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Cooper Molecules: Second Pairing of Cooper Pairs in Gapless Superconductor $CeCoIn_5$ *

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We establish a quantum field theory of phase transitions in gapless superconductor $CeCoIn_5$. It is found that uniform Cooper pair gases with pure gradient interactions with negative coefficient can undergo a Bardeen–Cooper–Schrieffer (BCS) condensation below a critical temperature. In the BCS condensation state, bare Cooper pairs with opposite wave vectors are bound into Cooper molecules, and uncoupled bare Cooper pairs are transformed into a new kind of quasiparticle, i.e., the dressed particles. The Cooper molecule system is a condensate or a superfluid, and the dressed particle system is a normal fluid. The critical temperature is derived analytically. The critical temperature of the superconductor $CeCoIn_5$ is obtained to be $T_c = 2.289\,K$, which approaches the experimental data. The transition from the BCS condensation state to the normal state is a first-order phase transition.

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Since the first heavy fermion superconductor $CeCu_2Si_2$ was discovered in 1979,[1] the Ce-based heavy fermion superconductors have gained extensive investigations both theoretically and experimentally. The superconductivity in $CeCu_2Si_2$ is gapped and the superconducting gap functions have been calculated recently.[2] The heavy fermion superconductor $CeCoIn_5$ has been under intensive study since its discovery in 2001.[3] The superconductor $CeCoIn_5$ possesses the highest T_c of all Ce-based superconductors. The experiments have observed that the superconductivity in this compound is gapless.[4] The superconductor $CeCoIn_5$ possesses four striking characteristics. The first characteristic is that elementary excitations in this superconductor are gapless and hence the gapless superconductor cannot be described by the theory of Bardeen, Cooper, and Schrieffer (BCS) on a fully gapped superconductor.[5,6] The second characteristic is that there are no magnetic impurities in this clean superconductor and thus the gapless property of this clean superconductor cannot be described by the theory of Abrikosov and Gor'kov (AG) on a gapless dirty superconductor.[7] The third characteristic is that the superconductor $CeCoIn_5$ has a short coherence length ξ and therefore the Cooper pair formed by conduction electrons is reasonably localized in the coordinate space. The fourth characteristic is that the superconductivity in this compound emerges out of a normal state with strong electronic correlations and hence Cooper pairs are preformed in the normal state. Based on the four characteristics, we develop a quantum field theory of phase transitions in gapless, clean superconductor $CeCoIn_5$. The detailed properties of phase transitions in gapless, clean superconductor $CeCoIn_5$ are expounded in the following.

It is well known that superconductivity and magnetism are generally competing ordered states in a superconductor. In a dirty superconductor, magnetic impurities have a drastic effect, and with increasing concentration the magnetic impurities lead quickly to a gapless superconductor and then the loss of order itself.[7] In a dirty superconductor, the reason for the pair-breaking is the spin-flip scattering on independent magnetic impurities. In the present work, we consider a clean superconductor $CeCoIn_5$, in which the mean free path greatly exceeds the coherence length $\xi_{ab} \approx 80\,Å$.[3] This is a nearly two-dimensional (with small 3D Fermi pockets) d-wave superconductor.[4] In the normal phase at $T > T_c = 2.3\,K$, the material is a paramagnet. It has been believed that the gapless superconductivity is caused by a strong pair-breaking scattering. However, the actual cause for pair-breaking is unknown. Because the superconductor $CeCoIn_5$ is extremely clean, there are no magnetic impurities inside it. Therefore, the pair-breaking effect is not due to magnetic scatterers. In this study, we believe that the reason for the pair-breaking is scattering on Cooper pairs in the heavy-fermion system. Based on this idea, we establish a new theory for gapless elementary excitations in the superconductor $CeCoIn_5$.

A Cooper pair in the heavy-fermion system behaves likes an atom. Scattering on Cooper pairs in the heavy-fermion system is characterized by wavelengths much larger than the van der Waals length of the microscopic potential between atoms. Such a separation of length scales implies that the associated low-energy interactions are determined mainly by the s-wave scattering length a. If $a < 0$, the interaction potential between Cooper pairs corresponds to a long-range attractive force. The presence of attractive interactions between Cooper pairs has a profound effect on the stability of a superconductive state, since a large enough attractive interaction will cause the superconductive state to become unstable and collapse in some way. In the present work, we consider the case of $a = 0$, when the superconductive state of the heavy-fermion system is highly stable. If the scattering length becomes zero, however, the interaction between Cooper pairs will be a momentum dependent interaction with negative coefficient. In the following we refer to a Cooper pair gas with pure gradient inter-

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actions with negative coefficient in momentum space. If the Cooper pair system can possess pure gradient interactions with negative coefficient, the Cooper pair system can undergo a BCS condensation below a critical temperature. The BCS condensation means a pairing state of two bosons with opposite momentums. The BCS condensation state of Cooper pairs possesses some peculiar properties. Firstly, the interaction between Cooper pairs is the pure gradient interaction with negative coefficient in momentum space. Secondly, Cooper pairs with opposite wave vectors are bound into pairs. Such a pair is called a Cooper molecule and a Cooper molecule is comprised of four heavy fermions. The Cooper molecule system is a condensate or a superfluid. Thirdly, the elementary excitations in the BCS condensation are dressed particles rather than phonons and a dressed particle is comprised of two heavy fermions. The dressed particle system is a normal fluid. Fourthly, the BCS condensate is highly stable. Fifthly, in the transition from the normal to the BCS condensation state, the Cooper pair system undergoes a first-order phase transition. The predicted properties of the BCS condensation state of Cooper pairs can be verified in the present-day physics laboratories.

We start our discussion by introducing the field operators of a conduction electron: $\hat{c}^{\dagger}_{\boldsymbol{k}\sigma}$ and $\hat{c}_{\boldsymbol{k}\sigma}$, where $\sigma = \pm\frac{1}{2}$, $\hat{c}^{\dagger}_{\boldsymbol{k}\sigma}$ creates an electron with wave vector $\boldsymbol{k}$ and helicity σ, and $\hat{c}_{\boldsymbol{k}\sigma}$ annihilates an electron with wave vector $\boldsymbol{k}$ and helicity σ. The system under study consists of interacting electrons in a superconducting crystal. The number of electrons in the system is conserved, and so the system possesses a chemical potential μ. The BCS Hamiltonian of the system is written as

$$\hat{H} = \sum_{\boldsymbol{k}\sigma}\Big(\frac{\hbar^2\boldsymbol{k}^2}{2m^*} - \mu\Big)\hat{c}^{\dagger}_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma',\boldsymbol{q}} V_{\boldsymbol{k},\boldsymbol{k}'}\hat{c}^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}/2,\sigma} \cdot \hat{c}^{\dagger}_{-\boldsymbol{k}+\boldsymbol{q}/2,-\sigma}\hat{c}_{-\boldsymbol{k}'+\boldsymbol{q}/2,-\sigma'}\hat{c}_{\boldsymbol{k}'+\boldsymbol{q}/2,\sigma'}, \tag{1}$$

where $\hbar$ is Planck's constant reduced, m^* is the effective mass of electrons, and $V_{\boldsymbol{k},\boldsymbol{k}'}$ is the attraction interaction between electrons. It is necessary to point out that the attraction interaction between electrons results from a virtual exchange of magnetic fluctuations in local f-electron moments, rather than the exchange of phonons that leads to superconductivity in ordinary metals. Because of the attraction interaction, two electrons form a zero-spin, bound Cooper pair, with a total mass $2m^*$ and a total charge $-2e$, where $-e$ is the charge of an electron. A Cooper pair has the composite wave vector $\boldsymbol{q}$. The corresponding creation operator of a Cooper pair is defined as[8]

$$\hat{a}^{\dagger}_{\boldsymbol{q}} = \sum_{\boldsymbol{k}\sigma}\phi_{\boldsymbol{k}}\hat{c}^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}/2,\sigma}\hat{c}^{\dagger}_{-\boldsymbol{k}+\boldsymbol{q}/2,-\sigma}, \tag{2}$$

where $\phi_{\boldsymbol{k}}$ is the internal wave function, extending over a characteristic distance ξ, and $\hat{a}_{\boldsymbol{q}}$ annihilates a Cooper pair with wave vector $\boldsymbol{q}$. If two Cooper pairs have only a small overlap, i.e., if the coherence length ξ approaches the lattice constant, the Cooper pairs can be treated as a gas of point bosons, whose internal orbital structure is irrelevant.

Since plane-wave modes constitute a complete orthonormal set, they can be used for the expansion of the field operators in coordinate space. In terms of the creation and annihilation operators $\hat{a}^{\dagger}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}$ of Cooper pairs with wave vector $\boldsymbol{k}$, the field operators of Cooper pairs in coordinate space are defined as

$$\hat{\Psi}(\boldsymbol{x}) = \sum_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\frac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\ \ \hat{\Psi}^{\dagger}(\boldsymbol{x}) = \sum_{\boldsymbol{k}}\hat{a}^{\dagger}_{\boldsymbol{k}}\frac{1}{\sqrt{V}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3}$$

where the crystal occupies a volume V, $\hat{\Psi}^{\dagger}(\boldsymbol{x})$ creates a Cooper pair at the point $\boldsymbol{x}$, and $\hat{\Psi}(\boldsymbol{x})$ annihilates a Cooper pair at the point $\boldsymbol{x}$. Hence, the Cooper pair state can be well described by a phenomenological local boson field $\hat{\Psi}(\boldsymbol{x})$.

It is prescribed that the system under study consists of N interacting Cooper pairs. Our goal is to determine the interaction between Cooper pairs. We have known that an attractive interaction between electrons results from a virtual exchange of local moment fluctuations. The magnetic pairing interaction can be so strong that the interaction between Cooper pairs is attractive in the whole space. Next, we must consider the fact that every Cooper pair carries a total charge $-2e$. However, the magnetic attractive interaction can overcome the Coulomb repulsive interaction. Therefore, one concludes that the interaction between Cooper pairs is a weak attractive interaction. Here $U(\boldsymbol{x}-\boldsymbol{x}')$ represents the attractive interaction potential between a Cooper pair located at $\boldsymbol{x}$ and another Cooper pair located at $\boldsymbol{x}'$.

We adopt the grand canonical ensemble, in which Cooper pairs have a chemical potential μ. In this study, we do not include external fields, and therefore we consider a uniform system of interacting Cooper pairs. Now one can write the Hamiltonian of the Cooper pair system in terms of the field operators $\hat{\Psi}(\boldsymbol{x})$ and $\hat{\Psi}^{\dagger}(\boldsymbol{x})$,

$$\hat{H} = \int\hat{\Psi}^{\dagger}(\boldsymbol{x})\Big(-\frac{\hbar^2}{4m^*}\nabla^2 - \mu\Big)\hat{\Psi}(\boldsymbol{x})d\boldsymbol{x} + \frac{1}{2}\int\int\hat{\Psi}^{\dagger}(\boldsymbol{x})\hat{\Psi}^{\dagger}(\boldsymbol{x}')U(\boldsymbol{x}-\boldsymbol{x}') \cdot \hat{\Psi}(\boldsymbol{x}')\hat{\Psi}(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{x}'. \tag{4}$$

On substituting Eq. (3) into Eq. (4), the Hamiltonian of the Cooper pair system is second quantized as

$$\hat{H} = \sum_{\boldsymbol{k}}(\varepsilon_{\boldsymbol{k}} - \mu)\hat{a}^{\dagger}_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}}U(\boldsymbol{q})\hat{a}^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}}\hat{a}^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q}}\hat{a}_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}}, \tag{5}$$

where $\varepsilon_{\boldsymbol{k}} = \hbar^2\boldsymbol{k}^2/4m^*$ is the kinetic energy of a Cooper pair with wave vector $\boldsymbol{k}$, and $U(\boldsymbol{q})$ is the Fourier transform of the interaction potential $U(\boldsymbol{x})$ of Cooper pairs. The first term of the Hamiltonian in Eq. (5) represents the energy of a free Cooper-pair system, and the second term denotes the energy of the interaction between Cooper pairs arising from the potential energy $U(\boldsymbol{q})$.

A single parameter, the s-wave scattering length a, characterizes all of the effects of the interaction on

the physical properties of the Cooper pair gas. If n denotes the number density of Cooper pairs, the condition of diluteness of the Cooper pair gas can be written as $|a| \ll n^{-1/3}$. The interaction potential $U(\boldsymbol{q})$ satisfies the symmetric property $U(-\boldsymbol{q}) = U(\boldsymbol{q})$, namely, $U(\boldsymbol{q})$ is an even function of $\boldsymbol{q}$. Since only small wave vectors are involved in the solution of the many-body problem, we can expand $U(\boldsymbol{q})$ as a Maclaurin series

$$U(\boldsymbol{q}) = U(\boldsymbol{0}) + \frac{1}{2!}U^{(2)}(\boldsymbol{0})\boldsymbol{q}^2 + \frac{1}{4!}U^{(4)}(\boldsymbol{0})\boldsymbol{q}^4 + \ldots, \quad (6)$$

where $U(\boldsymbol{0})$ is the $\boldsymbol{q} = \boldsymbol{0}$ value of the Fourier transform of $U(\boldsymbol{x})$. In terms of the scattering length a, one can easily express the parameter $U(\boldsymbol{0})$ as

$$U(\boldsymbol{0}) = \frac{2\pi\hbar^2 a}{m^* V}. \quad (7)$$

A noninteracting Cooper pair gas is characterized by the scattering length of $a = 0$, so that $U(\boldsymbol{0}) = 0$. By the pure gradient interactions with negative coefficient, we mean that in Eq. (6), $U(\boldsymbol{0}) = 0$ and the momentum dependent terms are negative. Here $U^{(n)}(\boldsymbol{0})$ represents the nth derivative of $U(\boldsymbol{q})$ at $\boldsymbol{q} = \boldsymbol{0}$. To the non-vanishing term of the lowest order, the interaction potential is given by

$$U(\boldsymbol{q}) = \frac{1}{2!}U^{(2)}(\boldsymbol{0})\boldsymbol{q}^2. \quad (8)$$

There are Cooper pair systems where $U^{(2)}(\boldsymbol{0}) < 0$ and further this sign can be determined in experiments.

We first notice that since the total momentum of the Cooper pair system is zero, the Cooper pairs of opposite wave vectors always appear simultaneously. We have pointed out that there is a magnetically mediated attractive interaction between Cooper pairs. The attractive interaction leads to second pairing of Cooper pairs. Such a pair is called a Cooper molecule and a Cooper molecule is comprised of four heavy fermions. In the standing-wave configuration a Cooper molecule is stable only if the two Cooper pairs have opposite wave vectors $\boldsymbol{k}$ and $-\boldsymbol{k}$. The Cooper molecule Hamiltonian is obtained by ignoring all interaction terms of Eq. (5) with the exception of the following three types: (a) $\boldsymbol{q} = \boldsymbol{0}$; (b) $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$, $(\boldsymbol{k}' \neq \boldsymbol{k})$; (c) $\boldsymbol{k}' = -\boldsymbol{k}$, $(\boldsymbol{q} \neq \boldsymbol{0})$. For the interaction terms of the first two types, we adopt the Hartree–Fock approximation. For the interaction terms of the third type, we change the index $\boldsymbol{q}$ of summation into $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{q}$. Consequently the Cooper molecule Hamiltonian is acquired as

$$\begin{aligned}\hat{H} = &\frac{1}{2}\sum_{\boldsymbol{k}} \tilde{\varepsilon}_{\boldsymbol{k}}(\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}) \\ &+ \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'} U(\boldsymbol{k} - \boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}^\dagger_{-\boldsymbol{k}'}\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}, \end{aligned} \quad (9)$$

where $\tilde{\varepsilon}_{\boldsymbol{k}}$ is the Hartree–Fock energy given by

$$\tilde{\varepsilon}_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} + NU(\boldsymbol{0}) + \xi_{\boldsymbol{k}} - \mu, \quad (10)$$

with $\xi_{\boldsymbol{k}}$ being the Fock self-energy defined by

$$\xi_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} U(\boldsymbol{k} - \boldsymbol{k}')\langle\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}'}\rangle, \quad (11)$$

where $\langle\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\rangle$ denotes thermal averages of the number operator of Cooper pairs $\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}$ for temperature T. The total Cooper pair number is $N = \sum_{\boldsymbol{k}}\langle\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\rangle$, and $NU(\boldsymbol{0})$ denotes the Hartree self-energy.

Now the Cooper pairs have the interaction potential given by

$$U(\boldsymbol{k} - \boldsymbol{k}') = -\alpha\frac{\hbar^2(\boldsymbol{k} - \boldsymbol{k}')^2}{4m^*}, \quad (12)$$

where α is a positive dimensionless parameter and represents the interaction strength between two Cooper pairs. The parameter α must be smaller than unity. The interaction potential has the symmetric property $U(\boldsymbol{k} - \boldsymbol{k}') = U(\boldsymbol{k}' - \boldsymbol{k})$. The symmetric property is used in our future treatment. Further, we remark that the Fock self-energy $\xi_{\boldsymbol{k}}$ of a Cooper pair takes its value mainly at the $\boldsymbol{k} = \boldsymbol{0}$ state, namely, $\xi_{\boldsymbol{k}} \approx \xi_{\boldsymbol{0}}$. At this point, the chemical potential is determined as $\mu = \xi_{\boldsymbol{0}}$. Concomitantly, the Hartree–Fock energy $\tilde{\varepsilon}_{\boldsymbol{k}}$ of a Cooper pair is reduced to $\tilde{\varepsilon}_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}}$. As a side remark, we mention that Beliaev studied the properties of Bose gases at finite temperatures when atomic interactions contain momentum dependence.[9]

Single uncoupled Cooper pairs in the Cooper pair system are transformed into a new kind of quasiparticle, i.e., the dressed particle. A dressed particle is an elementary excitation of the Cooper pair system with pure gradient interactions with negative coefficient, which carries all the information of the system of interacting Cooper pairs. A dressed particle is comprised of two heavy fermions. The diagonalization of the Cooper molecule Hamiltonian in Eq. (9) can be performed by the Bogoliubov transformation[10]

$$\begin{aligned}\hat{c}_{\boldsymbol{k}} &= U\hat{a}_{\boldsymbol{k}}U^\dagger = \hat{a}_{\boldsymbol{k}}\cosh\varphi_{\boldsymbol{k}} - \hat{a}^\dagger_{-\boldsymbol{k}}\sinh\varphi_{\boldsymbol{k}}, \\ \hat{c}^\dagger_{\boldsymbol{k}} &= U\hat{a}^\dagger_{\boldsymbol{k}}U^\dagger = \hat{a}^\dagger_{\boldsymbol{k}}\cosh\varphi_{\boldsymbol{k}} - \hat{a}_{-\boldsymbol{k}}\sinh\varphi_{\boldsymbol{k}}, \end{aligned} \quad (13)$$

where the parameter $\varphi_{\boldsymbol{k}}$ is assumed to be real and spherically symmetric $\varphi_{-\boldsymbol{k}} = \varphi_{\boldsymbol{k}}$, $\hat{c}^\dagger_{\boldsymbol{k}}$ and $\hat{c}_{\boldsymbol{k}}$ are the creation and annihilation operators of dressed particles in the Cooper pair system. The transition from the operators of Cooper pairs to those of dressed particles can be effected by a symplectic transformation

$$U = \exp\left[\frac{1}{2}\sum_{\boldsymbol{k}}\varphi_{\boldsymbol{k}}(\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}} - \hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}})\right]. \quad (14)$$

If we denote the vacuum state of Cooper pairs by $|0\rangle$, then $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$. The normalized state vector of Cooper molecules in the Cooper pair system may be constructed as $|G\rangle = U|0\rangle$, such that $\hat{c}_{\boldsymbol{k}}|G\rangle = 0$.

The mean-field approximation consists of the three steps.[11] The first step is to introduce the number operator of dressed particles $\hat{N}_k = \hat{c}^\dagger_{\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}$. The excitation energy of dressed particles is defined by $\tilde{\varepsilon}_{\boldsymbol{k}}(T) = \partial\langle\hat{H}\rangle/\partial\langle\hat{N}_{\boldsymbol{k}}\rangle$. The second step is to approximate the averages of products of number operators as

$$\langle\hat{N}_{\boldsymbol{k}}\hat{N}_{\boldsymbol{k}'}\rangle \approx \langle\hat{N}_{\boldsymbol{k}}\rangle\langle\hat{N}_{\boldsymbol{k}'}\rangle.$$

The third step is to let the second-order nondiagonal terms in $\hat{H}$ vanish. Under the mean-field approximation, the Cooper molecule Hamiltonian in Eq. (9) becomes

$$\hat{H} = E_{\rm m} + \sum_{\boldsymbol{k}} \tilde{\varepsilon}_{\boldsymbol{k}}(T)\hat{c}^{\dagger}_{\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}, \tag{15}$$

where $E_{\rm m}$ is the energy of the system of Cooper molecules, as given by

$$E_{\rm m} = \sum_{\boldsymbol{k}} \Big[\varepsilon_{\boldsymbol{k}} \sinh^2 \varphi_{\boldsymbol{k}} + \frac{1}{4}\sinh(2\varphi_{\boldsymbol{k}}) \sum_{\boldsymbol{k}'} \frac{U(\boldsymbol{k}-\boldsymbol{k}')}{2} \sinh(2\varphi_{\boldsymbol{k}'})\Big]. \tag{16}$$

The excitation energy of dressed particles is derived as

$$\tilde{\varepsilon}_{\boldsymbol{k}}(T) = [\varepsilon^2_{\boldsymbol{k}} - \Delta^2_{\boldsymbol{k}}(T)]^{1/2}, \tag{17}$$

where the excitation spectrum $\tilde{\varepsilon}_{\boldsymbol{k}}(T)$ has no gap, $\Delta_{\boldsymbol{k}}(T)$ is the order parameter for second pairing of Cooper pairs, and $\tilde{\varepsilon}_{\boldsymbol{k}}(T)$ represents the excitation energy of a dressed particle and is dependent on temperature. As we have seen, the Cooper molecule Hamiltonian in Eq. (9) can be solved only when the interaction potential $U(\boldsymbol{k}-\boldsymbol{k}')$ is negative. In this case, the order parameter is determined self-consistently by

$$\Delta_{\boldsymbol{k}}(T) = -\sum_{\boldsymbol{k}'} \frac{U(\boldsymbol{k}-\boldsymbol{k}')}{2} \frac{\Delta_{\boldsymbol{k}'}(T)}{[\varepsilon^2_{\boldsymbol{k}'} - \Delta^2_{\boldsymbol{k}'}(T)]^{1/2}} \times \coth \frac{[\varepsilon^2_{\boldsymbol{k}'} - \Delta^2_{\boldsymbol{k}'}(T)]^{1/2}}{2k_{\rm B}T}, \tag{18}$$

which is suitable for any temperature T, and $k_{\rm B}$ is Boltzmann's constant.

Below a critical temperature, Eq. (18) has the second pairing-state solution, namely, $\Delta_{\boldsymbol{k}}(T) \neq 0$, for all $\boldsymbol{k}$. However, above the critical temperature, the normal-state solution is $\Delta_{\boldsymbol{k}}(T) = 0$ for all $\boldsymbol{k}$. Because the energy $\tilde{\varepsilon}_{\boldsymbol{k}}(T)$ of dressed particles must be directly proportional to the square of wave vector $\boldsymbol{k}^2$, according to Eq. (17) this criterion demands that $\Delta_{\boldsymbol{k}}(T) \propto \boldsymbol{k}^2$. Correspondingly, we let $\Delta_{\boldsymbol{k}}(T) = \varepsilon_{\boldsymbol{k}}\Delta(T)$, where $\varepsilon_{\boldsymbol{k}} = \hbar^2\boldsymbol{k}^2/4m^*$, and $\Delta(T)$ is the order parameter that depends on temperature but not on wave vector. Consequently, the energy of dressed particles is acquired as $\tilde{\varepsilon}_{\boldsymbol{k}}(T) = \hbar^2\boldsymbol{k}^2/2m(T)$, where the mass of dressed particles reads

$$m(T) = \frac{2m^*}{\sqrt{1-\Delta^2(T)}}. \tag{19}$$

The order parameter $\Delta(T)$ is nonzero below the critical temperature and is meaningful only if $\Delta(T) < 1$. The critical temperature $T_{\rm c}$ is the characteristic parameter of a uniform Cooper pair gas with pure gradient interactions with negative coefficient.

To derive the equation for $m(T)$, the interaction potential $U(\boldsymbol{k}-\boldsymbol{k}')$ is taken into account. At low temperatures, only the Cooper pairs of small wave vectors can be coupled. Therefore, we must introduce an upper cutoff wave vector $k_{\rm c}$. Under a reasonable approximation, we set $k_{\rm c} = n^{1/3}$, where n is the number density of the Cooper pair gas. Consequently, the interaction potential in Eq. (12) can be reduced to

$$U(\boldsymbol{k}-\boldsymbol{k}') = \begin{cases} \frac{-\alpha\hbar^2(\boldsymbol{k}-\boldsymbol{k}')^2}{4m^*} & \text{if } |\boldsymbol{k}| \text{ and } |\boldsymbol{k}'| < n^{1/3}, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

At this point, we substitute Eq. (20) into Eq. (18) and obtain a calculable equation

$$\frac{2m^*}{m(T)} = \frac{\alpha}{2}{\sum_{\boldsymbol{k}'}}' \coth\Big[\frac{\hbar^2\boldsymbol{k}'^2/2m(T)}{2k_{\rm B}T}\Big], \tag{21}$$

where the prime on the summation symbol means $|\boldsymbol{k}'| < n^{1/3}$. In the derivation of Eq. (21), we have considered the fact that the main contribution to the summation comes from the wave vector $\boldsymbol{k}' = \boldsymbol{0}$. Equation (21) at zero temperature reduces to

$$\frac{2m^*}{m(0)} = \gamma = \frac{\alpha N}{12\pi^2}, \tag{22}$$

where $N = Vn$ is the total number of Cooper pairs in the system, and γ is a dimensionless constant that is characteristic of a uniform Cooper pair gas with pure gradient interactions with negative coefficient. The constant γ is meaningful only if $\gamma < 1$. The constant γ signifies the total interaction strength among N Cooper pairs. In what follows, we regard γ as an adjustable parameter.

The order parameter $\Delta(T)$ determined by Eqs. (19) and (21) is a monotonically decreasing function of temperature T, which vanishes at the critical temperature $T_{\rm c}$. Thus we first set $\Delta(T_{\rm c}) = 0$ in Eq. (19), then substitute $m(T_{\rm c}) = 2m^*$ into Eq. (21), and finally convert the summation into the integration over Cooper pair energies $\varepsilon = \hbar^2\boldsymbol{k}'^2/4m^*$,

$$1 = \frac{3\gamma}{2\varepsilon_{\rm c}^{3/2}} \int_0^{\varepsilon_{\rm c}} \varepsilon^{1/2} \coth\Big(\frac{\varepsilon}{2k_{\rm B}T_{\rm c}}\Big) d\varepsilon, \tag{23}$$

where $\varepsilon_{\rm c} = \hbar^2 n^{2/3}/4m^*$ is the upper limit of the integral, and $\varepsilon_{\rm c}$ represents the Debye cutoff energy of Cooper pairs. Making the substitution $x = \varepsilon/2k_{\rm B}T_{\rm c}$, one can cast Eq. (23) into the form

$$1 = \frac{3\gamma}{2x_{\rm c}^{3/2}} \int_0^{x_{\rm c}} x^{1/2} \coth x dx, \tag{24}$$

where $x_{\rm c} = \varepsilon_{\rm c}/2k_{\rm B}T_{\rm c}$ is the upper limit of the integral. It is convenient to introduce a numerical factor $C(\gamma) = k_{\rm B}T_{\rm c}/\varepsilon_{\rm c}$, and C represents a reduced critical temperature. We solve Eq. (24) for $C(\gamma)$. The variation of C with γ is revealed in Fig. 1. The numerical factor C is a monotonically decreasing function of γ. Now the critical temperature is given by $T_{\rm c} = C(\gamma)\varepsilon_{\rm c}/k_{\rm B}$. The essential characteristics are that the critical temperature is proportional to the Debye cutoff energy divided by Boltzmann's constant and that the proportional coefficient is a universal function of γ, independent of any particular property of the Cooper pair gas.

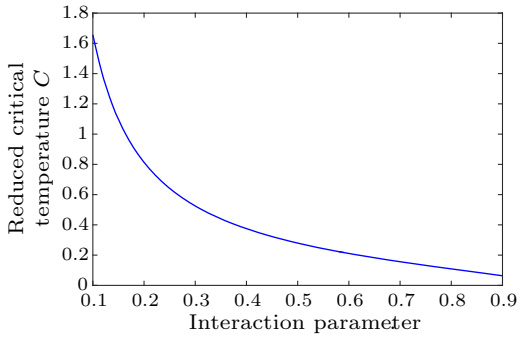

Fig. 1. According to Eq. (24), the variation of the reduced critical temperature $C = k_{\rm B}T_{\rm c}/\varepsilon_{\rm c}$ with the interaction parameter γ.

The topic of Bose–Einstein condensation (BEC) in a uniform noninteracting gas of bosons is treated in most textbooks on statistical mechanics. The critical temperature of BEC of such a gas is given by $T_{\rm c} = C\varepsilon_{\rm c}/k_{\rm B}$, where $C = 6.62$. From Fig. 1, one can see that $C(\gamma) < 0.44$ if $\gamma > 0.35$. Therefore, we conclude that the critical temperature of the BCS condensation of uniform boson gases with pure gradient interactions with negative coefficient is much lower than that of the BEC of a uniform, noninteracting boson gas. To give a numerical impression of $T_{\rm c}$, we take into account the superconductor $CeCoIn_5$. The parameters of the superconductor $CeCoIn_5$ are as follows: $m^* = 10m_{\rm e}$, where $m_{\rm e}$ is the bare electron mass, $\alpha = 1.6668 \times 10^{-19}$, $N = 4.8 \times 10^{20}$, and $n = 4.8 \times 10^{20}\,{\rm cm}^{-3}$. The value of γ is evaluated from Eq. (22) as $\gamma = 0.6755$. Figure 1 shows that at $\gamma = 0.6755$, $C = 0.1689$. The value of $\varepsilon_{\rm c}$ is evaluated as $\varepsilon_{\rm c} = 1.8711 \times 10^{-22}$ J. Thereby, the critical temperature of the superconductor $CeCoIn_5$ is obtained to be $T_{\rm c} = 2.289\,{\rm K}$, which approaches the experimental critical temperature of the superconductor $CeCoIn_5$. We address clearly that the BCS state discussed must be highly stable. At temperature $T > T_{\rm c}$, a uniform boson gas with pure gradient interactions with negative coefficient enters the normal state, which is also stable. Thus the superconductive state of the compound $CeCoIn_5$ is not the BEC state, because the critical temperature of BEC of such a Cooper pair gas will be $T_{\rm c} = 89.717\,{\rm K}$. This is the critical temperature of a high-temperature superconductor.

The BEC state of uniform boson gases with weak repulsion has been extensively investigated in literature. However, the BCS condensation state of uniform boson gases with pure gradient interactions with negative coefficient is studied only in the present work. There are the following contrasts between the BEC state and the BCS condensation state. (i) In the BEC state the interaction between atoms is repulsive, whereas in the BCS condensation state the interaction between Cooper pairs is attractive. (ii) In the original BEC a macroscopic number of atoms occupy the zero momentum state, whereas in the BCS condensation a macroscopic number of Cooper pairs with opposite momenta are bound into Cooper molecules. (iii) The low-energy elementary excitations in the BEC state are phonons, while the low-energy elementary excitations in the BCS condensation state are dressed particles rather than phonons. (iv) The BEC state is highly stable, irrespective of the interatomic interaction strength and the atom number, while the BCS condensation state is stable only if the interaction strength between Cooper pairs is small. (v) The transition from the BEC state to the normal state is a second-order phase transition, whereas the transition from the BCS condensation state to the normal state is a first-order phase transition. On the other hand, there is the following similarity: the BEC state and the BCS condensation state are all quantum statistical effects of many-boson systems. The general Hamiltonian in Eq. (4) was treated by Bogoliubov in the case of superfluid helium. It must be admitted, however, that the Bogoliubov treatment is radically different from the BCS treatment of superconductivity in that it does not determine the order parameter nor the depletion of the condensate in a self-consistent way. The off-diagonal long-range order in the one-particle density matrix in the BEC theory is associated with the macroscopic condensation of Bose atoms, while the off-diagonal long-range order in the two-particle density matrix in the BCS theory is more properly associated with the second pairing of Cooper pairs.[12,13]

In summary, we have proposed a BCS condensation state of uniform Cooper pair gases with pure gradient interactions with negative coefficient. The Cooper pair system in the BCS condensation state consists of Cooper molecules and individual dressed particles. The system of Cooper molecules is a superfluid (condensate) and the system of individual dressed particles is a normal fluid. The critical temperature is derived analytically. The critical temperature is a monotonically decreasing function of the interaction strength between Cooper pairs. The BCS condensation state will be stable only if the interaction strength between Cooper pairs is sufficiently small. The transition from the BCS condensation state to the normal state is a first-order phase transition.

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