Soliton, Breather and Rogue Wave Solutions for the Nonlinear Schrödinger Equation Coupled to a Multiple Self-Induced Transparency System *

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We derive an N-fold Darboux transformation for the nonlinear Schrödinger equation coupled to a multiple self-induced transparency system, which is applicable to optical fiber communications in the erbium-doped medium. The N-soliton, N-breather and Nth-order rogue wave solutions in the compact determinant representations are derived using the Darboux transformation and limit technique. Dynamics of such solutions from the first- to second-order ones are shown.

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In nonlinear optics, waves propagating in an erbium-doped fiber are described by the nonlinear Schrödinger (NLS) equation coupled to the self-induced transparency (SIT) system,[1] namely,

$$iE_z = E_{tt} + 2|E|^2E + \langle p\rangle, \tag{1a}$$
$$p_t = i(2NE - \omega p),\ N_t = -i(Ep^* - E^*p), \tag{1b}$$

where $E(z,t)$ is the complex propagating envelope of the electric field, $\langle p\rangle = p(z,t)$ denotes the averaged polarization, and $N = N(z,t)$ represents the population inversion, which are given by $p = \nu_1\nu_2^*$ and $N = |\nu_2|^2 - |\nu_1|^2$ with ν_1 and ν_2 being the normalized wave functions of the two ground and excited states. In recent years, there has been a surge of interest in the complete integrability, explicit analytical solutions and higher-order dispersion generalization for system (1).[2]

Remarkably, as pointed out by Kundu that, considering an important enhancement of the amplification and control of optical solitons, the single SIT system should be generalized to the multiple case, namely, the coupled NLS-multiple SIT system[3]

$$iE_z = E_{tt} + 2|E|^2E + \langle p\rangle, \tag{2a}$$
$$p_t = i(2NE - \omega p + e),\ N_t = -i(Ep^* - E^*p), \tag{2b}$$
$$e_t = i(2ME - \omega e),\ M_t = -i(Ee^* - E^*e), \tag{2c}$$

where the induced polarization e and the population inversion M are two new introduced variables. The integrability and the existence of soliton solutions for system (2) have been reported in Ref. [3].

As is well known, compared to the solitons and breathers, rogue waves (or freak waves) are the waves that have amplitudes usually three times larger than the average crest and are localized in both space and time.[4–11] Up to now, to the best of our knowledge, the general soliton, breather and rogue wave solutions with unified expressions for system (2) have not been studied. In this study, we derive N-soliton, N-breather and Nth-order rogue wave solutions for system (2) using the Darboux transformation and limit approach.[12–16] The explicit one-soliton, one-breather and the first-order rogue wave solutions are presented, and dynamics of the solitons, breather and rogue waves up to the second order are shown.

To begin with, we present the Lax pair of system (2) utilizing the AKNS scheme,[3,17–19] which admits

$$\Phi_t = U\Phi = (\zeta J + U_0)\Phi, \tag{3}$$
$$\Phi_z = V\Phi = \Big(\zeta^2V_0 + \zeta V_1 + V_2 + \frac{1}{4i(\zeta - \omega/2)}V_3 + \frac{1}{4(\zeta - \omega/2)^2}V_4\Big)\Phi, \tag{4}$$

where

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},\ U_0 = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix},$$
$$V_0 = -2J,\ V_1 = -2U_0,\ V_2 = i\begin{pmatrix} -|E|^2 & -E_t \\ -E_t^* & |E|^2 \end{pmatrix},$$
$$V_3 = \begin{pmatrix} -2N & -2ip \\ 2ip^* & 2N \end{pmatrix},\ V_4 = \begin{pmatrix} -iM & e \\ -e^* & iM \end{pmatrix}.$$

Next, using the standard DT of the AKNS spectral problem,[12] we know that if $\Phi_j = (\psi_j, \varphi_j)^T$ $(j = 1, 2, \ldots, N)$ are N basic solutions of the Lax pair Eqs. (3) and (4) at $E = E[0]$ and $\zeta = \zeta_j$, then the N-fold DT for system (2) can be given in the form

$$\Phi[N] = T[N]\Phi,\ T[N] = I - Y(SD)^{-1}Y^\dagger, \tag{5}$$
$$E[N] = E[0] + 2i\frac{\det(D_1)}{\det(D)}, \tag{6}$$

where $Y = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, $S = \zeta I -$

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$\mathrm{diag}(\zeta_1^*, \zeta_2^*, \ldots, \zeta_N^*)$,

$$D = \begin{pmatrix} \frac{\Phi_1^\dagger \Phi_1}{\zeta_1 - \zeta_1^*} & \frac{\Phi_1^\dagger \Phi_2}{\zeta_2 - \zeta_1^*} & \cdots & \frac{\Phi_1^\dagger \Phi_N}{\zeta_N - \zeta_1^*} \\ \frac{\Phi_2^\dagger \Phi_1}{\zeta_1 - \zeta_2^*} & \frac{\Phi_2^\dagger \Phi_2}{\zeta_2 - \zeta_2^*} & \cdots & \frac{\Phi_2^\dagger \Phi_N}{\zeta_N - \zeta_2^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Phi_N^\dagger \Phi_1}{\zeta_1 - \zeta_N^*} & \frac{\Phi_N^\dagger \Phi_2}{\zeta_2 - \zeta_N^*} & \cdots & \frac{\Phi_N^\dagger \Phi_N}{\zeta_N - \zeta_N^*} \end{pmatrix},$$

$$D_1 = \begin{pmatrix} \frac{\Phi_1^\dagger \Phi_1}{\zeta_1 - \zeta_1^*} & \frac{\Phi_1^\dagger \Phi_2}{\zeta_2 - \zeta_1^*} & \cdots & \frac{\Phi_1^\dagger \Phi_N}{\zeta_N - \zeta_1^*} & \varphi_1^* \\ \frac{\Phi_2^\dagger \Phi_1}{\zeta_1 - \zeta_2^*} & \frac{\Phi_2^\dagger \Phi_2}{\zeta_2 - \zeta_2^*} & \cdots & \frac{\Phi_2^\dagger \Phi_N}{\zeta_N - \zeta_2^*} & \varphi_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\Phi_N^\dagger \Phi_1}{\zeta_1 - \zeta_N^*} & \frac{\Phi_N^\dagger \Phi_2}{\zeta_2 - \zeta_N^*} & \cdots & \frac{\Phi_N^\dagger \Phi_N}{\zeta_N - \zeta_N^*} & \varphi_N^* \\ \psi_1 & \psi_2 & \ldots & \psi_N & 0 \end{pmatrix}.$$

Here I is the identity matrix, and $\dagger$ denotes the conjugate transpose. We note that, in this study, we focus on the characteristics of the nonlinear wave solutions of the electric field, namely, the E component. For the components of p, N, e, and M, we omit the results, since the nonlinear waves in them are similar to that in the E component under the same initial physical parameters.

To obtain the soliton solution for system (2), we choose the initial solution as $E[0] = 0$, $p[0] = 0$, $e[0] = 0$, $M[0] = 0$ and $N[0] = d$, then the basic solution of the Lax pair of Eqs. (3) and (4) can be calculated by

$$\Phi_j = \begin{pmatrix} e^{-A_j} \\ e^{A_j} \end{pmatrix}, \quad A_j = i\zeta_j t + i\frac{4\zeta_j^3 - 2\zeta_j^2\omega + d}{-2\zeta_j + \omega} z, \quad j = 1, 2, \ldots, N, \tag{7}$$

where d is an arbitrary real constant.

Substituting Eq. (7) into Eq. (6), we can readily derive the N-soliton solution

$$E[N] = 2i\frac{\det(D_1)}{\det(D)}, \tag{8}$$

where

$$D = \left(\frac{e^{A_j^* + A_l} + e^{-A_j^* - A_l}}{\zeta_l - \zeta_j^*}\right)_{1 \le j,l \le N},$$
$$D_1 = \begin{pmatrix} D & Y_2^\dagger \\ Y_1 & 0 \end{pmatrix},$$
$$Y_1 = (e^{-A_1}, e^{-A_2}, \ldots, e^{-A_N}),$$
$$Y_2 = (e^{A_1}, e^{A_2}, \ldots, e^{A_N}).$$

Particularly, one can explicitly obtain the one-soliton solution

$$E[1] = 2\eta_1 \mathrm{sech}(2\alpha_1) e^{-2i\beta_1}, \tag{9}$$

where

$$\alpha_1 = -\eta_1 t + \frac{2\eta_1(8\xi_1^3 - 8\xi_1^2\omega + 8\xi_1\eta_1^2 + 2\xi_1\omega^2 - d)}{(2\xi_1 - \omega)^2 + 4\eta_1^2} z,$$
$$\beta_1 = \xi_1 t + \frac{1}{(2\xi_1 - \omega)^2 + 4\eta_1^2}[-8\xi_1^4 + 8\xi_1^3\omega - 8\xi_1\eta_1^2\omega - 2\xi_1^2\omega^2 + 2\eta_1^2\omega^2 - 2\xi_1 d + d\omega + 8\eta_1^4] z,$$

here we set $\zeta_1 = \xi_1 + i\eta_1$. The maximum of the one-soliton solution is $|2\eta_1|$, and the trace of the soliton is given by

$$t = \frac{2(8\xi_1^3 - 8\xi_1^2\omega + 8\xi_1\eta_1^2 + 2\xi_1\omega^2 - d)}{(2\xi_1 - \omega)^2 + 4\eta_1^2} z.$$

Figures 1(a) and 1(b) depict the evolution plots of the stationary and non-stationary one-soliton waves described by Eq. (9), respectively. For the two-soliton solution, we omit presenting the more complicated expression of it and just show the interactional process of the two solitons, see Fig. 1(c). For the fixed α_1, we assume $z \to -\infty$, $e^{2\alpha_2} \sim 0$ and $z \to +\infty$, $e^{2\alpha_2} \sim +\infty$, then we have the following asymptotic behaviors[20–22]

$$E[2] \sim 2\eta_1 \frac{(\zeta_1 - \zeta_2)}{(\zeta_1 - \zeta_2^*)} \frac{|\zeta_1 - \zeta_2^*|}{|\zeta_1 - \zeta_2|} \cdot \mathrm{sech}\left(2\alpha_1 + \ln\frac{|\zeta_1 - \zeta_2^*|}{|\zeta_1 - \zeta_2|}\right) e^{-2i\beta_1},$$
$$E[2] \sim 2\eta_1 \frac{(\zeta_1 - \zeta_2)}{(\zeta_1 - \zeta_2^*)} \frac{|\zeta_1 - \zeta_2^*|}{|\zeta_1 - \zeta_2|} \cdot \mathrm{sech}\left(2\alpha_1 + \ln\frac{|\zeta_1 - \zeta_2|}{|\zeta_1 - \zeta_2^*|}\right) e^{-2i\beta_1}.$$

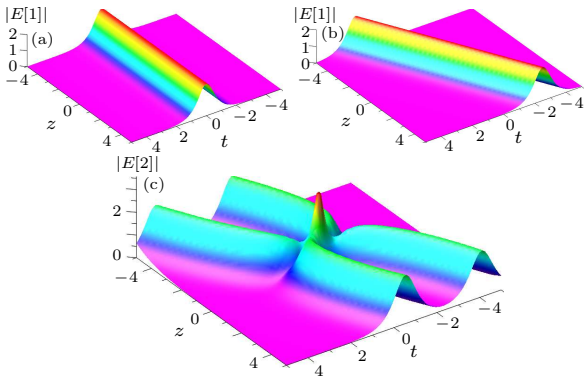

Fig. 1. Evolution plots of (a) stationary and (b) non-stationary one-soliton waves given by Eq. (9) with $\xi_1 = 0$, $\eta_1 = 1$, $d = 0$ and $\xi_1 = 0$, $\eta_1 = 1$, $d = 1$. (c) Evolution plot of the two-soliton wave with $\xi_1 = 0$, $\eta_1 = 1$, $\xi_2 = 0$, $\eta_2 = 1.1$, and $d = 1$. The other parameter is $\omega = 0$.

Next, we choose the following plane-wave solution as the initial solution to generate the breather solution

$$E[0] = ce^{i\theta}, \; p[0] = ce^{i\theta}(a^2 - b - 2c^2), \; \theta = at + bz,$$
$$e[0] = ce^{i\theta}(a^3 + a^2\omega - 2ac^2 - 2c^2\omega - ab - b\omega - 2d),$$
$$M[0] = (a^3 + a^2\omega - 2ac^2 - 2c^2\omega - ab - b\omega - 2d)(a + \omega)/2, \; N[0] = d. \tag{10}$$

By plugging Eq. (10) into Eqs. (3) and (4), we have the basic solution of the Lax pair of equations

$$\Phi_j = \begin{pmatrix} (\mu_j e^{A_j} + \nu_j e^{-A_j}) e^{\frac{i}{2}\theta} \\ (e^{A_j} + e^{-A_j}) e^{-\frac{i}{2}\theta} \end{pmatrix}, \ j = 1, 2, \ldots, N, \quad (11)$$

where

$$\mu_j = \frac{-\sqrt{-(2\zeta_j + a)^2 - 4c^2} + 2i\zeta_j + ia}{2c},$$
$$\nu_j = \frac{\sqrt{-(2\zeta_j + a)^2 - 4c^2} + 2i\zeta_j + ia}{2c},$$
$$A_j = \frac{\sqrt{-(2\zeta_j + a)^2 - 4c^2}}{2}(t + k_j z),$$

with

$$k_j = \frac{1}{(2\zeta_j - \omega)^2}(-8\zeta_j^3 + (4a + 8\omega)\zeta_j^2 + (-2a^2 - 4a\omega + 4c^2 - 2\omega^2 + 2b)\zeta_j + a^3 - 4c^2\omega + 2a^2\omega - 2ac^2 + a\omega^2 - 2d - ab - 2b\omega).$$

Then, we can give the N-breather solution

$$E[N] = \left[1 + 2i\frac{\det(D_1)}{\det(D)}\right] c e^{i\theta}, \quad (12)$$

where $D = (D_{j,l})_{1 \le j,l \le N}$,

$$D_{jl} = \frac{1}{\zeta_l - \zeta_j^*}[(\mu_j^* \mu_l + 1)e^{A_j^* + A_l} + (\mu_j^* \nu_l + 1)e^{A_j^* - A_l} + (\nu_j^* \mu_l + 1)e^{-A_j^* + A_l} + (\nu_j^* \nu_l + 1)e^{-A_j^* - A_l}],$$
$$D_1 = \begin{pmatrix} G & Y_2^\dagger \\ Y_1/c & 0 \end{pmatrix},$$
$$Y_1 = (\mu_1 e^{A_1} + \nu_1 e^{-A_1}, \mu_2 e^{A_2} + \nu_2 e^{-A_2}, \ldots, \mu_N e^{A_N} + \nu_N e^{-A_N}),$$
$$Y_2 = (e^{A_1} + e^{-A_1}, e^{A_2} + e^{-A_2}, \ldots, e^{A_N} + e^{-A_N}).$$

At this point, setting

$$\zeta_1 = \xi_1 + i\eta_1,$$
$$A_1 = (h_R + ih_I)[t + (k_R + ik_I)z] = A_R + iA_I,$$

we have the explicit one-breather solution

$$E[1] = \left(1 + 4\eta_1 \frac{G_1 + iH_1}{F_1}\right) c e^{i\theta}, \quad (13)$$

where

$$F_1 = (a^2/4 + a\xi_1 + c^2 + \eta_1^2 + h_R^2 + h_I^2 + \xi_1^2)\cosh(2A_R) + (2\eta_1 h_R - ah_I - 2h_I\xi_1)\sinh(2A_R) + (a^2/4 + a\xi_1 + c^2 + \eta_1^2 - h_R^2 - h_I^2 + \xi_1^2)\cos(2A_I) + (ah_R + 2\eta_1 h_I + 2h_R\xi_1)\sin(2A_I),$$
$$G_1 = -\eta_1\cosh(2A_R) - h_R\sinh(2A_R) - \eta_1\cos(2A_I) - h_I\sin(2A_I),$$
$$H_1 = (a/2 + \xi_1)\cosh(2A_R) - h_I\sinh(2A_R) + (a/2 + \xi_1)\cos(2A_I) + h_R\sin(2A_I),$$

with

$$A_R = h_R t + (h_R k_R - h_I k_I)z,$$
$$A_I = h_I t + (h_R k_I - h_I k_R)z.$$

Figures 2(a) and 2(b) show the Kuznetsov–Ma soliton and the Akhmediev breather dynamics described by Eq. (13) taking $\xi_1 = -a/2$ with $|\eta_1| > c$ and $|\eta_1| < c$, respectively. Furthermore, dynamics of two breathers interacting with each other can be presented, see Fig. 2(c).

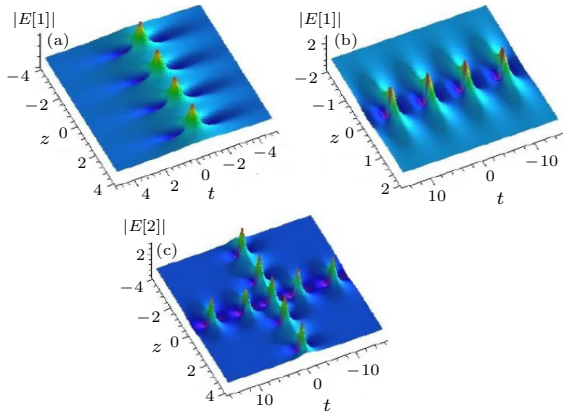

Fig. 2. Evolution plots of (a) the Kuznetsov–Ma soliton and (b) the Akhmediev breather with $\xi_1 = 0$, $\eta_1 = 1.1$ and $\xi_1 = 0$, $\eta_1 = 0.9$. (c) Evolution plot of the second-order breather with $\xi_1 = 0$, $\eta_1 = 1.1$, $\xi_2 = 0$, $\eta_2 = 0.9$. The other parameters are $a = 0$, $b = 1$, $c = 1$, $d = 0$, and $\omega = 0$.

More interesting, with the help of the limit approach, is that we can compute the Nth-order rogue wave solution for system (2) based on the plane-wave solution (10). For this end, we take $\zeta_1 = \zeta_1(\epsilon) = -a/2 - ic - 2i\epsilon^2$, where ϵ is a small complex parameter. In this circumstance, the basic solution of the Lax pair of Eqs. (3) and (4) can be calculated as

$$\Phi_1(\epsilon) = \begin{pmatrix} \psi_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} (C_1 e^A - C_2 e^{-A}) e^{\frac{i}{2}\theta} \\ (C_2 e^A - C_1 e^{-A}) e^{-\frac{i}{2}\theta} \end{pmatrix}, \quad (14)$$

where

$$C_1 = \frac{(1 + 2\epsilon^2 - 2\epsilon\sqrt{1 + \epsilon^2})^{\frac{1}{2}}}{2\epsilon\sqrt{1 + \epsilon^2}},$$
$$C_2 = \frac{(1 + 2\epsilon^2 + 2\epsilon\sqrt{1 + \epsilon^2})^{\frac{1}{2}}}{2\epsilon\sqrt{1 + \epsilon^2}},$$
$$A = 2c\epsilon\sqrt{1 + \epsilon^2}[t + k_1(\epsilon)z + \sum_{j=1}^{N-1} s_j \epsilon^{2j}],$$

with

$$k_1(\epsilon) = \frac{1}{(2\zeta_1 - \omega)^2}[-8\zeta_1^3 + (4a + 8\omega)\zeta_1^2 + (-2a^2 - 4a\omega + 4c^2 - 2\omega^2 + 2b)\zeta_1 + a^3 - 4c^2\omega + 2a^2\omega - 2ac^2 + a\omega^2 - 2d - ab - 2b\omega],$$

and s_j $(j = 1, 2, \ldots, N-1)$ being the complex arbitrary constants.

Hereby, one can verify that $\Phi_1(\epsilon)$ can be expressed as the Taylor series

$$\Phi_1 = \Phi_1^{[0]} + \Phi_1^{[2]}\epsilon^2 + \ldots + \Phi_1^{[N-1]}\epsilon^{2(N-1)} + O(\epsilon^{2N}),$$

where $\Phi_1^{[j]} = (\psi_1^{[i]}, \varphi_1^{[i]})^T = \lim\limits_{\epsilon\to 0} \frac{1}{2j!}\frac{\partial^{2j}\Phi_1}{\partial\epsilon^{2j}}$, $j = 0, 1, \ldots, N-1$. Moreover, it is easy to check that

$$\frac{\psi_1\psi_1^* + \varphi_1\varphi_1^*}{-2ic(1+\epsilon^2+\epsilon^{*2})} = \sum_{j,l=1}^{N} D^{[j,l]}\epsilon^{2(l-1)}\epsilon^{*2(j-1)} + O(|\epsilon|^{4N}),$$

where

$$D^{[j,l]} = \lim_{\epsilon,\epsilon^*\to 0} \frac{1}{2(j-1)!2(l-1)!}\frac{\partial^{2(j+l-2)}}{\partial\epsilon^{2(l-1)}\partial\epsilon^{*2(j-1)}} \cdot \frac{\psi_1\psi_1^* + \varphi_1\varphi_1^*}{-2ic(1+\epsilon^2+\epsilon^{*2})}.$$

In what follows, taking the limit approach of the N-fold DT, one can end up with the Nth-order rogue wave solution

$$E[N] = \Big[1 + 2i\frac{\det(D_1)}{\det(D)}\Big]ce^{i\theta}, \tag{15}$$

where

$$D = (D^{[j,l]})_{1\le j,l\le N},\ D_1 = \begin{pmatrix} D & Y_2^\dagger \\ Y_1/c & 0 \end{pmatrix},$$
$$Y_1 = (\psi_1^{[0]}, \psi_1^{[1]}, \ldots, \psi_1^{[N-1]}),$$
$$Y_2 = (\varphi_1^{[0]}, \varphi_1^{[1]}, \ldots, \varphi_1^{[N-1]}).$$

Specially, the first-order rogue wave solution can be represented as

$$E[1] = \Big[\frac{4\kappa^2 + 16ic^2\varsigma z}{\kappa^2 + 4c^2(\kappa t + 2\vartheta z)^2 + 16c^4\varrho^2 z^2} - 1\Big]ce^{i\theta}, \tag{16}$$

where $\kappa = (a+\omega)^2 + 4c^2$,

$$\begin{aligned}
\vartheta =& (2a^5 + 7a^4\omega + 6a^3c^2 + 9a^3\omega^2 + 10a^2c^2\omega + 5a^2\omega^3 \\
&+ 16ac^4 + 2ac^2\omega^2 + a\omega^4 - 2c^2\omega^3 - a^3b - 3a^2b\omega \\
&- 3ab\omega^2 - b\omega^3 - a^2d - 2ad\omega + 4c^2d - d\omega^2)/\kappa, \\
\varrho =& (2a^4 + 2a^3\omega - 10a^2c^2 - 3a^2\omega^2 - 28ac^2\omega - 4a\omega^3 \\
&- 24c^4 - 14c^2\omega^2 - \omega^4 - 3a^2b - 6ab\omega - 4bc^2 \\
&- 3b\omega^2 - 4ad - 4d\omega)/\kappa, \\
\varsigma =& 2a^4 + 2a^3\omega - 10a^2c^2 - 3a^2\omega^2 - 28ac^2\omega - 4a\omega^3 \\
&- 24c^4 - 14c^2\omega^2 - \omega^4 - 3a^2b - 6ab\omega - 4bc^2 \\
&- 3b\omega^2 - 4ad - 4d\omega.
\end{aligned}$$

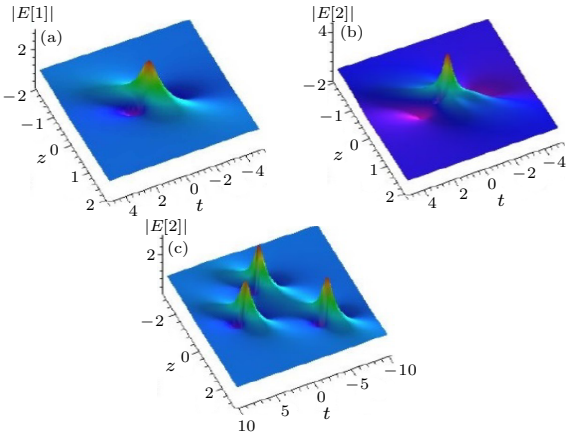

Fig. 3. (a) Evolution plot of the first-order rogue wave with $a = 1$, $b = 0$, $c = 1$, $\omega = 0$, and $d = 1$; and evolution plots of the second-order rogue waves with (b) $s_1 = 0$ and (c) $s_1 = 100$ with the other parameters the same as those in (a).

Choosing adequate parameters, we can display dynamics of the first-order rogue wave as shown in Fig. 3(a). The maximum amplitude of the first-order rogue wave is 3 and arrives at $(0, 0)$, and the minimum amplitude is 0 and occurs at $(\pm\frac{\sqrt{3}}{2}, 0)$. Further, dynamics of the fundamental and triangular patterns of the second-order rogue waves can be shown by choosing different values of the arbitrary parameters, see Figs. 3(b) and 3(c).

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