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Effect of Hyper-Resistivity on Nonlinear Tearing Modes *

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We analytically investigate nonlinear tearing modes with the anomalous electron viscosity or, as it is normally called, hyper-resistivity. In contrast to the flux average method used by previous work, we employ the standard singular perturbation technique and a quasilinear method to obtain the time evolution equation of tearing modes. The result that the magnetic flux grows with time in a scaling as $t^{2/3}$ demonstrates that nonlinear tearing modes with the hyper-resistivity effect alone have a weaker dependence on time than that of the corresponding resistive case.

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Resistive magnetohydrodynamics instability can be observed in a huge range of areas, from astrophysical to laboratorial plasma. As one of the most important phenomena in plasma physics, the tearing mode instability has the potential to cause a major disruption or to decrease plasma confinement in magnetic fusion devices, thus it is necessary to research tearing modes.

The previous work of Furth *et al.*[1] has given a glimpse of various resistive instabilities. After that, the development of resistive tearing modes has been investigated including the linear phase, the nonlinear phase and the saturation phase.[2–4] However, in contrast to sufficient studies on the tearing modes caused by plasma resistivity, researches about the tearing modes with the anomalous electron viscosity, or what is preferentially called hyper-resistivity, are rare.

The physical mechanisms that lead to hyper-resistivity can be explained in two ways. One is that hyper-resistivity is produced by tearing mode turbulence.[5] Hyper-resistivity describes the process in which current gradients cause tearing modes and then they are flattened as tearing modes grow. The other is concerned with braided field lines.[6,7] Perpendicular electron viscosity is enhanced because of the perpendicular transport of parallel momentum along wandering field lines. As a result, this effect appears in the form of hyper-resistivity in the parallel Ohm law.

In analytical researches, Furth *et al.*[2] examined the influence of hyper-resistivity on tearing modes for the first time. Then obtained by the flux average method, the hyper-resistivity effect on nonlinear tearing modes was investigated by Kaw *et al.*[6] with the conclusion that the width of magnetic island grows as $t^{1/3}$. On the other hand, nonlinear simulations of peeling–ballooning (P–B) modes with hyper-resistivity by Xu *et al.*[8] demonstrate that the P–B modes trigger magnetic reconnection and the pedestal collapse is limited to the edge region. In astrophysical plasma, Huang *et al.*[9] and Vekstein[10] studied the hyper-resistivity effect on magnetic reconnection in linear and nonlinear phases.

The configuration is studied in a cylindrical tokamak of an incompressible plasma with uniform resistivity. With hyper-resistivity, Ohm's law should take the term $-m\mu_{\rm e}/ne^2\nabla^2\boldsymbol{J}$ into account, where $\mu_{\rm e}$ is the anomalous electron viscosity coefficient and is approximately equal to the coefficient of anomalous perpendicular electron thermal conductivity. According to the orderings, the inertia term and the Hall term should be neglected and only dominated terms should be reserved.[2,6,11] Thus the set of magnetohydrodynamic (MHD) equations with hyper-resistivity in rational cgs units with $c = 1$ are

$$\rho\frac{d\boldsymbol{v}}{dt} = \nabla p + \boldsymbol{J}\times\boldsymbol{B}, \tag{1}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E}, \tag{2}$$

$$\boldsymbol{J} = \nabla\times\boldsymbol{B}, \tag{3}$$

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J} - \frac{m\mu_{\rm e}}{ne^2}\nabla^2\boldsymbol{J}, \tag{4}$$

where the mass density ρ and the resistivity η are assumed to be constants, d/dt is the convective time derivative, p, $\boldsymbol{J}$, $\boldsymbol{B}$ and $\boldsymbol{E}$ are the pressure, current density, magnetic and electric field, respectively.

The poloidal flux function $\psi(r,\theta,z,t)$ and the stream function $\phi(r,\theta,z,t)$ are used to express the vector fields

$$B = \nabla z\times\nabla\psi + B_z\hat{z}, \tag{5}$$

$$V_\perp = (1/B_z)\nabla\phi\times\hat{z}, \tag{6}$$

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where B_z is the Z-axis component. In the large aspect ratio ($\varepsilon = a_0/R_0 \ll 1$, where a_0 and R_0 are the minor radius and the major radius, respectively) and low-beta limits, Eqs. (1)–(4) can be reduced to the dimensionless scalar form

$$\begin{aligned}&\frac{\partial \psi}{\partial t} + \frac{1}{r}\Big(\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial \phi}{\partial r}\Big)\\ &= \eta J_z - \eta_{\rm H}\nabla_\perp^2 J_z - \frac{\partial \phi}{\partial z}, \qquad (7)\\ &\frac{\partial U}{\partial t} + \frac{1}{r}\Big(\frac{\partial U}{\partial r}\frac{\partial \phi}{\partial \theta} - \frac{\partial U}{\partial \theta}\frac{\partial \phi}{\partial r}\Big)\\ &= -S^2\Big[\frac{1}{r}\Big(\frac{\partial J_z}{\partial \theta}\frac{\partial \psi}{\partial r} - \frac{\partial J_z}{\partial r}\frac{\partial \psi}{\partial \theta}\Big) + \frac{\partial J_z}{\partial z}\Big], \qquad (8)\end{aligned}$$

where r, θ and z refer to the radial, poloidal, and Z-axis coordinates, $J_z = \nabla_\perp^2\psi$ and $U = \nabla_\perp^2\phi$ are the Z-axis current density and vorticity, respectively. The Laplacian operator is defined as $\nabla_\perp^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$. The dimensionless parameter Lundquist number $S = \tau_{\rm R}/\tau_{\rm A}$ is the ratio of the resistive time $\tau_{\rm R} = \mu_0 a_0^2/\eta$ to the Alfven time $\tau_{\rm A} = (\mu_0\rho)^{1/2} a R_0/B_\varphi$, and $\eta_{\rm H} = m\mu_{\rm e}/ne^2$ is defined as the hyper-resistivity coefficient.[5,12] In this work, we employ the basic model in Eqs. (7) and (8) as the starting point for the present model.

Since the equilibrium flux will be quasilinearly modified as the flux perturbation $\tilde{\psi}$ increases,[12] the total flux function is assumed as

$$\psi(r,\theta,z,t) = \psi_0(r) + \delta\psi(r,t) + \tilde{\psi}(r,\theta,z,t), \qquad (9)$$

where $\psi_0(r)$ is the equilibrium flux, $\tilde{\psi}(r,\theta,z,t)$ is the linear perturbation, and $\delta\psi(r,t)$ is the quasilinearly modified flux function due to the nonlinear ExB convection of the perturbed flux in Ohm's law. The equilibrium flow is not considered in this study, thus the stream function can be assumed as

$$\phi(r,\theta,z,t) = \tilde{\phi}(r,\theta,z,t). \qquad (10)$$

For the flux and stream functions, the perturbation functions are assumed as

$$\tilde{\psi}(r,\theta,z,t) = \sum_{l=1}^{\infty}\psi_l(r,t)\cos l(m\theta - nz), \qquad (11)$$

$$\tilde{\phi}(r,\theta,z,t) = \sum_{l=1}^{\infty}\phi_l(r,t)\sin l(m\theta - nz), \qquad (12)$$

where m and n denote the poloidal and toroidal mode numbers, respectively.

The diffusion equation for the quasilinearly modified flux function $\delta\psi(r,t)$, obtained by substituting the expressions of ψ and ϕ into Eq. (7) and taking the poloidal average operation $(m/2\pi)\int_0^{2\pi/m} f(r,\theta,z,t)d\theta$ over it, is

$$\frac{\partial \delta\psi}{\partial t} + \frac{m}{2r}\sum_{l=1}^{\infty}\frac{\partial(\psi_l\phi_l)}{\partial r} = \eta\delta J_z - \eta_{\rm H}\nabla_\perp^2\delta J_z, \qquad (13)$$

where the equilibrium current density J_{z0} is ignored. Then, subtracting Eq. (13) from Eq. (7), the induction equation for the lth harmonic of the flux perturbation is

$$\begin{aligned}&\frac{\partial \psi_l}{\partial t} + \frac{lm}{r}\Big(\frac{d\psi_0}{dr} + \frac{\partial \delta\psi}{\partial r}\Big)\phi_l\\ &= \eta J_{zl} - \eta_{\rm H}\nabla_\perp^2 J_{zl} + ln\phi_l. \qquad (14)\end{aligned}$$

Similarly, the vorticity equation for the lth harmonic of the stream function is

$$\begin{aligned}\frac{\partial U_l}{\partial t} = &-S^2\Big\{\frac{lm}{r}\Big[\Big(\frac{dJ_{z0}}{dr} + \frac{\partial \delta J_z}{\partial r}\Big)\psi_l\\ &-\Big(\frac{d\psi_0}{dr} + \frac{\partial \delta\psi}{\partial r}\Big)J_{zl}\Big] + lnJ_{zl}\Big\}, \qquad (15)\end{aligned}$$

where the lth harmonic of the perturbed toroidal current density and the vorticity are $J_{zl} = \nabla_\perp^2\psi_l$ and $U_l = \nabla_\perp^2\phi_l$, respectively, and the quasilinearly modified current density is $\delta J_z = \nabla_\perp^2\delta\psi$.

This work only concentrates on the fundamental mode, as Rutherford's method has proved that the fundamental is linearly unstable while all other harmonics are relatively strongly stable, thus the higher harmonics are all ignored.[3] In the following based on Eqs. (13)–(15), the hyper-resistivity effect on nonlinear tearing modes will be discussed (in reduced MHD) by employing the boundary-layer theory.

Following the standard singular perturbation technique for resistive instabilities, the boundary-layer theory will be used and the outer and inner regions are separated.

The outer region is considered firstly. The two kinds of resistivity and the inertia should be neglected in the absence of singularity. Equations (13) and (14) are thus omitted and Eq. (15) is reduced to

$$\frac{m}{r}\frac{dJ_{z0}}{dr}\Psi_1 + \Big(n - \frac{m}{q_0}\Big)J_{z1} = 0, \qquad (16)$$

where $q_0 = r[d\psi_0(r)/dr]^{-1}$ is the safety factor. The quasilinear terms should also be neglected here because the perturbation terms are much smaller than the equilibrium terms in the outer region. Using variables separation, the solution of this equation is accomplishable, as the form of $\psi_1(r,t) = \psi_1(r_{\rm s},t)Z(r)$.

Define Δ and Δ' as the discontinuities of ψ_1 in inner and outer regions, respectively, which represent the jump in the logarithmic derivative of the magnetic field perturbation across the singular layer.[1] It is easy to find that Δ' is a function of r, independent of time in the outer region. For example, a new expression of instability criterion for tearing modes is derived for arbitrary magnetic shear configuration in the low beta and large aspect ratio limits.[12]

Next, the inner region is investigated. In the linear phase, the width of magnetic island and the linear mode width are approximately equal, that is, $w \sim \delta_0$. The linear and nonlinear terms in Eqs. (13)–(15) have the same order based on the linear tearing mode order scalings $\gamma_0 \sim \eta^{3/5}$ and $\delta_0 \sim \eta^{2/5}$. In the nonlinear phase, however, the island width exceeds the

linear resistive layer thickness. The nonlinear $\boldsymbol{J}\times B$ terms play an increasingly significant role and they slow down the growth of the island because of their stabilizing effect. As a result, the linear tearing mode scaling is no longer valid. Here ψ_1 and ϕ_1 are assumed to be of order of δ^2 in the inner region of the singular layer at $r=r_{\rm s}$.[12]

To retain the dominant terms, Eqs. (13)–(15) are simplified in the vicinity of the singular surface

$$\frac{\partial\psi_1}{\partial t}=\eta\frac{\partial^2\psi_1}{\partial x^2}-\eta_{\rm H}\frac{\partial^4\psi_1}{\partial x^4}+\mu_0 x\phi_1, \tag{17}$$

$$\frac{\partial\delta\psi}{\partial t}-\eta\frac{\partial^2\delta\psi}{\partial x^2}+\eta_{\rm H}\frac{\partial^4\delta\psi}{\partial x^4}=-\frac{m}{2rs}\frac{\partial(\psi_1\phi_1)}{\partial x}, \tag{18}$$

$$\frac{\partial^3\phi_1}{\partial x^2\partial t}=-S^2\Big[\frac{m}{r_{\rm s}}\frac{\partial^3\delta\psi}{\partial x^3}\psi_1+\mu_0 x\frac{\partial^2\psi_1}{\partial x^2}\Big], \tag{19}$$

where $x=r-r_{\rm s}$, and $\mu_0=[m/q_0^2(r_{\rm s})][q_0(r_{\rm s})/dr]$.

For simplicity, $\delta\psi$ is assumed to be independent of time since it does not affect the main nonlinear behavior and the inertial terms in Eq. (19) will not be considered at this moment.[3,12] With the familiar constant-ψ assumption, $\psi_1(x,t)$ in the first term of Eq. (17) should be approximated to $\psi_1(0,t)$ because the spatial partial derivations of ψ_1 is $O(\delta)$ larger than itself.[1] The inner region is so thin that the derivatives of flux and stream function are very sensitive to themselves. Thus, in the boundary layer theory, it is a basic technique to enlarge the coordinate to research the fine structure of the singular layer. Introducing the transforms $x=\delta(\tau)X$ and $t=\tau$, the partial derivatives are $\partial/\partial x=\partial/\delta(\tau)\partial X$, where $\delta(\tau)$ is the expression for the mode width but it would increase, which is different from the linear phase. Hence, Eqs. (13)–(15) become

$$\begin{aligned}&\frac{\partial\psi_1(0,\tau)}{\partial\tau}=\eta\frac{\partial^2\psi_1}{\delta^2(\tau)\partial X^2}-\eta_{\rm H}\frac{\partial^4\psi_1}{\delta^4(\tau)\partial X^4}\\&+\mu_0\delta(\tau)X\phi_1,\end{aligned} \tag{20}$$

$$\begin{aligned}&-\eta\frac{\partial^2\delta\psi}{\delta^2(\tau)\partial X^2}+\eta_{\rm H}\frac{\partial^4\delta\psi}{\delta^4(\tau)\partial X^4}\\&=-\frac{m\psi_1(0,\tau)}{2rs}\frac{\partial\phi_1}{\delta(\tau)\partial X},\end{aligned} \tag{21}$$

$$\frac{m}{r_{\rm s}}\frac{\partial^3\delta\psi}{\delta^3(\tau)\partial X^3}\psi_1(0,\tau)+\mu_0\delta(\tau)X\frac{\partial^2\psi_1}{\delta^2(\tau)\partial X^2}=0. \tag{22}$$

Because of the transform $x=\delta(\tau)X$, the range of inner variable X has been extended to $(-\infty,+\infty)$, which forms the basis of the following Fourier transforms

$$\begin{Bmatrix}\psi_1(k)\\ \phi_1(k)\end{Bmatrix}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\begin{Bmatrix}\psi_1(X)\\ \phi_1(X)\end{Bmatrix}e^{-ikX}dX. \tag{23}$$

Then the transformed equations are

$$\begin{aligned}&\frac{\partial\psi_1(0,\tau)}{\partial\tau}\delta(k)=-\frac{\eta}{\delta^2(\tau)}k^2\psi 1-\frac{\eta_{\rm H}}{\delta^4(\tau)}k^4\psi_1\\&+i\mu_0\delta(\tau)\frac{\partial\phi_1}{\partial k},\end{aligned} \tag{24}$$

$$-\frac{\eta}{\delta^2(\tau)}k^2\delta\psi-\frac{\eta_{\rm H}}{\delta^4(\tau)}k^4\delta\psi=i\frac{m\psi_1(0,\tau)}{2rs\delta(\tau)}k\phi_1, \tag{25}$$

$$\frac{imk^3}{r_{\rm s}\delta^3(\tau)}\delta\psi\psi_1(0,\tau)+\frac{i\mu_0}{\delta(\tau)}\frac{d(k^2\psi 1)}{dk}=0. \tag{26}$$

The expressions of ψ_1 and $\delta\psi$ can be derived from Eqs. (24) and (25) and substituted into Eq. (26). With the transform $\phi_1(k,\tau)=\frac{1}{\mu_0\delta}\frac{\partial\psi_1(0,\tau)}{\partial\tau}H(k,\tau)$, one can obtain

$$\begin{aligned}\frac{\partial}{\partial k}\Big(\frac{1}{\frac{\eta_{\rm H}}{\eta\delta^2}k^2+1}\frac{\partial H}{\partial k}\Big)&=\frac{m^2\psi_1(0,\tau)}{2rs^2\mu_0^2\delta^4(\frac{\eta_{\rm H}}{\eta\delta^2}k^2+1)}k^2H\\&-i\frac{\partial}{\partial k}\Big(\frac{\delta(k)}{\frac{\eta_{\rm H}}{\eta\delta^2}k^2+1}\Big).\end{aligned} \tag{27}$$

When $k\neq 0$, Eq. (27) becomes a homogeneous equation

$$\frac{\partial}{\partial k}\Big(\frac{1}{\frac{\eta_{\rm H}}{\eta\delta^2}k^2+1}\frac{\partial H}{\partial k}\Big)=\frac{1}{\psi_1(0,\tau)(\frac{\eta_{\rm H}}{\eta\delta^2}k^2+1)}k^2H. \tag{28}$$

For convenience, we set $m^2\psi_1^2(0,\tau)/2rs^2\mu_0^2\delta^4(\tau)=1$ here.

In the inner region, the discontinuity is deified as

$$\Delta=\frac{\psi_1'(+\infty)-\psi_1'(-\infty)}{\psi_1}=\frac{1}{\psi_1(0,t)}\int_{\infty-}^{\infty+}\frac{\partial^2\psi_1}{\partial x^2}dx. \tag{29}$$

Then the asymptotic matching between the solutions of outer and inner regions can be obtained as[1,12,13]

$$\begin{aligned}\Delta'&=\frac{2\pi}{\delta(\tau)\psi_1(0,\tau)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-k^2\psi_1(k,\tau)e^{ikX})dkdX\\&=-\frac{\pi\delta(\tau)}{\psi_1(0,\tau)\eta}\frac{\partial\psi_1(0,\tau)}{\partial\tau}\Big[\frac{1}{1+\frac{\eta_{\rm H}}{\eta\delta^2}k^2}\frac{\partial H}{\partial k}\Big]_{k=0}\frac{1}{H(0^+)}.\end{aligned} \tag{30}$$

This equation can describe the general evolution of nonlinear tearing modes with both plasma resistivity and hyper-resistivity effect, especially the transition between these two situations. Though it is difficult to obtain its analytical solution directly, some discussions about limiting cases in different parameters will be presented in the following.

There are two limiting cases of interests. The first one is $\eta_{\rm H}k^2/\eta\delta^2\ll 1$, which means that the hyper-resistivity effect is so small that the plasma resistivity still dominates in the inner region. Thus, it is expected that the hyper-resistivity effect would not change the time evolution equation for the nonlinear tearing mode. Equation (28) is reduced to

$$\frac{\partial}{\partial k}\Big(\frac{\partial H}{\partial k}\Big)=\frac{1}{\psi_1(0,\tau)}k^2H. \tag{31}$$

We can easily obtain its solution

$$H(k,\tau) = D_{-\frac{1}{2}}[\sqrt{2}\psi_1^{-1/4}k], \tag{32}$$

where D is the parabolic cylinder function. This solution satisfies the boundary condition $H(k,\tau) \to 0$ as $k \to \infty$. With the asymptotic matching, for this limiting case one can obtain the time evolution equation

$$\frac{\partial \psi_1}{\partial t} = \frac{2^{3/4}\Gamma[1/4]}{\pi\Gamma[3/4]}\eta\Delta'\sqrt{\frac{r_s\mu_0}{m}}\psi_1^{1/2}. \tag{33}$$

It is obvious that ψ_1 scales as t^2, which recovers the previous result.[3,12] In this case where the hyper-resistivity effect is very small, the time evolution equation remains the same as that without hyper-resistivity.

The second limiting case is $\eta_{\rm H}k^2/\eta\delta^2 \gg 1$, which implies that the hyper-resistivity effect is the dominated term in the inner region. Then Eq. (28) is reduced to

$$\frac{\partial}{\partial k}\Big(\frac{1}{k^2}\frac{\partial H}{\partial k}\Big) = \frac{1}{\psi_1(0,\tau)}H. \tag{34}$$

Similarly, we can obtain the time evolution equation

$$\frac{\partial \psi_1}{\partial t} = \frac{2^{-1/4}\Gamma[3/4]\eta_{\rm H}\Delta'}{\pi\Gamma[5/4]}\Big(\frac{m}{\mu_0 rs}\Big)^{-3/2}\psi_1^{-1/2}. \tag{35}$$

This result that ψ_1 scales as $t^{2/3}$ demonstrates that the limiting case with the hyper-resistivity effect alone has a weaker dependence on time than that of the corresponding resistive case.

It is very difficult to solve Eq. (28) to obtain the whole time evolution equation at the moment. A direct approach without Fourier transforms is developing and the result will be hopefully obtained in the near future.

By introducing hyper-resistivity into Ohm's law, its effects on nonlinear tearing modes are investigated in a cylindrical geometry. The time evolution equation in the reduced MHD model is derived by the boundary-layer approach, and is analyzed in the constant-ψ approximation and in two limiting cases, respectively.

The present model reproduces the result of the classical nonlinear theory,[3] which can be recovered exactly when the hyper-resistivity term is ignorable. On the other hand, when the hyper-resistivity effect overpowers plasma resistive, tearing modes grow much slower than the corresponding resistive tearing modes, where the magnetic flux scales as $t^{2/3}$ in the nonlinear hyper-resistivity case in contrast to t^2 in the nonlinear resistive case.

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