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An Exponential Shape Function for Wormholes in Modified Gravity *

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We propose a new exponential shape function in wormhole geometry within modified gravity. The energy conditions and the equation-of-state parameter are obtained. The radial and tangential null energy conditions, and also the weak energy condition are validated, which indicates the absence of exotic matter due to modified gravity allied with such a new proposal.

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Wormholes (WHs) are asymptotically flat tube-like structures. They are said to be useful for interstellar travel because they may be able to connect two different points in the same universe or two points in different universes.[1,2]

WHs arise from the solutions of general relativity (GR). Schwarzschild's WH was the first WH-like solution obtained.[3] It was later found that it would collapse very quickly, preventing it being traversable.[4]

This issue was delved deeper in Refs. [1,5], in which a static and spherically symmetric metric was suggested to describe WHs and the required energy constraints to make them traversable were discussed. This analysis led to the violation of the null energy condition (NEC), so that for the GRWH to be traversable, it should be filled by exotic matter (matter violating the NEC).

The issue is that finding suitable contenders for exotic matter has never been carried out. Therefore, modified gravity theories (MGTs), which input some extra degrees of freedom to GR in a fundamental level, appear as a possibility to treat this issue by addressing the question of whether or not it is possible to have stable WH solutions with no need for exotic matter. In regard to MGTs, we recommend the important reviews.[6–8]

MGTs have been used to address not only the exotic matter issue but also other several issues of current observational astrophysics and cosmology.[9–16]

Due to the lack of WH observations so far, despite all of the efforts and proposals[17–23], some geometrical and material features of WHs, such as the shape function and equation of state (EoS), are still not precisely known. In particular, several forms for the shape function $b(r)$ have been proposed and analyzed so far, such as in Refs. [24–26]. In this Letter, we propose a new form for the WH shape function. As $b(r)$ is not arbitrary and has to obey several conditions, and therefore our proposal must be in accordance with these conditions.

We also aim at the obedience of the WH energy conditions (ECs). To attain this aim, allied to the proposed shape function, we will underline our model with a particular MGT, which is named $f(R,T)$ theory.[27] The $f(R,T)$ theory starts from a gravitational action that substitutes the Ricci scalar R in the usual Einstein–Hilbert action by a general function of R and T, with T being the trace of the energy momentum-tensor T_{ij}. The motivation to insert some material terms in the gravitational action is related to the possible existence of imperfect fluids in the universe. Since WHs' material content is described by an anisotropic fluid, their investigation in such a theory of gravity is well motivated.

The $f(R,T)$ gravity authors have argued that, due to the coupling of matter and geometry, this gravity model depends on a source term, which is nothing but the variation of the matter stress-energy tensor.[27] This source term could be related to quantum effects because it could lead to a particle creation scenario.[28] Consequently, the motion of test particles in $f(R,T)$ gravity is not along geodesic path due to the presence of an extra force perpendicular to the four velocities.

To obtain our WH solutions, we consider the $f(R,T)$ modified theory of gravity,[27] where the gravitational Lagrangian is given by an arbitrary function of R and T. The gravitational action for this theory is defined as[27]

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}f(R,T)+\int d^4x\sqrt{-g}\mathcal{L}_m. \quad (1)$$

In Eq. (1), $f(R,T)$ is an arbitrary function of R and T, g denotes the determinant of the metric g_{ij} and $\mathcal{L}_m$ is the matter Lagrangian. Moreover, we take natural units.

By varying Eq. (1) with respect to the metric g_{ij}, we can obtain the field equations

$$f_R(R,T)R_{ij}-\frac{1}{2}f(R,T)g_{ij}+(g_{ij}\Box-\nabla_i\nabla_j)f_R(R,T) = 8\pi T_{ij}-f_T(R,T)T_{ij}-f_T(R,T)\Theta_{ij}. \quad (2)$$

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Here

$$f_R(R,T)=\frac{\partial f(R,T)}{\partial R},\quad f_T(R,T)=\frac{\partial f(R,T)}{\partial T},$$
$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g})\mathcal{L}_m}{\delta g^{\mu\nu}}, \tag{3}$$
$$\Theta_{ij}=-2T_{ij}-pg_{ij}, \tag{4}$$

if we choose $\mathcal{L}_m=-p$ with p being the total pressure of the fluid. $\mathcal{L}_m=-p$ represents the matter Lagrangian density of a perfect fluid, which is not uniquely defined.[27,29–31] It is quite usual to see choices such as $\mathcal{L}_m=\rho$, with ρ being the matter-energy density, and $\mathcal{L}_m=-p$, but even $\mathcal{L}_m=T$ was already used.[30] It is known that geometry-matter coupling gravity theories, such as the $f(R,T)$ gravity, predict the existence of an extra force acting orthogonally to the four velocities in a (non-)geodesic motion. This extra force remarkably depends on the matter Lagrangian density and vanishes if $\mathcal{L}_m=-p$,[29,32] which is the reason why we have assumed so.

We will consider $f(R,T)=R+2\lambda T$, which was assumed as the functional form for the function $f(R,T)$ in several approaches such as Refs. [33–38] among many others. The considered form is the simplest one, which reduces to general relativity for the choice of $\lambda=0$. For this choice, one can easily correlate the obtained results with the most successful Einstein's general relativity. The $f(R,T)$ gravity field equations, for this case, read

$$G_{ij}=8\pi T_{ij}+\lambda T g_{ij}+2\lambda(T_{ij}+pg_{ij}), \tag{5}$$

with G_{ij} being the Einstein tensor.

To describe the geometry of WHs' spacetime, we use the modified version of the spherically symmetric spacetime metric as

$$ds^2=-dt^2+\left[\frac{dr^2}{1-\frac{b(r)}{r}}+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right]. \tag{6}$$

Here the redshift function was normalized. Constant redshift functions were assumed in several references, such as Refs. [39–41]. The radial coordinate r decreases from ∞ to a minimum value r_0, called the WH throat, and then increases to ∞. The shape function $b(r)$ needs to satisfy the following conditions:

- At the throat, $b(r_0)=r_0$ and for $r>r_0$, $1-\frac{b(r)}{r}>0$;
- $b'(r_0)<1$ (flaring-out condition);
- $\lim_{r\to\infty}\frac{b(r)}{r}=0$ (asymptotically flatness condition); with primes indicating radial derivatives.

The Ricci scalar for the WH metric is obtained as

$$R=\frac{2b'}{r^2}. \tag{7}$$

The non-zero Einstein tensor components for the WH metric are

$$G_{00}=\frac{b'}{r^2},$$
$$G_{11}=-\frac{b}{r^2(r-b)},$$
$$G_{22}=\frac{b-rb'}{2r},$$
$$G_{33}=\frac{\sin^2\theta\,(b-rb')}{2r}. \tag{8}$$

The field Eq. (5) for the metric (6) and anisotropic energy-momentum tensor are, then, written explicitly as

$$\frac{b'}{r^2}=(8\pi+\lambda)\rho-\lambda(p_r+2p_t),$$
$$-\frac{b}{r^3}=(8\pi+3\lambda)p_r+\lambda(\rho+2p_t),$$
$$\frac{b-rb'}{2r^3}=(8\pi+4\lambda)p_t+\lambda(\rho+p_r), \tag{9}$$

with p_r and p_t being, respectively, the radial and transverse pressures of the WH, such that the WH total pressure is $p=(p_r+2p_t)/3$. One can obtain the values in GR by making $\lambda=0$ in this equation.

In GR, the energy conditions are a set of inequalities that are required to prove various important theorems such as those related to WHs and black holes. It is well-known that static traversable GRWHs violate the energy conditions near the WH throat.[1,42]

The ECs have significant theoretical applications, such as the Hawking Penrose singularity conjecture, which is based on the (strong energy condition) SEC[43] while the (dominant energy condition) DEC is applicable to proof the positive mass theorem.[44] Furthermore, the NEC is a basic requirement to derive the second law of black hole thermodynamics.[45] The cosmological terms suchlike deceleration, look back time, distance modulus and statefinder parameters are seen in terms of redshift using ECs in Ref. [46]

The ECs have been studied in MGTs, such as $f(R)$ gravity, Brans-Dicke theory, $f(G)$ gravity, $f(G,\mathcal{T})$ gravity[47–50] with $\mathcal{T}$ being the torsion scalar. The generalized ECs are analyzed in MGTs considering the degrees of freedom related to scalar fields and curvature invariants.[51,52] In particular, the ECs were derived for a power law solution in $f(R,T)$ gravity and the stability of the same were established.[53]

We will take into consideration the following energy conditions for a perfect fluid[2]

1. Weak energy condition (WEC): $\rho\geq0$; $\rho+p_i>0$;
2. NEC: $\rho+p_i\geq0$;
3. SEC: $\rho+p_r+2p_t\geq0$;
4. DEC: $\rho\geq|p_i|$,

where $i=r,t$.

In the following, we will construct these energy conditions for the $f(R,T)$ WHs presented above for a

new proposal for the shape function, namely an exponential shape function.

In this section we propose a new exponential form for the shape function as

$$b(r) = r_0 \cdot e^{1-\frac{r}{r_0}}. \tag{10}$$

In Fig. 1, we can see some features of $b(r)$ (10).

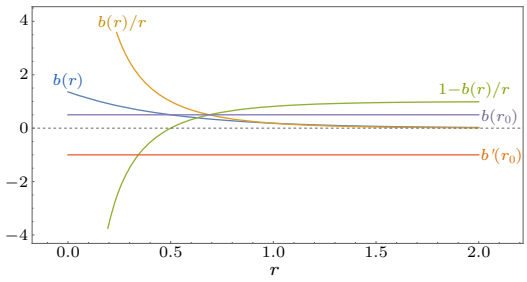

Fig. 1. Features of the shape function (10) with $r_0 = 0.5$.

One can observe from Fig. 1 that the shape function satisfies all the basic requirements given in the above paragraph. Using the above shape function in the field Eq. (9), we obtain

$$\begin{aligned}
\rho &= -\frac{e^{1-\frac{r}{r_0}}}{2(\lambda+4\pi)r^2}, \\
p_r &= -\frac{e^{1-\frac{r}{r_0}}[4(\pi-1)r+(\lambda+4\pi)r_0]}{2(\lambda+4\pi)^2 r^3}, \\
p_t &= \frac{e^{1-\frac{r}{r_0}}(r+r_0)}{4(\lambda+4\pi)r^3}.
\end{aligned} \tag{11}$$

We plot the radial EoS as well as the energy conditions for the present WH model with the exponential shape function.

Here it is determined that λ must remain in the range from -80 to -13 to validate NEC, WEC and DEC. In this work we have considered $\lambda = -30$. The respective behavior in GR is plotted in Fig. 4.

It is quite clear that in the absence of the $f(R,T)$ gravity extra terms, most of the energy conditions are no longer respected. In fact, only one of the energy conditions is respected, even within the assumption of such a promising and fruitful proposal for the WH shape function.

The $f(R,T)$ gravity, departing from many alternative gravity theories in the literature, allows one to modify the effective energy-momentum tensor worked out in GR. This is made clear in Eq. (5), in which even if one assumes the energy-momentum tensor of a perfect fluid, the effective energy-momentum tensor of the theory, namely,

$$T_{ij}^{\text{eff}} = T_{ij} + \frac{\lambda}{8\pi}[Tg_{ij} + 2(T_{ij} + pg_{\text{ig}})], \tag{12}$$

presents "imperfect" fluid terms, which may be related to viscosity or anisotropy. For such a strong modification, one can see the effective fluid permeating a particular astrophysical or cosmological system, such as in the present case the wormhole makes it possible to obtain significantly different material features and in our case. Therefore, it is possible to respect the energy conditions, as a consequence of the description of viscosity/anisotropy disguised in terms proportional to λ.

WHs are tube-like structures that, as shortcuts, connect two distant regions in the universe (or even in different universes). If their geometrical structure was not singular enough, according to GR formalism, WHs are expected to be filled by exotic (negative mass) matter.

The lack of observations of WHs so far means that we are unable to predict exactly some of their geometrical and material properties, such as the shape function and the EoS. In the present study we have proposed a novel functional form for the shape function, which depends only exponentially on r. We did not need to assume any particular form for the WH EoS, which was obtained from the model, rather than imposed to it, as happens in some cases in the literature.[54–57]

Before going any further discussion of the WH EoS obtained, we should mention that for the exponential shape function presented in Eq. (10), Fig. 1 has shown that it satisfies all the requirements needed to have traversable asymptotically flat WHs. Consequently, we are allowed to obtain the EoS parameter solution ω_r, as well as to construct the WH energy conditions.

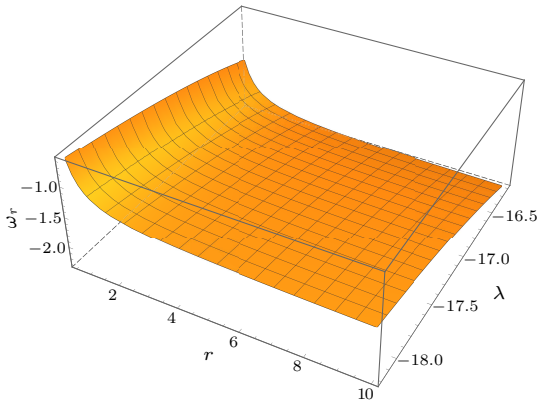

Fig. 2. Radial equation-of-state parameter $\omega_r = p_r/\rho$ as functions of r and λ with $r_0 = 0.5$.

Figure 2 shows that the WH EoS is in the phantom region; that is, $\omega_r < -1$. It is well-known that a phantom EoS parameter < -1 for the universe will imply in the so-called Big Rip[58] though some alternatives to evade such a catastrophic scenario have appeared.[59–62] Phantom WHs have also appeared in the literature,[54–65] though it is important to remark that in these cases the phantom EoS was invoked rather than obtained from the model (as in the present case).

Figure 3 show the energy conditions of the present WHs' scenario. They show a properly obedience of NEC and WEC, departing from standard GR solutions. The DEC is also satisfied while SEC is not.

Similar approaches to WHs in $f(R,T)$ gravity can be seen in the literature, though with non-exponential shape functions.[41,57,66–73] In comparison, in our approach the important role of the exponential shape function becomes clear because none of them presents

WHs fully satisfying NEC, WEC and DEC, as our model does (recall Fig. 1).

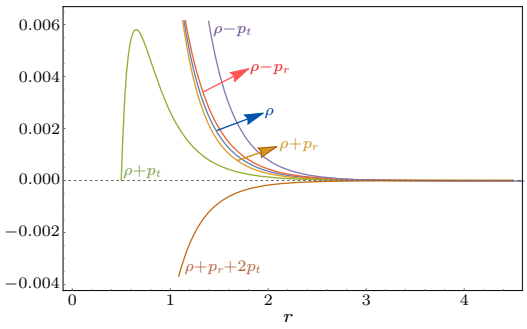

Fig. 3. The energy conditions as a function of r with $\lambda = -30$ and $r_0 = 0.5$.

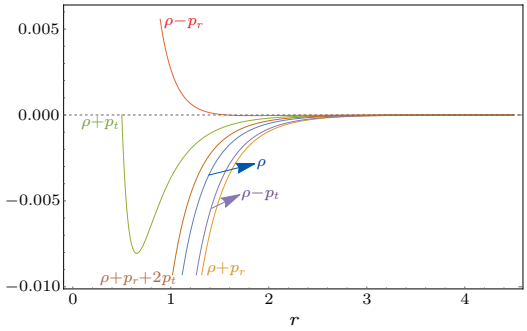

Fig. 4. The energy conditions as a function of r with $\lambda = 0$ and $r_0 = 0.5$.

The SEC (which is not obeyed in the present study, as well as in many others) has been discussed for some time. For instance, the SEC must be violated during the inflationary epoch and the need for this violation is the reason why inflationary models are typically driven by scalar inflation fields.[74] Furthermore, the recent observational data regarding the accelerating universe[75–77] means that the SEC is violated on cosmological scales rat the moment.[78]

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