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# Nonlinear Dicke Quantum Phase Transition and Its Quantum Witness in a Cavity-Bose–Einstein-Condensate System *

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*We investigate nonlinear Dicke quantum phase transition (QPT) induced by inter-atomic nonlinear interaction and its quantum witness in a cavity-Bose–Einstein-condensate (BEC) system. It is shown that inter-atomic nonlinear interaction in a cavity BEC system can induce first-order Dicke QPT. It is found that this nonlinear Dicke QPT can happen in an arbitrary coupling regime of the cavity and atoms. It is demonstrated that the quantum speed limit time can witness the Dicke QPT through its sudden change at the critical point of the QPT.*

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The Dicke model[1] describes a large number of two-level atoms interacting with a single cavity field mode, and predicts the existence of the Dicke quantum phase transition (QPT)[2–4] from the normal phase to the super-radiant phase. However, it is difficult to observe the Dicke QPT in the standard Dicke model, since the critical collective atom-field coupling strength needs to be of the same order as the energy separation between the two atomic levels. Fortunately, strong collective atom-field coupling has realized experimentally in a Bose–Einstein-condensate (BEC) coupling with a ultrahigh-finesse cavity field.[5,6] Emary and Brandes[7] first indicated that the Dicke model exhibits a zero-temperature QPT from the normal phase to the super-radiant phase in the thermodynamic limit. Then, Nagy *et al.*[8] pointed out that the Dicke QPT corresponds to the self-organization of atoms from the homogeneous into a periodically patterned distribution. Soon after this, the Dicke QPT was experimentally observed in the sense of the self-organization of atoms using the cavity-BEC system.[9] In the Dicke QPT experimental realization,[9] the normal phase corresponds to the BEC being in the ground state associated with vacuum cavity field state while both the BEC and cavity field have collective excitations in the super-radiant phase. A few extended Dicke models[10–17] have been proposed to reveal rich phase diagrams and exotic QPTs, which are different from those in the original Dicke model. Recently, the connection between the quantum speed limit (QSL) time[18–21] and quantum criticality of an open system has been revealed and it has been found that the QSL time displays a sudden transition at the QPT critical point.[22] In this Letter, motivated by the experimental relevance on inter-atomic nonlinear interaction, we introduce and analyze a generalized Dicke model Hamiltonian which is relevant to current experimental setups in the cavity-BEC system. We study Dicke QPT induced by inter-atomic nonlinear interaction in a cavity-BEC system, called a nonlinear Dicke QPT, and show that quantum speed limit (QSL) time can witness the nonlinear Dicke QPT.

The physical system under consideration is a cavity-BEC system shown in Fig. 1. The BEC with $N$ identical two-level $^{87}$Rb atoms is confined in a ultrahigh-finesse optical cavity. The atoms interact with a single cavity model of frequency $\omega_{\rm c}$ and a transverse pump field of frequency $\omega_{\rm p}$. We consider a situation where the frequencies $\omega_{\rm c}$ and $\omega_{\rm p}$ are detuned far from the atomic resonance frequency $\omega_{\rm A}$ so that the detunings far exceed the rate of atomic spontaneous emission, the atoms only scatter photons either along or transverse to the cavity axis. Before the pump field turns on, atoms in the BEC are supposed to be in the zero-momentum state $|p_x, p_z\rangle = |0, 0\rangle$. As soon as one turns on the pump field, via the photon scattering of the pump and cavity fields, some atoms are excited into nonzero momentum sates $|p_x, p_z\rangle = |\pm k, \pm k\rangle = \sum_{\upsilon_1,\upsilon_2=\pm1} |\upsilon_1 k, \upsilon_2 k\rangle$ (hereafter, we take $\hbar = 1$) due to the conservation of momentum, where $k$ is the wave-vector, which is approximated to be equal on the cavity and pump fields. The zero and nonzero momentum states denoted by $|0, 0\rangle$ and $|\pm k, \pm k\rangle$ are regarded as two effective levels of the $^{87}$Rb atom with energy separation $\omega_0 = k^2/m$, where $m$ is the mass of $^{87}$Rb atom. One can introduce the collective operators $\hat{J}_z = \sum_i |\pm k, \pm k\rangle_i\langle \pm k, \pm k|$, $\hat{J}_+ = \hat{J}_-^\dagger = \sum_i |\pm k, \pm k\rangle_i\langle 0, 0|$ with the index $i$ labelling the atoms, Hamiltonian of the cavity-BEC system is described by the generalized Dicke model,[9,23]

$$\hat{H}_{\rm s} = \omega_{\rm a}\hat{a}^\dagger\hat{a} + \omega\hat{J}_z + \frac{\lambda}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)(\hat{J}_+ + \hat{J}_-) + \frac{\chi}{N}\hat{J}_z^{\,2}, \tag{1}$$

where $\omega_{\rm a} = -\Delta_{\rm c} + NU_0/2$ is the effective frequency of the cavity field, the atomic effective transition frequency is given by $\omega = 2\omega_{\rm r} + \chi'$, where $\Delta_{\rm c} = \omega_{\rm p} - \omega_{\rm c}$,

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$\omega_{\rm p}$ ($\omega_{\rm c}$) is the frequency of the pump (cavity) field, $N$ is the number of atoms, $U_0 = g_0^2/\Delta_{\rm a}$ is the light shift induced by the atom where $g_0$ is the atom-cavity coupling strength, $\Delta_{\rm a} = \omega_{\rm p} - \omega_0$, $\omega_0$ is the atomic resonance frequency, $\omega_{\rm r}$ is the single photon recoil frequency and $\chi' = (N-1)(\chi_1 - \chi_0)/2$, where $\chi_{1(0)} = s\int dxdy|\Phi_{1(0)}(x,y)|^4$ is the intraspecies coupling strength with $s$ being proportional to the s-wave scattering length,[23] $\Phi_1(x,y)$ and $\Phi_0(x,y)$ being the wave function of the BEC in $k$-momentum state and zero-momentum state, respectively. Here we make $\chi' \approx 0$. The coupling strength induced by the cavity field and pump field is given by $\lambda = \sqrt{N}g_0\Omega_{\rm p}/2\Delta_{\rm a}$ where $\Omega_{\rm p}$ denotes the maximum pump Rabi frequency which can be adjusted by the pump power, and the nonlinear interaction strength $\chi = N[(\chi_0+\chi_1)/2 - \chi_{01}]$ with $\chi_{01} = s\int dxdy|\Phi_0(x,y)|^2|\Phi_1(x,y)|^2$ being the interspecies coupling strength.

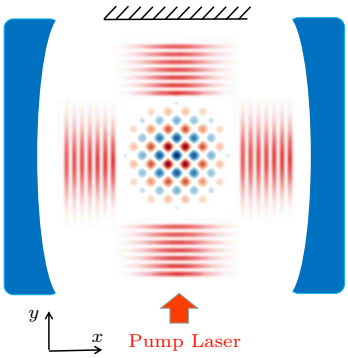


**Fig. 1.** (Color online) The cavity-BEC system under our consideration. The system will be in super-radiant phase when the cavity-atom coupling strength is larger than a certain critical value.

Hamiltonian (1) can be expressed as a two-mode bosonic Hamiltonian by employing a bosonization technique based upon the Holstein–Primakoff transformation of the angular momentum algebra

$$\hat{J}_+ = \hat{c}^\dagger\sqrt{N-\hat{c}^\dagger\hat{c}},\ \hat{J}_- = \sqrt{N-\hat{c}^\dagger\hat{c}}\hat{c},\ \hat{J}_z = \hat{c}^\dagger\hat{c} - \frac{N}{2}, \tag{2}$$

where the collective boson operators $\hat{c}$ and $\hat{c}^\dagger$ obey the commutation relation $[\hat{c},\hat{c}^\dagger] = 1$. Then, Hamiltonian (1) can be rewritten in the form

$$\begin{aligned}\hat{H} =& \omega_{\rm a}\hat{a}^\dagger\hat{a} + (\omega-\chi)\hat{c}^\dagger\hat{c} + \frac{\chi}{N}(\hat{c}^\dagger\hat{c})^2 \\ &+ \lambda(\hat{a}+\hat{a}^\dagger)\Big(\hat{c}^\dagger\sqrt{1-\frac{\hat{c}^\dagger\hat{c}}{N}} + \sqrt{1-\frac{\hat{c}^\dagger\hat{c}}{N}}\hat{c}\Big),\end{aligned} \tag{3}$$

which indicates that the interatomic nonlinear interaction leads to a frequency shift of the condensed atoms. It should be noted that we have neglected a constant term in the above Hamiltonian.

In the super-radiant phase, both the field and the atomic ensemble acquire macroscopic occupations. To describe the collective behaviors of the condensed atoms and the photon, one can introduce displaced bosonic operators $\hat{a}_1 = \hat{a} + \sqrt{N}\alpha$, $\hat{c}_1 = \hat{c} - \sqrt{N}\beta$, where $\alpha$ and $\beta$ are real numbers. Substituting bosonic operators $\hat{a}_1$ and $\hat{c}_1$ into the Hamiltonian (3) and neglecting terms with $N$ in the denominator, Hamiltonian (3) can be expanded as

$$\hat{H} = NE_0 + \sqrt{N}\hat{H}_1 + \hat{H}_2, \tag{4}$$

where $E_0$, $\hat{H}_1$ and $\hat{H}_2$ are defined by

$$E_0 = \omega_{\rm a}\alpha^2 + (\omega-\chi)\beta^2 + \chi\beta^4 - 4\lambda\sqrt{1-\beta^2}\alpha\beta, \tag{5a}$$

$$\begin{aligned}\hat{H}_1 =& \Big[2\lambda\alpha\frac{1-2\beta^2}{\sqrt{1-\beta^2}} - (\omega-\chi)\beta - 2\chi\beta^3\Big](c_1^\dagger + c_1) \\ &+ [\omega_{\rm a}\alpha - 2\lambda\sqrt{1-\beta^2}\beta](a_1^\dagger + a_1),\end{aligned} \tag{5b}$$

$$\begin{aligned}\hat{H}_2 =& \omega_{\rm a}a_1^\dagger a_1 + \Big[(\omega-\chi) + 2\chi\beta^2 + \frac{2\lambda\alpha\beta}{\sqrt{1-\beta^2}}\Big]c_1^\dagger c_1 \\ &+ \Big[\chi\beta^2 + \frac{\lambda\alpha\beta(2+\beta^2)}{2(1-\beta^2)\sqrt{1-\beta^2}}\Big](c_1^\dagger + c_1)^2 \\ &- \frac{\lambda\alpha\beta}{\sqrt{1-\beta^2}} + \frac{\lambda(1-2\beta^2)(a_1^\dagger + a_1)(c_1^\dagger + c_1)}{\sqrt{1-\beta^2}}.\end{aligned} \tag{5c}$$

The collective excitation parameters $\alpha$ and $\beta$ can be determined from the equilibrium conditions $\partial E_0/\partial\alpha = 0$ and $\partial E_0/\partial\beta = 0$, which leads to the following two equations

$$\begin{aligned}\omega_{\rm a}\alpha - 2\lambda\sqrt{1-\beta^2}\beta =& 0, \\ (\omega - \chi + 2\chi\beta^2)\sqrt{1-\beta^2}\beta - 2\lambda\alpha(1-2\beta^2) =& 0,\end{aligned} \tag{6}$$

from which we can obtain an equation governing the fundamental features of the QPT

$$\beta[(2\chi\omega_{\rm a} + 8\lambda^2)\beta^2 + \omega_{\rm a}(\omega-\chi) - 4\lambda^2] = 0. \tag{7}$$

For the convenience of discussion, we choose the range of interatomic nonlinear interaction $\chi \in [0, +\infty)$. When $\omega_{\rm a}(\omega-\chi) \geq 4\lambda^2$, from Eq. (7) we can find $\alpha = \beta = 0$ due to $\lambda^2 > 0$. This means that both the BEC and the cavity field have no excitations. Hence the cavity-BEC system is in the normal phase with the energy $E_0 = 0$.

However, when $\omega_{\rm a}(\omega-\chi) < 4\lambda^2$, from Eqs. (6) and (7) we can find that the BEC and the cavity field exhibit nonzero population with collective excitation parameters

$$\alpha^2 = \frac{\lambda^2}{\omega_{\rm a}^2} - \frac{\lambda^2\omega^2}{(4\lambda^2 + \chi\omega_{\rm a})^2},\ \beta^2 = \frac{1}{2} - \frac{\omega_{\rm a}\omega}{8\lambda^2 + 2\chi\omega_{\rm a}}, \tag{8}$$

which implies that there exists a macroscopic quantum population of the collective excitations of the condensed atoms and the photon. Therefore, the cavity-BEC system is in the super-radiant phase in this situation.

From the QPT equation given by Eq. (7), we can see that there exist two independent QPT parameters, the interatomic interaction strength $\chi$ and the cavity-field-atom coupling parameter $\lambda$. The interatomic interaction leads to the nonlinear Dicke QPT while the cavity-field-atom coupling creates the conventional

Dicke QPT, which was investigated previously with the critic coupling constant $\lambda_c = \sqrt{\omega_a(\omega - \chi)}/2$ and shown to be a second-order QPT.[23]

From the QPT Eq. (7), we can find that the critical value of the nonlinear interaction strength $\chi_c$ at the QPT point with the following expression

$$\chi_c = \omega - \frac{4\lambda^2}{\omega_a}, \tag{9}$$

which indicates that the critical value of the interatomic nonlinear interaction depends not only on the effective frequency of the cavity field and the condensed atoms $\omega$ and $\omega_a$, but also on the coupling between the cavity field and the BEC $\lambda$. The stronger the coupling between the cavity field and the BEC, the smaller the QPT critic value of the interatomic nonlinear interaction becomes. It is interesting to note that the nonlinear Dicke QPT can occur for an arbitrary value of the coupling strength between the cavity field and the BEC while the conventional Dicke QPT can happen only in the strong coupling regime of the cavity field and atoms.

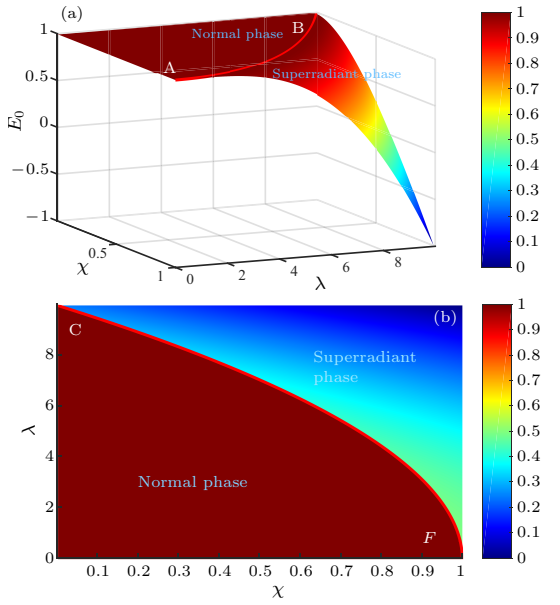


**Fig. 2.** (Color online) Phase diagrams of $E_0$ and $(\partial E_0/\partial\chi)|_{\chi=\chi_c}$ (their values denoted by colors). (a) Phase diagram of $E_0$ with respect to $\chi$ and $\lambda$. (b) Phase diagram of $(\partial E_0/\partial\chi)|_{\chi=\chi_c}$ with respect to $\chi$ and $\lambda$. The curves AB and CD are the critical curves. The related parameters are taken as $\omega_a = 20\,\text{MHz}$, $\omega = 0.05\,\text{MHz}$, $N = 10^5$ from the experimental parameters in Ref. [9] and the labels of coordinate axis are in units of $\omega$.

We can now discuss the order of the QPT in the cavity-BEC system. Following the usual criterion,[4] we can obtain the fundamental information by analyzing the behavior of the ground state energy and their derivatives with respect to QPT parameters. One can determine the type of QPTs through investigating the nonanalyticity of the scaled energy $E_0$ at the critical point in the thermodynamic limit $N \to \infty$. If the $n$th derivative of $E_0$ shows nonanalytic behavior then it is an $n$th order phase transition. In the present cavity-BEC system, it can be demonstrated that both $\lambda$ and $\chi$ can be chosen as QPT parameters. It was indicated that the QPT induced by the cavity-field-BEC coupling is a kind of the second-order Dicke QPT.[23] Here we choose the interatomic nonlinear interaction strength as the QPT parameter.

In this situation, when the cavity-BEC system is in the normal phase, the arbitrary order derivative of the scaled energy with respect to the QPT parameter $\chi$ is zero since the scaled energy $E_0 = 0$. However, in the super-radiant phase, we find the scaled energy to be

$$E_0 = \frac{4\omega_a\omega - 8\lambda^2 - 2\omega_a\chi}{8\omega_a} - \frac{\lambda^2\omega^2\omega_a + \chi\omega_a^2\omega^2/4}{(4\lambda^2 + \chi\omega_a)^2}, \tag{10}$$

which leads to the first derivative with respect to the QPT parameter $\chi$ in the super-radiant phase

$$\Big(\frac{\partial E_0}{\partial\chi}\Big)|_{\chi=\chi_c} = -\frac{1}{2}\Big(1 + \frac{\lambda^2}{\omega\omega_a}\Big), \tag{11}$$

which is always nonzero. This implies that the first derivative of the scale ground-state energy is discontinuous at the quantum critical point $\chi_c$. Therefore, we can conclude that the nonlinear Dicke QPT induced by the interatomic interaction is the first-order QPT.

To intuitively show the nonlinear Dicke QPT to be the first-order QPT, in Fig. 2 we plot the ground state energy $E_0$ and its first derivative $(\partial E_0/\partial\chi)|_{\chi=\chi_c}$ with respect to the QPT parameter $\chi$ as functions of coupling strength $\lambda$ and the atomic interaction $\chi$. Figure 2(b) indicates that the first derivative of the scale ground-state energy to the atomic interaction is discontinuous and the first-order QPT happens at the critical curve CD. From Fig. 2(b) we can see that the nonlinear Dicke QPT induced by the inter-atomic nonlinear interaction is different from the second-order Dicke QPT induced by the atom-field interaction. The nonlinear Dicke QPT induced by the inter-atomic nonlinear interaction may happen for arbitrary atom-field interaction strength while the second-order Dicke QPT nduced by the atom-field interaction occurs only in the strong coupling regime.

More generally, when two QPT parameters $\chi$ and $\lambda$ vary independently, collective excitations of the cavity-BEC system are given by Eq. (8). In this case, the critical point of the QPT with respect to a specific QPT parameter becomes a critical curve related to two QPT parameters. In the thermodynamic limit $N \to \infty$ we can obtain the scaled population inversion of the BEC $\langle\hat{J}_z\rangle/N$ as

$$\frac{\langle\hat{J}_z\rangle}{N} = -\frac{\omega_a\omega}{8\lambda^2 + 2\chi\omega_a}. \tag{12}$$

We have plotted the phase diagram of the scaled population inversion of the BEC with respect to the independent QPT parameters in Fig. 3. The Dicke

QPT happens at the critical curve EF in the phase diagram indicated in Fig. 3. The critical curve in the phase diagram appears as the intersection of the two phase regimes of the normal and super-radiant phases, and it can be described by the critical equation $4\lambda^2 + \omega_a\chi - \omega_a\omega = 0$. In Fig. 3 we have taken the related parameters as $\omega = 0.05\,\mathrm{MHz}$, $\omega_a = 20\,\mathrm{MHz}$ corresponding to the experimental parameters in Ref. [9], then the critical curve EF described by the critical equation $\lambda^2+100\chi-100 = 0$. The cavity-BEC system is in normal-phase in the regime of $\lambda^2+100\chi-100 < 0$ while the system is in the super-radiant phase when $\lambda^2 + 100\chi - 100 > 0$.

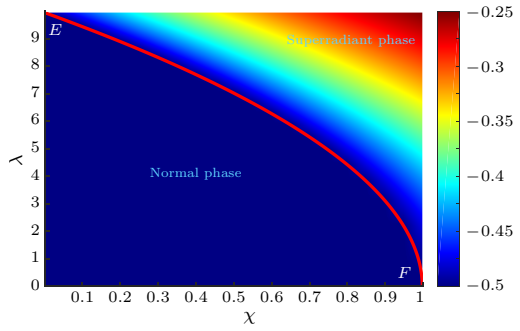


**Fig. 3.** (Color online) Phase diagrams described by $\langle J_z\rangle/N$ (its values denoted by colors) with respect to $\chi$ and $\lambda$. The red dashed line EF is the critical curve of normal phase (NP) and superradiant phase (SP). The related parameters are the same as Fig. 2.

We then turn to how to witness the nonlinear Dicke QPT induced by interatomic nonlinear interaction through the QSL time. The QSL time is the minimal evolution time of a quantum system from an initial state to a given target state. Especially, the QSL time of a quantum system between two orthogonal states with each other is given by the following simple expression

$$\tau_{\mathrm{QSL}} = \max\left\{\frac{\pi}{2}\frac{\hbar}{\Delta H}, \frac{\pi}{2}\frac{\hbar}{\langle\hat{H}\rangle}\right\}, \tag{13}$$

where $\Delta H$ and $\langle\hat{H}\rangle$ are the variance and the mean value of the Hamiltonian $\hat{H}$ with respect to an initial state under consideration, respectively.

In what follows we investigate the QSL time of the cavity-BEC system when the system is in an entangled coherent state[24–29]

$$|\psi\rangle = \frac{1}{\sqrt{2+2\exp(-4|\gamma|^2)}}(|\gamma,\gamma\rangle + |-\gamma,-\gamma\rangle), \tag{14}$$

where $|\gamma\rangle = \exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}(\gamma^n/\sqrt{n!})|n\rangle$ is a coherent state of a single mode boson.

To calculate the QSL time, we need the Hamiltonian of the cavity-BEC system in the normal and super-radiant phase. In the normal phase, at the thermodynamic limit $N \to \infty$, since $\hat{c}^\dagger\hat{c}/N \approx 0$, $(\hat{c}^\dagger\hat{c})^2/N \approx 0$, from Eq. (3) we obtain the Hamiltonian in the normal phase

$$\hat{H}_{\mathrm{np}} = \omega_a\hat{a}^\dagger\hat{a} + (\omega-\chi)\hat{c}^\dagger\hat{c} + \lambda(\hat{a}^\dagger+\hat{a})(\hat{c}^\dagger+\hat{c}). \tag{15}$$

From Eqs. (4) and (5), we can obtain the Hamiltonian in the super-radiant phase with the following form

$$\begin{aligned}\hat{H}_{\mathrm{sp}} =& \omega_a\hat{a}_1^\dagger\hat{a}_1 + \tilde{\omega}_0\hat{c}_1^\dagger\hat{c}_1 + \lambda_1(\hat{a}_1^\dagger+\hat{a}_1)(\hat{c}_1^\dagger+\hat{c}_1)\\ &+ \lambda_2(\hat{c}_1^\dagger+\hat{c}_1)^2,\end{aligned} \tag{16}$$

where the parameters $\tilde{\omega}_0$, $\lambda_1$ and $\lambda_2$ are given by

$$\tilde{\omega}_0 = \frac{2\lambda^2(1+\mu)}{\omega_a},\ \lambda_1 = \frac{\lambda\mu}{\sqrt{1+\mu}}, \tag{17}$$

$$\lambda_2 = \frac{\lambda^2}{2\omega_a}\frac{(1-\mu)(3+\mu)}{1+\mu} + \frac{\chi}{2}(1-\mu), \tag{18}$$

where we have introduced the parameter $\mu = \omega_a\omega/(4\lambda^2+\chi\omega_a)$.

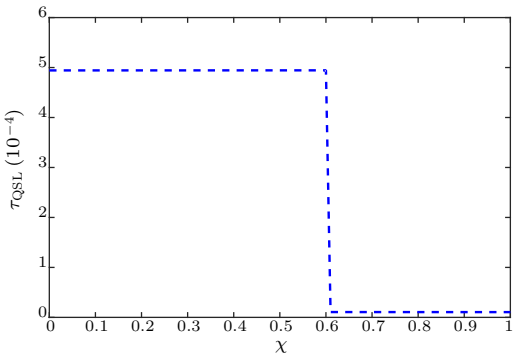


**Fig. 4.** (Color online) The QSL time of the system as a function of the interatomic nonlinear interaction strength $\chi$. The related parameters are taken as $\omega_a = 20\,\mathrm{MHz}$, $\omega = 0.05\,\mathrm{MHz}$, $\lambda = \sqrt{40}\,\mathrm{MHz}$, $N = 10^5$, $\chi_c = 0.6\omega_0$, and $\gamma = 5$.

We now numerically investigate the effect of the nonlinear Dicke QPT on the QSL time when the cavity-BEC system is in the entangled coherent state given by Eq. (14). The QSL time of the system as a function of $\chi$ is plotted in Fig. 4 for the entangled coherent state with the state parameter $\gamma = 5$. In Fig. 4, the other related parameters are taken as $\omega_a = 20\,\mathrm{MHz}$, $\omega = 0.05\,\mathrm{MHz}$, $\lambda = \sqrt{40}\,\mathrm{MHz}$, $N = 10^5$. It is easy to know the critical value of QPT parameter is $\chi_c/\omega_0 = 0.6$. The system is in normal phase when $\chi_c/\omega_0 < 0.6$ and in super-radiant phase for $\chi_c/\omega_0 > 0.6$. From Fig. 4 one can see that the QSL time changes sharply at the critical point of $\chi_c = 0.6\omega_0$. Obviously, the anomalous change of the QSL time at the QPT critical point reveals the critical characteristics of the nonlinear Dicke QPT. Therefore, this sudden change of the QSL time at the critical point can witness the nonlinear Dicke QPT.

In conclusion, we have studied nonlinear Dicke QPT induced by inter-atomic nonlinear interaction and its quantum witness in a cavity-BEC system. We have shown that experimentally relevant inter-atomic nonlinear interaction can change the nature of super-radiant phase transition in the traditional Dicke QPT. It has been shown that the nonlinear Dicke QPT is a first-order Dicke QPT, which can happen in an

arbitrary coupling regime of the cavity and atoms. The QSL time can witness the nonlinear Dicke QPT through the sudden change of the QSL time at the critical point of the QPT. Our results highlight the importance of the inter-atomic nonlinear interaction and open up an interesting question regarding the possible competition between the traditional Dicke QPT induced by the cavity-field-atom coupling and the nonlinear Dicke QPT induced by the inter-atomic nonlinear interaction.

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