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A Closure for Isotropic Turbulence Based on Extended Scale Similarity Theory in Physical Space

Chu-Han Wang(王楚涵)[1], Le Fang(方乐)[1,2**]

[1]*Laboratory of Mathematics and Physics, Ecole Centrale de Pékin, Beihang University, Beijing 100191*
[2]*Co-Innovation Center for Advanced Aero-Engine, Beihang University, Beijing 100191*

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The closure of a turbulence field is a longstanding fundamental problem, while most closure models are introduced in spectral space. Inspired by Chou's quasi-normal closure method in spectral space, we propose an analytical closure model for isotropic turbulence based on the extended scale similarity theory of the velocity structure function in physical space. The assumptions and certain approximations are justified with direct numerical simulation. The asymptotic scaling properties are reproduced by this new closure method, in comparison to the classical Batchelor model.

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Nonlinearity is the origin of many complicated multiscale phenomena. To quantitatively investigate the nonlinear multiscale mechanism, an analytical model is often considered as the most powerful tool. In the case of homogeneous isotropic turbulence, i.e., a simple and ideal turbulent flow to study the physical mechanism of turbulence, analytical models have been developed. Dated from Chou's closure with a quasi-normal (QN) assumption,[1] the closure models have been developed for more than 70 years, aiming at providing analytical tools to describe turbulent flows in spectral space. The assumption of an eddy-damping quasi-normal Markovian (EDQNM)[2,3] is usually regarded as one of the most accurate analytical models of turbulence.

The nonlinear multiscale behaviour of isotropic turbulence is characterized by the scaling laws either on the energy spectrum in spectral space or on the second order structure function in physical space. As the closure in spectral space is employed on an energy spectrum, we phenomenally guess that a closure method on the velocity structure function in physical space is possible. A closure model in physical space has the potential to be employed in complicated turbulent flow, which is impossible with the present models in spectral space.

The assumption applied to equation closure in spectral space is a quasi-normal assumption, which is a link between second- and fourth-order spectral correlations. In physical space, instead, we use the extended scale similarity (ESS) theory,[4–6] which describes the relation between the fourth-order and second-order longitudinal structure functions.

In our recent work,[7] we proposed an analytical framework for equation closure in physical space of isotropic turbulence. The asymptotic scaling behavior at small and large scales is qualitatively produced, yet we observe an inappropriate oscillation around the inertial scale. Moreover, the scaling information of the turbulence, such as the length of the dissipation scale, was not involved in the model, which makes it impossible to predict the multiscale behavior of a specific isotropic turbulence. Finally, the assumptions applied in the closure process were not validated with numerical experiment data, which may lead to a risk in equation derivation.

In this Letter, we try to resolve the above problems with the help of direct numerical simulation (DNS). The key assumptions are validated, and the modeling process is considerably improved.

Since DNS is used not only for model verification but also for parameterized modeling, we have to explain the mathematical derivation and DNS observations at the same time. A pseudo-spectral code is employed in DNS with the grid number of 512^3. The length of computation domain is 12. All the DNS observations are carried out in a statistically stationary homogeneous turbulence field whose dissipation scale η is 3.56×10^{-2} and whose Reynolds number of Taylor scale Re_λ is 93.53.

The statistical stationarity implies for any statistical quantity there is $\partial_t\langle\bullet\rangle = 0$, where t is time and $\langle\rangle$ is the ensemble average. Neglecting the time-dependent term, as is proved by DNS in Ref. [8], the classical Kolmogorov equation can be simply written as[9–12]

$$D_{111} = 6\nu D'_{11} - \frac{4}{5}\epsilon r, \tag{1}$$

with ν being the kinematic viscosity, r the two-point distance, ϵ the dissipation rate, $D_{11}(r) = \langle(u_1(r\boldsymbol{e}_1) - u_1(\boldsymbol{0}))^2\rangle$ and $D_{111}(r) = \langle(u_1(r\boldsymbol{e}_1) - u_1(\boldsymbol{0}))^3\rangle$ are the second- and third-order longitudinal velocity structure functions respectively, with $\boldsymbol{e}_1$ being the unit vector in the x_1 axis.

Clearly, this equation shows the relation between second- and third-order structure functions. The relation between third- and fourth-order structure functions can be found in Ref. [13]. Neglecting the non-stationary term $\partial_t D_{1111}$, we obtain

$$D'_{1111} + \frac{2}{r}(D_{1111} - 3D_{1122}) - 2\nu(C - Z_{111}) = -T_{111}, \tag{2}$$

**Corresponding author. Email: le.fang@buaa.edu.cn
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where $D_{1111}(r) = \langle (u_1(r\boldsymbol{e}_1) - u_1(\boldsymbol{0}))^4 \rangle$ and $D_{1122}(r) = \langle (u_1(r\boldsymbol{e}_1) - u_1(\boldsymbol{0}))^2 (u_2(r\boldsymbol{e}_1) - u_2(\boldsymbol{0}))^2 \rangle$ are the fourth-order longitudinal and cross velocity structure functions, respectively, $T_{111}(r)$ is a velocity-pressure correlation function, $Z_{111}(r)$ is a third-order velocity-velocity derivation function and

$$C = \underbrace{-\frac{4}{r^2} D_{111}}_{\mathrm{C}_1} + \underbrace{\frac{4}{r} D'_{111}}_{\mathrm{C}_2} + \underbrace{D''_{111}}_{\mathrm{C}_3}, \tag{3}$$

where the three components are respectively named as C_1, C_2 and C_3 for simplicity.

To close the fourth-order structure functions in Eq. (2), we employ the ESS theory which describes the relation between the fourth- and second-order longitudinal structure functions, and the quasi-normal assumption which considers the relation between fourth-order longitudinal and cross structure functions.

The ESS theory implies that in a wide range, the fourth- and second-order longitudinal structure functions can be expressed by a constant scale-similarity fractal scaling. From Table 2 of Ref. [5], this scaling can be estimated to be $p = 1.28/0.70 \approx 1.83$. McComb *et al.* showed that the ESS theory is valid at almost all scales in homogeneous isotropic turbulence (see Fig. 1 of Ref. [14]), which directly supports the present closure idea. Therefore, we can obtain

$$D_{1111} = c_p D_{11}^p, \tag{4}$$

with c_p being constant.

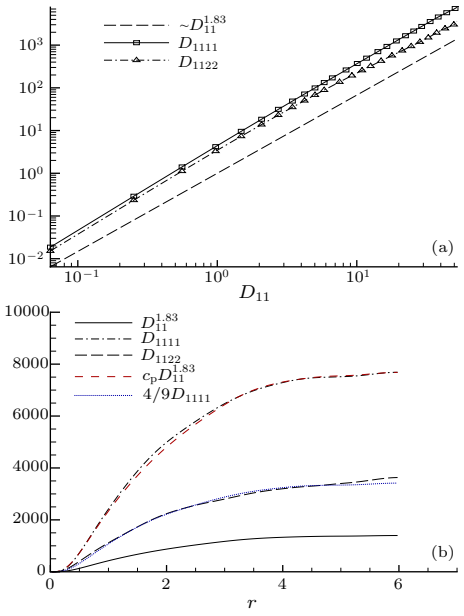

Fig. 1. ESS theory applied in Eqs. (4) and (5) validated with DNS (a) in logarithmic coordinates as a function of D_{11} and (b) in linear coordinates as a function of r. Here $c_p = 5.5$.

On the other hand, there are hardly any investigations on different fourth-order structure functions, therefore the quasi-normal assumption may be the only choice in the present model. From Eq. (3) in Ref. [6], using the isotropy condition $D_{22} = D_{11} + \frac{r}{2} D'_{11}$ and the scaling $D_{11}(r) \propto r^{2/3}$ in the inertial range, we can obtain

$$D_{1122} = \frac{4}{9} D_{1111} = \frac{4}{9} c_p D_{11}^p. \tag{5}$$

There are numerical evidences that imply the rationality of using quasi-normal assumption to consider the relation between fourth-order longitudinal and cross structure functions. An example is given in Fig. 4(a) of Ref. [6]. The quasi-normal (i.e., the Gaussian state) approximation works better than the restricted Euler (RE) approximation[15,16] in many situations in equilibrium turbulence and low Mach turbulence. However, to consider the non-equilibrium procedures, one needs to consider the Gaussian-RE line[6] instead of the quasi-normal assumption in Eq. (5). The Gaussian-RE line involves an additional parameter on the non-equilibrium property.[17] In the present work we only focus on the statistically stationary turbulence where the flow cannot be non-equilibrium, thus the quasi-normal assumption is acceptable.

The ESS theory applied in Eqs. (4) and (5) is demonstrated with DNS in Fig. 1. Generally, the ESS theory is well validated, despite a minor divergence of Eq. (5) in a large scale. Concerning the latter, this DNS validation demonstrates the possibility for the closure of fourth-order longitudinal and cross structure functions with the fourth-order longitudinal structure function, which may be regarded as a potential closure method in future researches. Moreover, the identification of c_p is important, because it is shown in Ref. [7] that c_p is responsible for scaling information. This parameter is applied in the model validation.

The next step is to model the viscous term $2\nu(C - Z_{111})$ in Eq. (2). In our previous work, an incorrect Taylor approximation was made, which leads to the inappropriate performance of the model on a large scale. Our strategy is to model the term Z_{111} with the components of C, based on DNS observations in Fig. 2(a). In Ref. [13], Hill and Boratav stated that $Z_{111}(r) = C(r)$ for $r \to 0$ in the dissipation scale with an analysis of convergence properties of Eq. (2). Numerical computation of Z_{111} was not performed due to the limit of the calculation. This relation of the dissipation scale property is now numerically supported by our observation in Fig. 2(a). Furthermore, from DNS in Ref. [13], the term $2\nu C$ is sufficiently negligible in the inertial range compared with the other terms, while it is observed in Fig. 2(a) that $|Z_{111}(r) - C(r)| < 2|C(r)|$ for any r. Thus our modeling objective can be simplified to modeling precisely Z_{111} in the small scale, on maintaining $Z_{111}(r) = C(r)$ for $r \to 0$.

We start with taking Eq. (1) into the definition of C in Eq. (3), and we obtain

$$C = \nu\Big(-\frac{24}{r^2} D'_{11} + \frac{24}{r} D''_{11} + 6 D'''_{11} \Big), \tag{6}$$

where the dissipation rate ϵ is eliminated, and the

third-order structure function is replaced by the second-order structure function. It is rational in this case that Z_{111} is modeled by the components of C, because they contain simply the second-order structure function. The model proposed is

$$Z_{111} = \left(1 + z_1\left(\frac{r}{\eta}\right)^m\right)(C_1 + C_2) + \left(1 + z_2\left(\frac{r}{\eta}\right)^m\right)C_3. \tag{7}$$

where z_1, z_2 and m are the parameters to be identified. The relation $Z_{111}(r) = C(r)$ for $r \to 0$ can be satisfied with this model. The reason why we choose the same converging coefficient for C_1 and C_2 is that the dissipation rate ϵ, which is impossible to be modeled with the second-order structure function, cannot be eliminated without this assumption. It is also necessary to use another converging coefficient for the term C_3, because C in the inertial range is dominated with C_3, leading to an oscillation behavior. This oscillation could be amplified and lead to the oscillation in the inertial range, which was observed in our previous work. The parameters in the model are identified for the purpose of performing the behavior of Z_{111} in a small range in Fig. 2(b) and maintaining the order of magnitude in other ranges. The parameter z_2 identified is much less than z_1, conforming to our discussions on oscillation in the inertial range. The model for Z_{111} is well consistent with DNS in the dissipation range, while no divergence is found in other ranges. Thus we adapt Eq. (7) for modeling of Z_{111} and Eq. (6) for order reduction of C into Eq. (2).

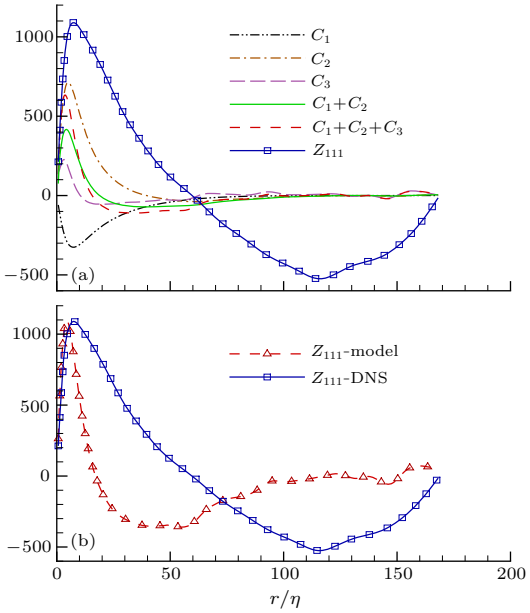

Fig. 2. Modeling of Z_{111} with DNS observations: (a) DNS observation of components of C and Z_{111}; (b) Z_{111} with DNS and model for Z_{111}. Here $z_1 = 0.5$, $z_2 = 0.1$ and $m = 0.5$.

The last challenging task turns out to be modeling the velocity-pressure correlation term $-T_{111}$ on the right-hand side of Eq. (2), which cannot be simply neglected. A proportional relation was proposed with observations from Figs. 1–2 of Ref. [13] that $-T_{111}$ is approximately proportional to both $D'_{1111} + \frac{2}{r}D_{1111}$ and $\frac{6}{r}D_{1122}$ at all scales,

$$\begin{aligned} -T_{111} &= \alpha\left(D'_{1111} + \frac{2}{r}D_{1111}\right) \\ &= \alpha\left(pc_pD_{11}^{p-1}D'_{11} + \frac{2c_p}{r}D_{11}^p\right), \end{aligned} \tag{8}$$

with α being a model constant. The assumption from the ESS theory in Eq. (5) and the isotropic condition are applied in this relation. We proved in Ref. [7] that from the Kolmogorov 2/3 scaling, the values of α can be analytically obtained, which yields $\alpha = 0.1718$. Note that the intermittency at small scales[18,19] can be considered here which leads to different values of α. This would change the asymptotic scaling of the present model in the inertial range, but the results are not qualitatively changed.

To this state, we have discussed the closure of the fourth order structure function and the modeling strategy for the viscous term and the velocity-pressure correlation term. Comparisons among our new closure model, the Batchelor model[20] and DNS data for the scaling exponent of the second-order structure function are shown in Fig. 3. In the new closure model, we apply the α formula in Eq. (8) to model the velocity-pressure correlation term. We adapt the theoretical result that $p = 1.83$ and $\alpha = 0.1718$. Other parameters are identified from DNS, including $c_p = 5.5$ from Fig. 1(b), with $z_1 = 0.5$, $z_2 = 0.1$ and $m = 0.5$ from Fig. 2(b). The initial condition of the new closure model is set to guarantee the scaling exponent $n = 2$ in the dissipation range.

Generally, the new closure model gives an appropriate description of the multiscale behavior of isotropic turbulence. The oscillations observed in a previous work are thoroughly eliminated, thanks to the new model of the viscous term. The information of scale is involved into the new model and well matches the numerical experiment with the help of the parameter c_p identified with DNS. The model shows a better prediction of the appearance of the inertial range than the Batchelor model while a worse estimation of the length of the dissipation range. We guess that the latter is because of a wrong approximation of certain terms in the dissipation range.

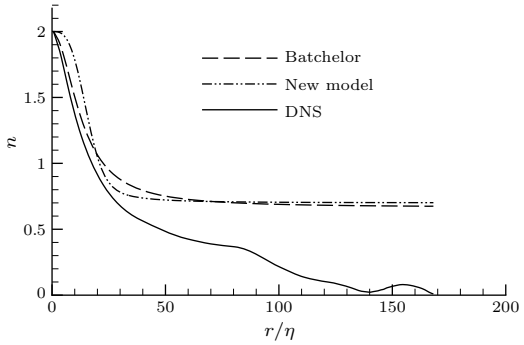

Fig. 3. Scaling exponent n calculated with the Batchelor model, the new closure model in this work and DNS data respectively.

In the present study, the approximations and assumptions through the closure process are verified by DNS. The ESS theory applied in model closure is completely validated. We identify from DNS that the inappropriate oscillation observed in our previous work is due to an incorrect Talyor expansion on the viscous term of the next-order equation, and a parameterized model for the viscous term is proposed based on DNS data. The scaling information is then included into the model by identification of the proportional constant in ESS theory with DNS data, which makes it possible for a comparison of our new model with the classical Batchelor model and numerical experiment. It is shown that the model is considerably improved to perform the correct multiscale behaviors of isotropic turbulence.

Although some artificial factors are involved, the whole modeling process is rational in the sense of physics. Thus the validation of this closure method demonstrates theoretically the application of ESS theory in physical space, while DNS proves directly the correctness of the two key assumptions $D_{1111} = c_p D_{11}^p$ and $D_{1122} = \frac{4}{9} D_{1111} = \frac{4}{9} c_p D_{11}^p$. Particularly, the latter, which associates D_{1122} with D_{11}, has not been exclusively studied and applied for the purpose of equation closure. With the theoretical and numerical evidence on these closure assumptions, this work can inspire the development of new theoretical tools for research on the nonlinear multiscale mechanism of turbulence.

Another interesting topic in the future would be the closure on more complex turbulent flows with external energy inputs, such as the von Kármán swirling flow, the bulk flow in thermally driven turbulence, and the turbulence under mean shear, as it is known that these energy inputs at large scales will change the scaling behaviors at small scales. Indeed, this may be inspired by a previous study, in which we have introduced a 10/9 scaling for the cross structure function under homogeneous mean shear via a closure, which simply assumes constant velocity gradient skewness.[21] We therefore expect that the present approach will also be extended by considering the energy source term to provide theoretical support to the scalings of complex turbulent flows in the future.

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