2+1)-dimensional fifth-order KdV system descried by Eq. (2) with the same parameter selections of Eq. (9) except for $\phi_2 = -5$; (d) $x$-curve of the asymmetric soliton with $t = 0$; (e) $y$-curve of asymmetric soliton with $t = 0$.

We can know from Eqs. (7) and (8) that two solitons are bounded to generate a soliton molecule or an asymmetric soliton depending on the selections on the solution parameters. To visually describe this phenomenon, we take $N = 2$ as an example. For convenience, we assign the parameters in Eq. (2) as follows:

$$k_1 = -\frac{4}{5},\ k_2 = \frac{7\sqrt{7}\sqrt{2}}{10},\ p_1 = -\frac{6}{5},\ p_2 = \frac{21\sqrt{7}\sqrt{2}}{20},$$
$$\phi_1 = 0,\ \phi_2 = 10. \tag{9}$$

The two-soliton solution with parameter selections of Eq. (9) exhibits one soliton molecule structure under the velocity resonance Eq. (6). It can be observed from (a) and (b) in Fig. 1 that the two solitons in molecules are obviously different because of $k_i \neq k_j, p_i \neq p_j$ although the velocities of the two solitons are the same. Compared (a) and (c) in Fig. 1, we can find that an asymmetric soliton can be obtained by changing the distance between two solitons in the molecule. The height of the asymmetric soliton is between the heights of the two solitons, and the wave width of asymmetric soliton is widened.

It is well-known that the interactions among solitons are elastic in the (2+1)-dimensional fifth-order KdV system Eq. (1). Soliton molecules and asymmetric solitons, specially the cases of soliton solutions to Eq. (2), are also elastic in their interactions. To clearly describe these phenomenons, let us assign the following values to the parameters in Eq. (2):

$$N = 4,\ k_1 = -\frac{5}{4},\ k_2 = \frac{1}{4}\sqrt{103},\ k_3 = -\frac{2}{3},$$
$$k_4 = -\frac{1}{3}\sqrt{41},\ p_1 = -2,\ p_2 = \frac{2}{5}\sqrt{103},\ p_3 = -\frac{2}{3},$$
$$p_4 = -\frac{1}{3}\sqrt{41},\ \phi_1 = 10,\ \phi_2 = 0,\ \phi_3 = -3,\ \phi_4 = 12. \tag{10}$$

As can be seen from Fig. 2, the height of wave peaks and the velocities of wave peaks does not change, except during the phase after the collision of asymmetric solitons with asymmetric solitons.

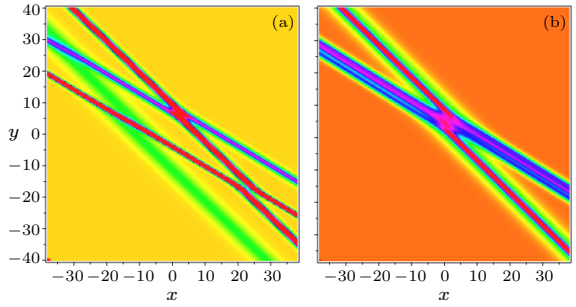


**Fig. 2.** The solutions $u$ to Eq. (1) at $t = 0$: (a) Elastic interaction property between two soliton molecules for Eq. (1) described by Eq. (2) with parameter selections of Eq. (10); (b) elastic interaction property between two asymmetric solitons for Eq. (1) descried by Eq. (2) with parameter selections of Eq. (10) except for $\phi_2 = -32$ and $\phi_3 = 5$.

Breather solutions, lump solutions and soliton molecules are popular topics. Hybrid solutions consisting soliton molecules, breathers and lumps are also the focus of attention. In the following content, we study the interactions between molecular solitons and other waves via partial velocity resonance.

In the four-soliton solutions to Eq. (1), two solitons satisfy the condition of velocity resonance Eq. (6), and the other two solitons satisfy the resonance condition $\eta_i = \overline{\eta_j}$. Then, we can obtain hybrid solutions consisting a soliton molecule and a breather wave. Combined with Eq. (6), we may assign the values in Eq. (2) as follows:

$$\begin{aligned}
&N = 4,\ k_1 = -\frac{4}{5},\ k_2 = \frac{7\sqrt{7}\sqrt{2}}{10},\ k_3 = \frac{2}{7} - \frac{2}{7}i,\\
&k_4 = \frac{2}{7} + \frac{2}{7}i,\ p_1 = -\frac{6}{5},\ p_2 = \frac{21\sqrt{7}\sqrt{2}}{20},\\
&p_3 = \frac{1}{8} + \frac{i}{2},\ p_4 = \frac{1}{8} - \frac{i}{2},\ \phi_1 = 0, \phi_2 = 25,\\
&\phi_3 = 0,\ \phi_4 = 0. \qquad (11)
\end{aligned}$$

As shown in Fig. 3, four-soliton solution to Eq. (2) with parameter selections of Eq. (11) exhibits the interaction between a soliton molecule and a breather wave under the partial velocity resonance. The interactions of soliton molecules, asymmetric solitons and breather solutions are also elastic.

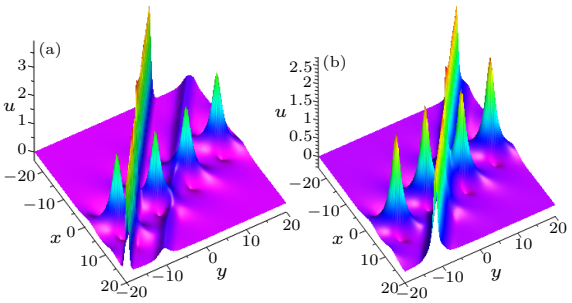


**Fig. 3.** The solutions $u$ to Eq. (1) at $t = 0$: (a) elastic interaction property between a soliton molecule and a breather wave for Eq. (1) described by Eq. (2) with parameter selections of Eq. (11); (b) elastic interaction property between an asymmetric soliton and a breather wave for Eq. (1) descried by Eq. (2) with parameter selections of Eq. (11) except for $\phi_2 = -5$.

Similarly, we can perform a partial velocity resonance and a partial long wave limit[11,12] on the $N$-soliton solution to Eq. (1) to obtain hybrid solutions consisting of lump waves and soliton molecules or asymmetric solitons depending on the selections on the solutions parameters. To describe the hybrid solutions consisting of soliton molecules and lumps, we take $N = 4$ as an example. However, we could also choose the following values for the parameters in Eq. (2):

$$\begin{aligned}
&k_1 = -\frac{4}{5},\ k_2 = \frac{7\sqrt{7}\sqrt{2}}{10},\ p_1 = -\frac{6}{5},\ p_2 = \frac{21\sqrt{7}\sqrt{2}}{20},\\
&\phi_1 = 6,\ \phi_2 = 20,\\
&k_3 = \left(\frac{1}{2} + i\right)\epsilon,\ k_4 = \left(\frac{1}{2} - i\right)\epsilon,\ p_3 = -2\epsilon,\\
&p_4 = -2\epsilon,\ \phi_3 = \pi i,\ \phi_4 = \pi i. \qquad (12)
\end{aligned}$$

Then, we can obtain hybrid solutions to Eq. (1) by taking a long wave limit ($\epsilon \mapsto 0$). As shown in Fig. 4, the four-soliton solution exhibits the interaction between a lump wave and one soliton molecule under partial velocity resonance and a partial long wave limit. The collision between the soliton molecule and the lump wave is also elastic, and the height of the lump wave does not change before or after the collision. Before the collision, the trajectory equation of the lump peak is $x = \frac{4}{9}t - \frac{350211120}{458061841}$, $y = \frac{2}{9}t + \frac{61380000}{458061841}$, and the trajectory equation of the peak after the collision is $x = \frac{111477030\sqrt{7}\sqrt{2}}{150403549} + \frac{4}{9}t$, $y = \frac{58891875\sqrt{7}\sqrt{2}}{601614196} + \frac{2}{9}t$.

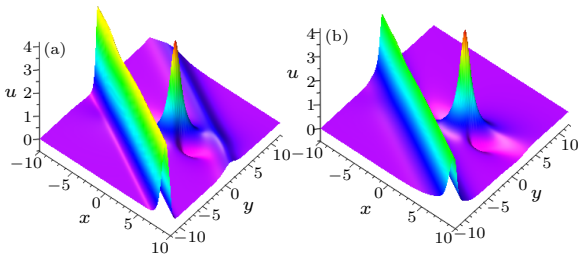


**Fig. 4.** The solutions $u$ to Eq. (1) at $t = 0$: (a) elastic interaction property between a soliton molecule and a lump wave for Eq. (1) described by Eq. (2) with parameter selections of Eq. (12); (b) elastic interaction property between an asymmetric soliton and a lump wave for Eq. (1) descried by Eq. (2) with parameter selections of Eq. (12) except $\phi_1 = -8$.

More generally, we can make the following constraints on the parameters in Eq. (2) to obtain the hybrid solutions consisting of $m$ soliton molecules, $n$ breather waves and $q$ lump waves,

$$\begin{aligned}
&\frac{k_1}{k_2} = \frac{p_1}{p_2} = \frac{w_1}{w_2}, \ldots,\ \frac{k_{2m-1}}{k_{2m}} = \frac{p_{2m-1}}{p_{2m}} = \frac{w_{2m-1}}{w_{2m}},\\
&\eta_{2m+1} = \overline{\eta_{2m+2}}, \ldots,\ \eta_{2m+2n-1} = \overline{\eta_{2m+2n}},\\
&k_{2m+2n+1} = K_{2m+2n+1}\epsilon,\ k_{2m+2n+2} = \overline{K_{2m+2n+1}}\epsilon, \ldots,\\
&k_{2m+2n+2q-1} = K_{2m+2n+2q-1}\epsilon,\\
&k_{2m+2n+2q} = \overline{K_{2m+2n+2q-1}}\epsilon,\\
&p_{2m+2n+1} = P_{2m+2n+1}\epsilon,\ p_{2m+2n+2} = \overline{P_{2m+2n+1}}\epsilon, \ldots,\\
&p_{2m+2n+2q-1} = P_{2m+2n+2q-1}\epsilon,\\
&p_{2m+2n+2q} = \overline{P_{2m+2n+2q-1}}\epsilon,\\
&\phi_{2m+2n+1} = \pi i,\ \phi_{2m+2n+2} = \pi i,\ \ldots,\\
&\phi_{2m+2n+2q} = \pi i. \qquad (13)
\end{aligned}$$

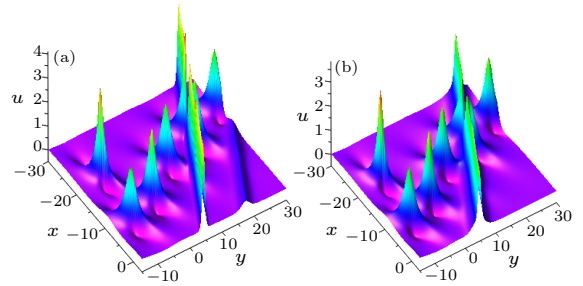


**Fig. 5.** The solutions $u$ to Eq. (1) at $t = -40$: (a) elastic interaction property among a soliton molecule, a lump wave and a breather wave for Eq. (1) described by Eq. (2) with parameter selections of Eq. (14); (b) elastic interaction property among an asymmetric solitons, a lump wave and a breather wave for Eq. (1) descried by Eq. (2) with parameter selections of Eq. (14) except $\phi_1 = -8$.

Then, we can get interactions between soliton molecules, lumps and breathers by taking $\epsilon \mapsto 0$. To

clearly describe their interaction, let us take a simple example of $N = 6$. Figure 5(a) displays the interaction between a soliton molecule, a lump wave and a breather wave descried by Eq. (2) with the parameter selections:

$$\begin{aligned}
&k_1 = -\frac{4}{5},\ k_2 = \frac{7\sqrt{7}\sqrt{2}}{10},\ p_1 = -\frac{6}{5},\ p_2 = \frac{21\sqrt{7}\sqrt{2}}{20},\\
&\phi_1 = 6,\ \phi_2 = 20,\\
&k_3 = \Big(\frac{1}{2} + i\Big)\epsilon,\ k_4 = \Big(\frac{1}{2} - i\Big)\epsilon,\ p_3 = -2\epsilon,\\
&p_4 = -2\epsilon,\ \phi_3 = \pi i,\ \phi_4 = \pi i,\\
&k_5 = \frac{2}{7} - \frac{2}{7}i,\ k_6 = \frac{2}{7} + \frac{2}{7}i,\ p_5 = \frac{1}{8} + \frac{i}{2},\\
&p_6 = \frac{1}{8} - \frac{i}{2},\ \phi_5 = 0,\ \phi_6 = 0. \qquad (14)
\end{aligned}$$

We can get the hybrid solution consisting of a lump wave, a breather wave and a soliton molecule shown in Fig. 5(b) by changing the distance between the two solitons in the molecule. The interaction between these waves is also elastic. The trajectory equation of the peak of the lump wave before the collision is $x = \frac{4}{9}t - \frac{350211120}{458061841}, y = \frac{2}{9}t + \frac{61380000}{458061841}$, and the trajectory equation of the peak after the collision is $x = \frac{4}{9}t + \frac{111477030\sqrt{7}\sqrt{2}}{150403549} + \frac{6086738213051863864723821557 76}{787908271930619137424839513 45}$, $y = \frac{2}{9}t + \frac{58891875\sqrt{7}\sqrt{2}}{601614196} + \frac{4939649329526381559768329011 2}{1575816543861238274849679026 9}$. Before and after the collision, the height of the peak is $\frac{64}{15}$.

In summary, the soliton molecules and asymmetric solitons of a (2+1)-dimensional fifth-order KdV system are theoretically obtained by velocity resonance. Although Eq. (1) is well known in the literature, the soliton molecules and hybrid solutions consisting of soliton molecules, lump waves and breathers have not yet been found. Thus, soliton molecules are also highly likely to exist in fluid systems, although they were first discovered in optical experiments. Furthermore, according to Eq. (13), we can obtain interactions between $m$ soliton molecules, $n$ breather waves and $q$ lump waves, and the interactions between them are elastic. The method that we used to obtain the hybrid solutions by taking partial velocity resonance and partial long wave limits can be applied to other integrable equations as well. Because the equations with variable coefficients can more accurately describe some nonlinear models in the physical domain, how to construct the soliton molecules of the (2+1)-dimensional integrable equation with variable coefficients is an important issue that we will study in future. We also hope that our results will provide some valuable information in the field of nonlinear science.

The authors would like to express their sincere thanks to Professor S. Y. Lou for his guidance and encouragement.

## References

[1] Wazwaz A M 2017 *Nonlinear Dyn.* **87** 1685
[2] Peng W Q, Tian S F and Zhang T T 2018 *Phys. Lett. A* **382** 2701
[3] Lou S Y and Lin J 2018 *Chin. Phys. Lett.* **35** 050202
[4] Liu Y K and Li B 2017 *Chin. Phys. Lett.* **34** 010202
[5] Chen M D, Li X, Wang Y and Li B 2017 *Commun. Theor. Phys.* **67** 595
[6] Chen J C, Ma Z Y and Hu Y H 2017 *Chin. Phys. Lett.* **34** 010201
[7] Syam M, Jaradat H M and Alquran M 2017 *Nonlinear Dyn.* **90** 1363
[8] Wazwaz A M 2017 *Math. Methods Appl. Sci.* **40** 4128
[9] Batwa S and Ma W X 2018 *Adv. Math. Phys.* **2018** 2062398
[10] Hong W P and Jung Y D 1999 *Z. Naturforsch.* **54a** 375
[11] Cao Y L, He J S and Mihalache D 2018 *Nonlinear Dyn.* **91** 2593
[12] Liu Y K, Li B and An H L 2018 *Nonlinear Dyn.* **92** 2061
[13] Lakomy K, Nath R and Santos L 2012 *Phys. Rev. E* **86** 013610
[14] Herink G, Kurtz F, Jalali B, Solli D and Ropers C 2017 *Science* **356** 50
[15] Liu X M, Yao X K and Cui Y D 2018 *Phys. Rev. Lett.* **121** 023905
[16] Lou S Y 2019 arXiv:1909.03399[nlin.SI]