Gravitational constant in $f(R)$ theories of gravity with non-minimal coupling between matter and geometry *

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We study the effect of the non-minimal coupling between matter and geometry on the gravitational constant in the context of $f(R)$ theories of gravity on cosmic scales. For a class of $f(R)$ models, the result shows that the value of the gravitational constant not only changes over time but also has the dampened oscillation behavior. Compared with the result of the standard ΛCDM model, the consequence suggests that the coupling between matter and geometry should be weak.

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To explain the observational phenomenon of the accelerating cosmic expansion,[1,2] $f(R)$ theories of gravity[3,4] are attractive candidates, and their capability to realize the late-time cosmic acceleration has been extensively studied (see Refs. [5–8] and references therein). For successful $f(R)$ theories of gravity, the so-called viability criteria should be satisfied.[9–13] The gravitational constant is one of the most important aspects, since its value determines the expansion rate of our universe. Due to corrections of the Ricci scalar R, the gravitational constant is time-dependent, which is one of the most essential differences between $f(R)$ theories of gravity and general relativity (GR). The issue of the gravitational constant in $f(R)$ theories of gravity without coupling has been studied in Ref. [14]. For $f(R)$ gravity with coupling, the compatibility with solar system tests has been studied in Ref. [15]. In this Letter, alternatively to solar system tests, we investigate the effect of the non-minimal coupling between matter and geometry on the gravitational constant on cosmic scales.

The action of our considered context is[16]

$$S=\int\left\{\frac{1}{2}f_1(R)+[1+\lambda f_2(R)]L_{\rm m}\right\}\sqrt{-g}d^4x, \quad (1)$$

where $f_i(R)$ $(i=1,2)$ are arbitrary functions of the Ricci scalar R, $L_{\rm m}$ is the Lagrangian density of matter, and λ is a constant to denote the coupling strength between matter and geometry. When $\lambda=0$, action (1) reduces to the context of $f(R)$ theories of gravity without coupling. Furthermore, by taking $f_1(R)=R$, the framework of GR can be reproduced. The units $8\pi G=c=1$ are used here and will be used throughout this work.

Varying action (1) with respect to the metric $g^{\mu\nu}$, the field equations are

$$\begin{aligned}&f_{1R}(R)R_{\mu\nu}-\frac{1}{2}f_1(R)g_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_{1R}(R)\\&=-2\lambda f_{2R}(R)L_{\rm m}R_{\mu\nu}+2\lambda(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box)L_{\rm m}f_{2R}(R)\\&\quad+[1+\lambda f_2(R)]T_{\mu\nu}, \quad (2)\end{aligned}$$

where $f_{iR}(R)\equiv df_i(R)/dR$ $(i=1,2)$, $\Box\equiv\nabla_\alpha\nabla^\alpha$, ∇_μ is the covariant derivative with respect to the Christoffel connections, and $T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_{\rm m})}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter.

The flat FLRW metric with the conformal time and scalar perturbations in the longitudinal gauge is

$$ds^2=-a^2(\eta)[(1+2\Phi)d\eta^2-(1-2\Psi)(dr^2+r^2d\Omega_2^2)], \quad (3)$$

where $a(\eta)$ is the scale factor, η denotes the conformal time, and $\Phi\equiv\Phi(\eta,\boldsymbol{x})$ and $\Psi\equiv\Psi(\eta,\boldsymbol{x})$ are scalar perturbations. For a perfect fluid, the energy-momentum tensor of matter is

$$T_{\mu\nu}=(\rho+p)U_\mu U_\nu+pg_{\mu\nu}, \quad (4)$$

where ρ, p, and U_μ are the energy density, the pressure, and the four-velocity of the fluid, respectively. Then, the first-order expressions for components of the energy-momentum tensor of matter are[17]

$$T_0^0=-(\rho+\delta\rho), \quad (5a)$$
$$T_i^0=(\rho+p)v_i=-T_0^i, \quad (5b)$$
$$T_j^i=(p+\delta p)\delta_j^i, \quad (5c)$$

where $\delta\equiv\delta\rho/\rho$, v, and δp are the density contrast, the potential for velocity perturbations, and the pressure perturbation, respectively.

According to the background field Eq. (2), one can obtain the background dynamical equations and the first-order perturbation equations in the Fourier space for $f(R)$ gravity with non-minimal coupling between

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matter and geometry as follows:[13] the $0-0$ component

$$\begin{aligned}
&3(f_{1R}+2\lambda f_{2R}L_{\rm m})\mathcal{H}^2\\
=&3\frac{a''}{a}(f_{1R}+2\lambda f_{2R}L_{\rm m})-\frac{1}{2}a^2 f_1\\
&-3\mathcal{H}(f_{1R}+2\lambda f_{2R}L_{\rm m})'+(1+\lambda f_2)a^2\rho, \qquad (6a)\\
&-3\mathcal{H}' f_{1RR}\delta R-f_{1R}[2k^2\Psi+6\mathcal{H}(\mathcal{H}\Phi+\Psi')]\\
&+k^2 f_{1RR}\delta R+3\mathcal{H}(f_{1RR}\delta R)'-3(\Psi'+2\mathcal{H}\Phi)f'_{1R}\\
=&6\mathcal{H}'\lambda f_{2RR}\delta R L_{\rm m}+6\mathcal{H}'\lambda f_{2R}(\delta L_{\rm m}-2\Phi L_{\rm m})\\
&-2\lambda f_{2R}L_{\rm m}[3\Psi''-k^2\Phi+3\mathcal{H}(\Phi'+\Psi')]\\
&-6\mathcal{H}\lambda(\delta L_{\rm m}f_{2R}+L_{\rm m}f_{2RR}\delta R)'-2\lambda k^2(\delta L_{\rm m}f_{2R}\\
&+L_{\rm m}\delta R f_{2RR})+6\lambda(2\mathcal{H}\Phi+\Psi')(L_{\rm m}f_{2R})'\\
&+a^2\rho[\lambda f_{2R}\delta R+(1+\lambda f_2)\delta], \qquad (6b)
\end{aligned}$$

The $0-i$ component

$$\begin{aligned}
&2A(\Psi'+\mathcal{H}\Phi)-[(f_{1RR}\delta R)'-\Phi f'_{1R}-\mathcal{H}f_{1RR}\delta R]\\
=&2\lambda[(\delta L_{\rm m}f_{2R})'+(L_{\rm m}\delta R f_{2RR})'-\Phi(L_{\rm m}f_{2R})'\\
&-\mathcal{H}\delta L_{\rm m}f_{2R}-\mathcal{H}L_{\rm m}\delta R f_{2RR}]\\
&-(1+\lambda f_2)[a^2(\rho+p)v]. \qquad (7)
\end{aligned}$$

The $i-j$ $(i\neq j)$ component

$$\begin{aligned}
&6\mathcal{H}^2(f_{1R}+2\lambda f_{2R}L_{\rm m})=a^2 f_1+3(f_{1R}+2\lambda f_{2R}L_{\rm m})''\\
&+(1+\lambda f_2)a^2(\rho+3p), \qquad (8a)\\
&A(\Psi-\Phi)=f_{1RR}\delta R+2\lambda(\delta L_{\rm m}f_{2R}+L_{\rm m}f_{2RR}\delta R), \qquad (8b)
\end{aligned}$$

where a prime denotes the derivative with respect to conformal time, $\mathcal{H}\equiv a'/a$, k is a comoving wave number, $A\equiv f_{1R}+2\lambda f_{2R}L_{\rm m}$, and δR is given as

$$\begin{aligned}
\delta R=&\frac{2}{a^2}[k^2(\Phi-2\Psi)-3\Psi''-6\Phi(\mathcal{H}'+\mathcal{H}^2)\\
&-3\mathcal{H}(\Phi'+3\Psi')]. \qquad (9)
\end{aligned}$$

From the field Eq. (2), the covariant divergence of the energy-momentum tensor of matter can be obtained as follows:[18]

$$\nabla^\mu T_{\mu\nu}=\frac{\lambda f_{2R}}{1+\lambda f_2}[g_{\mu\nu}L_{\rm m}-T_{\mu\nu}]\nabla^\mu R, \qquad (10)$$

where the Bianchi identities and the geometrical identities, $(\Box\nabla_\nu-\nabla_\nu\Box)f_{iR}=R_{\mu\nu}\nabla^\mu f_{iR}$, are used. It is clear that since non-minimal coupling exists, the conservation law of matter no longer holds and energy exchange occurs between matter and geometry. However, when the Lagrangian density of matter is taken to be opposite to the energy density of a perfect fluid, i.e. $L_{\rm m}=-\rho$, the conservative context can be achieved.[19–21] Then, in the Fourier space, the first-order expressions for the temporal and spatial components of the perturbed covariant conservation equations are[9]

$$3\Psi'(1+c_s^2)-\delta'+(1+c_s^2)k^2 v=0, \qquad (11)$$

$$\Phi+\frac{c_s^2}{1+c_s^2}\delta+v'+\mathcal{H}v(1-3c_s^2)=0, \qquad (12)$$

where the equation of state for both perturbed and unperturbed matter are supposed to be the same, i.e. $\delta p/\delta\rho\equiv c_s^2\equiv p/\rho$. Assuming the cold dark matter as the only kind of matter in the universe and focusing the case in the subscale approximation,[22,23] under which modes of perturbations are deep inside the Hubble radius, i.e. $k\gg\mathcal{H}$, and then combining Eqs. (11) and (12), the equation of matter density perturbations can be obtained as follows:

$$\delta''_{\rm m}+\mathcal{H}\delta'_{\rm m}+k^2\Phi\simeq 0, \qquad (13)$$

where the expression of the gravitational potential Φ can be obtained from equations. (6b) and (8b) through

$$\Phi\simeq-\frac{-2\lambda k^2 f_{2R}+(1-2Q)(1+\lambda f_2)a^2}{(2-3Q)k^2 A}\rho_{\rm m}\delta_{\rm m}, \qquad (14)$$

where $Q\equiv-2k^2B/a^2A$, and $B\equiv f_{1RR}-2\lambda f_{2RR}\rho_{\rm m}$. In this circumstance, one can define the effective gravitational constant $G_{\rm eff}$ as

$$G_{\rm eff}\simeq\frac{-2\lambda k^2 f_{2R}+a^2(1-2Q)(1+\lambda f_2)}{8\pi a^2 A(1-\frac{3}{2}Q)}, \qquad (15)$$

and then the equation of matter density perturbations, i.e. Eq. (13), and the expression of the gravitational potential, i.e. Eq. (14), can be rewritten as

$$\delta''_{\rm m}+\mathcal{H}\delta'_{\rm m}-4\pi a^2 G_{\rm eff}\rho_{\rm m}\delta_{\rm m}\simeq 0, \qquad (16)$$

$$\Phi\simeq-4\pi G_{\rm eff}\frac{a^2}{k^2}\rho_{\rm m}\delta_{\rm m}. \qquad (17)$$

The latter corresponds to a Poisson equation.

It is worth stressing that when the coupling strength between matter and geometry vanishes, the background dynamical equations, the perturbation equations, and the expression of the effective gravitational constant in $f(R)$ theories of gravity without coupling can be reproduced, which are consistent with the ones given in Ref. [24].

From the expression of the effective gravitational constant, i.e. Eq. (15), it is obvious that in our considered scenario the gravitational constant varies not only with time but also with the coupling strength constant λ, which is totally different from the case in GR. To explore the effect of the non-minimal coupling between matter and geometry on the effective gravitational constant, a class of cosmological models are taken into consideration, in which arbitrary functions of $f_1(R)$ and $f_2(R)$ are taken as[5,13,25–29]

$$f_1(R)=R+\alpha R^n, \qquad (18)$$

$$f_2(R)=R^2, \qquad (19)$$

where α and n are constants.

It is well known that the standard ΛCDM model is the best cosmological model and fits most present observational data. Therefore, following Ref. [13], to make the background evolution to be similar to that of the standard ΛCDM model and also guarantee the density contrast to be irrelevant to the comoving wave number, the values of parameters and the initial condition in the considered case are taken as $n = 0.00001$, $\alpha = -4.3H_0^2$, $H_0 = 75\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$, $\lambda = -10^{-12}\,\mathrm{Gyr}^4$, and $a = 0.01$, respectively.

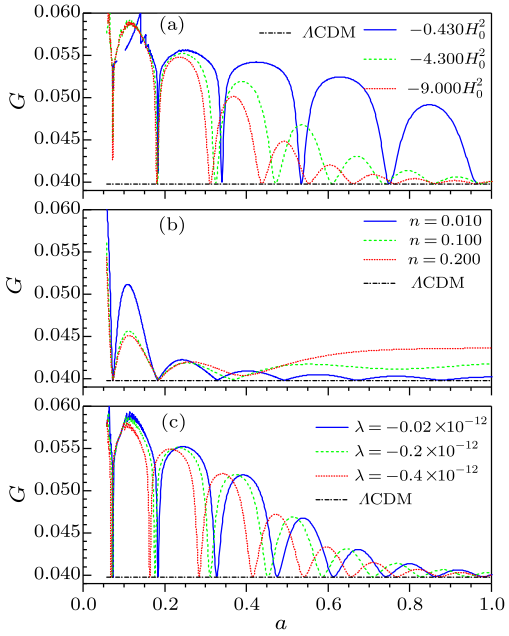

Fig. 1. (a)–(c) Evolutionary trajectories of the effective gravitational constant with parameters α and n and the coupling strength constant λ, at $k = 0.04h\,\mathrm{Mpc}^{-1}$ for both our considered case and the standard ΛCDM model.

Based on the above discussion, the evolution of the effective gravitational constant with parameters α and n and the coupling strength constant λ are illustrated in Fig. 1, where we also present the result of the standard ΛCDM model for convenience. For the parameter n, when $0 < a \leq 0.25$, evolutionary trajectories are dampened oscillations and the larger the value of parameter n, the smaller the amplitude of the dampened oscillations. However, when $0.25 < a \leq 0.5$, the larger the value of parameter n, the slower the evolution of dampened oscillations. When $a > 0.5$, the phenomenon of oscillations gradually disappears with increasing value of parameter n, and the value of the effective gravitational constant is positively correlated with the value of parameter n.

For the coupling strength constant λ, evolutionary trajectories are also dampened oscillations. Moreover, the larger the absolute value of the coupling strength constant λ, the smaller the amplitude of dampened oscillations and the faster the evolution of dampened oscillations. As the value of the coupling strength constant λ comes close to zero, evolutionary trajectories of the effective gravitational constant approach that of the standard ΛCDM model, which is a straight line, implying that the gravitational constant does not change with time. However, for the parameter α, the situation is the same as that of the coupling strength constant λ.

Since the obtained effective gravitational constant varies with time, the parameter n can be constrained by the observational consequence of $\dot{G}/G$. The constrained result shows that the interval of the parameter n is $n > -0.66$, when $\dot{G}/G > -0.55 \times 10^{-11}\,\mathrm{yr}^{-1}$.[30]

In summery, the gravitational constant has been studied in the context of $f(R)$ theories of gravity with non-minimal coupling between matter and geometry on cosmic scales. Considering the longitudinal gauge, the condition for the conservation of the energy-momentum tensor of matter and the subscale approximation, the expression of the effective gravitational constant is obtained. Furthermore, to investigate the effect of the non-minimal coupling between matter and geometry on the effective gravitational constant, a class of $f(R)$ models have been taken into consideration. The corresponding results have been illustrated in Fig. 1. Due to the non-minimal coupling term, the feature of the effective gravitational constant is not only time variation but also dampened oscillations with parameters. For parameter n, when $0 < a \leq 0.25$, the larger its value, the smaller the amplitude of dampened oscillations. When $0.25 < a \leq 0.5$, the larger the value of parameter n, the slower the evolution of dampened oscillations. However, when $a > 0.5$, the phenomenon of oscillations gradually disappears as the value of parameter n increases, and the value of the effective gravitational constant is positively correlated with the value of parameter n. According to the observational result of $\dot{G}/G$, parameter n is constrained as $n > -0.66$.

For the coupling strength constant λ and the parameter α, the larger the absolute value of one of them is, the smaller the amplitude of dampened oscillations is and the faster the evolution of dampened oscillations. As the values of the coupling strength constant λ and the parameter α come close to zero, the value of the effective gravitational constant approaches to that of the standard ΛCDM model, which does not change with time. It is obvious that the strength of the non-minimal coupling between matter and geometry affects the value and the evolution of the effective gravitational constant, and then finally influences the progress of the Universe.

Comparing the result of the gravitational constant in our considered context with that of the standard ΛCDM model, which is the best cosmological model to fit most present observational data, we see that to make numerical solutions compatible with observa-

tional ones, the coupling between matter and geometry should be weak.

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