CHIN. PHYS. LETT. Vol. 35, No. 10 (2018) 104701

Viscous Slip MHD Flow over a Moving Sheet with an Arbitrary Surface Velocity

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(Received 17 June 2018)

The magnetohydrodynamic (MHD) flow induced by a stretching or shrinking sheet under slip conditions is studied. Analytical solutions based on the boundary layer assumption are obtained in a closed form and can be applied to a flow configuration with any arbitrary velocity distributions. Seven typical sheet velocity profiles are employed as illustrating examples. The solutions to the slip MHD flow are derived from the general solution and discussed in detail. Different from self-similar boundary layer flows, the flows studied in this work have solutions in explicit analytical forms. However, the current flows require special mass transfer at the wall, which is determined by the moving velocity of the sheet. The effects of the slip parameter, the mass transfer at the wall, and the magnetic field on the flow are also demonstrated.

PACS: 47.10.ad, 47.15.Cb, 52.30.Cv DOI: 10.1088/0256-307X/35/10/104701

The flow over a continuously stretching or shrinking surface is important in many traditional applications such as the plastic and metal industries[1–3] and thus has been studied extensively.[4–16] Recently, significant attention has been drawn to the micro-scale flow in micro-electro-mechanical systems (MEMS).[17] The behavior of fluid flow in the micro-scale belongs to the slip flow regime and greatly differs from the traditional flow.[17] For the flow in the slip regime, the fluid motion still obeys the Navier–Stokes equations, but with slip velocity or temperature as its boundary conditions (BCs). In addition, partial velocity slip over a moving surface occurs for fluids with particulates such as emulsions, suspensions, foams, and polymer solutions.[18] Studies on magnetohydrodynamic (MHD) flow have been conducted under various conditions for an electrically conductive fluid flowing over a moving sheet.[19–24] Andersson[19] proposed an exact solution of the Navier–Stokes equations for MHD flow. MHD flow over a stretching sheet[20–24] has been investigated under conditions such as permeable surface, non-Newtonian fluid, and slip viscous flow. Flow over a stretching or shrinking sheet considering second order velocity slip was investigated.[25,26] It should be noted that all the previous works focused on a prescribed wall moving velocity. Flow over a stretching surface with an arbitrary surface velocity was first studied by Weidman and Magyari.[26] Following this interesting work, the studies were extended to shrinking surface,[27] second grade non-Newtonian fluid,[28] and micropolar fluid.[29] The MHD slip flow, however, over a moving sheet including stretching or shrinking with an arbitrary surface velocity has not been reported. The objective of this Letter is to study the moving sheet problem with an arbitrary stretching or shrinking velocity of MHD fluids with slip conditions at the wall. The analytical solutions will be presented in a closed form, and different moving velocity distributions of the sheet will be demonstrated as examples to show the flow characteristics. The impact of mass transfer parameter, magnetic field, and slip parameter on the flow will be discussed as well.

Consider a steady two-dimensional laminar flow over a continuously shrinking/stretching sheet in an electrically conductive quiescent fluid. Let the x-axis run along the surface in the same direction as the movement of the sheet and the y-axis perpendicular to the moving surface. The sheet moves at the velocity $\lambda u_{\rm w}(x)$, where $u_{\rm w}(x)$ is positive for all values of x, and λ is the shrinking/stretching parameter with $\lambda > 0$ for stretching and $\lambda < 0$ for shrinking. The flow is within the slip regime with a slip velocity at the wall of $\gamma\frac{\partial u}{\partial y}$. The mass transfer velocity at the wall is $v_{\rm w} = v_{\rm w}(x)$, which will be determined in the following, and the subscript w denotes the conditions at the wall. A magnetic field with strength B is applied in the vertical direction (y-axis) and the induced magnetic field is neglected. The governing equations based on the boundary layer assumption for the slip flow of MHD fluids read[19,23,24]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}u, \tag{2}$$

with the BCs

$$u(x,0)=\lambda u_{\rm w}(x)+\gamma\frac{\partial u}{\partial y}, \tag{3a}$$

$$v(x,0)=v_{\rm w}(x), \tag{3b}$$

$$u(x,\infty)=0, \tag{3c}$$

where u and v are the velocity components of the fluid in the x and y directions, respectively, ν is the kinematic viscosity, ρ is the fluid density, σ is the electrical conductivity of the fluid, and γ is a positive proportional constant of the velocity slip. Equation

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(3a) specifies the surface moving velocity with a slip velocity. Equation (3b) specifies the corresponding fluid velocity in the vertical direction (y-direction) at the wall. Equation (3c) is the ambient fluid velocity, which is assumed to be zero for the current flow. As $u_{\rm w}(x)$ and $v_{\rm w}(x)$ depend on specific velocity distributions used for different flows, the general terms are used in BCs. Specific velocity distributions will be discussed in the examples. This group of equations is valid for the MHD flow with weak magnetic fields. In the following a closed form solution will be given firstly, and then examples of several typical sheet velocity distributions will be reported and discussed in detail.

The governing equations and the associated BCs (Eqs. (1)–(3)) can be transformed into a dimensionless form using a constant characteristic velocity U_0 and a length scale L. Given $U(X,Y) = u(x,y)/U_0$, $V(X,Y) = v(x,y)/U_0$, $X = x/L$, $Y = y/L$ with $L = \nu/U_0$, the governing equations become dimensionless as follows:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{4}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} - MU, \tag{5}$$

with the BCs becoming

$$U(X,0) = \lambda U_{\rm w}(X) + \delta\frac{\partial U}{\partial Y}, \tag{6a}$$

$$V(X,0) = V_{\rm w}(X), \tag{6b}$$

$$U(X,\infty) = 0, \tag{6c}$$

where $\delta = \gamma/L$ is the dimensionless slip parameter, and $M = \frac{\sigma B^2 L}{\rho U_0}$ is the dimensionless parameter to characterize the MHD force.

The momentum and the continuity equations admit the following solutions

$$U(X,Y) = \Big(\frac{1}{1+\delta\beta}\Big)\lambda U_{\rm w}(X)e^{-\beta Y}, \tag{7}$$

$$V(X,Y) = \Big(-\beta + \frac{M}{\beta}\Big) + \frac{1}{(1+\delta\beta)\beta}\frac{\lambda dU_{\rm w}(X)}{dX}e^{-\beta Y}, \tag{8}$$

where β will be determined by the boundary condition of Eq. (6b), and it must be a positive number in order for the fluid flow to have a finite motion. Note that the analytical solution given in this study may not be the only solution to the problem and other solution branches may exist under the same BCs. In addition, in order for the solution to be valid, the wall mass transpiration condition must match the requirement of the vertical velocity at the wall.

The results in this work for the first time provide analytical solutions to the MHD fluid flow over a moving surface under the slip flow regime with an arbitrary boundary velocity. They not only generalize the results, but also provide many new solutions to the related problems. In addition, fluid flow problems with arbitrary BCs are very rare. Analytical solutions to these problems can provide a unique feature to construct benchmark problems for numerical code validations. The solutions provided in this work are different from the commonly available similarity solutions. The flows with an arbitrary boundary velocity provide a new flow field feature different from similarity solutions.

For the constant speed case $U_{\rm w}(X) = A$. The solutions, i.e., Eqs. (7) and (8), become

$$U(X,Y) = \Big(\frac{1}{1+\delta\beta}\Big)\lambda A e^{-\beta Y}, \tag{9}$$

$$V(X,Y) = -\beta + \frac{M}{\beta}. \tag{10}$$

Thus the flow becomes the asymptotic velocity profile over a plate with mass transfer in a direction perpendicular to the moving sheet. The fluid velocity near the wall along the sheet $U(X,Y)$ decreases with the increase of the slip parameter δ. The mass transfer velocity at the wall is always constant, thus let the mass transfer velocity at the wall be a constant s and Eq. (10) becomes

$$V(X,Y) = -\beta + \frac{M}{\beta} = s, \tag{11}$$

which is equivalent to

$$\beta^2 + s\beta - M = 0. \tag{12}$$

The solution for β is

$$\beta = \frac{-s \pm \sqrt{s^2+4M}}{2}. \tag{13}$$

Because only positive β is physically meaningful, only one solution exists. The asymptotic suction solution without magnetic field can be derived by having $M = 0$ in Eqs. (9) and (10). For the current flow configuration, due to the effects of the magnetic field, the velocity damps faster with less penetration to the fluid. Also unlike the conventional asymptotic flow profiles, solutions exist for both mass suction and mass injection. A special solution for an impermeable sheet is given by $\beta = \sqrt{M}$. Further, wall mass transpiration is required for the solutions to exist compared with the Sakiadis flow.[4]

For the linear velocity case $U_{\rm w}(X) = X$. The solution will be

$$U(X,Y) = \Big(\frac{1}{1+\delta\beta}\Big)\lambda X e^{-\beta Y}, \tag{14}$$

$$V(X,Y) = \Big(-\beta + \frac{M}{\beta}\Big) + \frac{\lambda}{(1+\delta\beta)\beta}e^{-\beta Y}. \tag{15}$$

The mass suction velocity at the wall is a constant based on Eq. (15) and it gives

$$V_{\rm w}(X) = \Big(-\beta + \frac{M}{\beta}\Big) + \frac{\lambda}{(1+\delta\beta)\beta} = s, \tag{16}$$

where s is the mass transfer parameter. The solution for a slip MHD flowing over a stretching sheet ($\lambda = 1$) has been discussed[23] and is consistent with the solution presented here. For the stretching sheet problem, only one solution exists; for the shrinking sheet problem ($\lambda = -1$), multiple solutions can exist under the condition of mass suction at the wall.[24] An interesting case exists for an impermeable moving sheet with $s = 0$. For this case, it requires

$$\delta\beta^3 + \beta^2 - \delta M\beta - M - \lambda = 0. \tag{17}$$

The solution only exists for a certain domain of the three control parameters. Equation (17) is a complete cubic algebraic equation, which can be converted to an incomplete cubic equation

$$\theta^3 + r\theta + t = 0, \tag{18}$$

where $\theta = \beta + \frac{1}{3\delta}$, $r = -\frac{1}{3}(\frac{1}{\delta})^2 - M$, and $t = \frac{2}{27}(\frac{1}{\delta})^3 + \frac{M}{3\delta} - \frac{\lambda+M}{\delta}$. The roots for the incomplete cubic solution are

$$\begin{aligned} \theta_1 &= A + B, \\ \theta_{2,3} &= -\frac{1}{2}(A+B) \pm i\frac{\sqrt{3}}{2}(A-B), \end{aligned} \tag{19}$$

where $A = \sqrt[3]{-\frac{t}{2} + \sqrt{D}}$, $B = \sqrt[3]{-\frac{t}{2} - \sqrt{D}}$, $i^2 = -1$, and $D = (\frac{r}{3})^3 + (\frac{t}{2})^2$. Based on the flow configuration, only positive real roots of β are physically feasible solutions. The number of real roots of Eq. (11) depends on the value of: $D > 0$, one real root and two complex conjugate roots; $D < 0$, three real roots; and $D = 0$, one simple real and two two-fold real roots, or if $r = t = 0$, one three-fold real root. Based on the above discussion, we have

$$D = -\Big(\frac{\frac{1}{3}(\frac{1}{\delta})^2 + M}{3}\Big)^3 + \Big(\frac{\frac{2}{27}(\frac{1}{\delta})^3 + \frac{M}{3\delta} - \frac{\lambda+M}{\delta}}{2}\Big)^2. \tag{20}$$

The solution domain of β for different combinations of M and δ as a function of the wall moving parameter is shown in Fig. 1. Some interesting findings are demonstrated in Fig. 1. For $M = 0$, one solution exists for a stretching sheet with different slip parameters and there is no solution for a shrinking sheet. For any positive value of M, solutions exist for both stretching and shrinking. For a stretching sheet, β decreases with the increase of the slip parameter but increases with M and λ. It is interesting that at $\lambda = 0$, the effects of slip disappear and $\beta = \sqrt{M}$. For a shrinking sheet, the solution domain becomes quite complicated. For $-M < \lambda < 0$, there is one solution; for $\lambda_{\rm Cr} < \lambda < -M$, where $\lambda_{\rm Cr}$ is a critical shrinking parameter determined by M and δ, two solutions exist. When $\lambda = \lambda Cr$ there is one solution, and no solution for $\lambda < \lambda_{\rm Cr}$. The critical value of $\lambda_{\rm Cr}$ is determined by

$$\begin{aligned} D = &-\Big(\frac{\frac{1}{3}(\frac{1}{\delta})^2 + M}{3}\Big)^3 + \Big(\frac{\frac{2}{27}(\frac{1}{\delta})^3 + \frac{M}{3\delta} - \frac{\lambda+M}{\delta}}{2}\Big)^2 \\ &= 0. \end{aligned} \tag{21}$$

Solving Eq. (21) yields

$$\lambda_{\rm Cr} = -\frac{2}{27}\frac{9\delta^2 M + \sqrt{(1+3\delta^2 M)}^3 - 1}{\delta^2}. \tag{22}$$

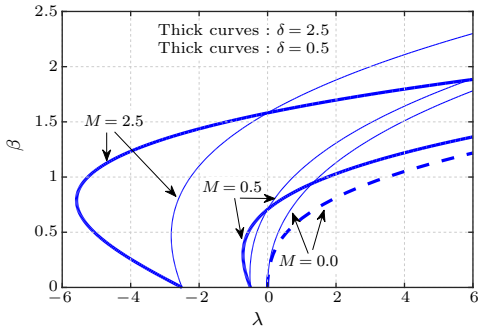

Fig. 1. Solution domain of β for the impermeable sheet with different slip and magnetic parameters as a function of the wall stretching/shrinking parameter.

For the quadratic velocity case $U_{\rm w}(X) = X + AX^2$. The solutions read

$$U(X,Y) = \Big(\frac{1}{1+\delta\beta}\Big)\lambda(X + AX^2)e^{-\beta Y}, \tag{23}$$

$$\begin{aligned} V(X,Y) = &\Big(-\beta + \frac{M}{\beta}\Big) + \frac{\lambda}{(1+\delta\beta)\beta}(1 \\ &+ 2AX)e^{-\beta Y}. \end{aligned} \tag{24}$$

The mass transfer velocity at the wall is reduced from Eq. (24) to

$$\begin{aligned} V_{\rm w}(X) &= \Big(-\beta + \frac{M}{\beta}\Big) + \frac{\lambda}{(1+\delta\beta)\beta}(1+2AX) \\ &= s(X), \end{aligned} \tag{25}$$

where s is the parameter to characterize mass transfer at the wall, and β is obtained from Eq. (25). Equation (25) shows that the mass transfer at the wall depends on the distance from the slot, the strength of the magnetic field, and the slip parameter. Mass suction or mass injection can occur for either a positive value or negative value of A. The magnetic field can change the direction of the mass transfer from mass suction to mass injection. Although there are multiple solutions for the linear velocity case with a shrinking sheet, which is a special case of the quadratic velocity, multiple solutions generally do not exist due to the following simple proof. Assuming that there are two solutions for the same mass transfer $s(X)$ as β_1 and β_2, it requires

$$\begin{aligned} &\Big(-\beta_1 + \frac{M}{\beta_1}\Big) + \frac{\lambda}{(1+\delta\beta_1)\beta_1}(1+2AX) \\ &= \Big(-\beta_2 + \frac{M}{\beta_2}\Big) + \frac{\lambda}{(1+\delta\beta_2)\beta_2}(1+2AX). \end{aligned} \tag{26}$$

Matching the coefficients of X and the constant, we obtain

$$\Big(-\beta_1+\frac{M}{\beta_1}\Big)+\frac{\lambda}{(1+\delta\beta_1)\beta_1} =\Big(-\beta_2+\frac{M}{\beta_2}\Big)+\frac{\lambda}{(1+\delta\beta_2)\beta_2}, \tag{27}$$

$$\frac{1}{(1+\delta\beta_1)\beta_1}=\frac{1}{(1+\delta\beta_2)\beta_2}. \tag{28}$$

Equation (27) may have two solutions for a shrinking sheet problem with a negative λ. However, the two distinct solutions cannot satisfy Eq. (28) because function $f(t)=t+\delta t^2$ is a single-valued monotonically function for a positive value of t. Moreover, for this case, there is no solution for an impermeable moving sheet due to the dependence of mass transfer on the distance from the slot.

For the power law case $U_{\rm w}(X)=X^n$. The solution will be

$$U(X,Y)=\Big(\frac{1}{1+\delta\beta}\Big)\lambda X^n e^{-\beta Y}, \tag{29}$$

$$V(X,Y)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}nX^{n-1}e^{-\beta Y}. \tag{30}$$

For this flow configuration, the mass transfer velocity at the wall is $V_{\rm w}(X)=(-\beta+\frac{M}{\beta})+\frac{\lambda}{(1+\delta\beta)\beta}nX^{n-1}=s(X)$. As reported for the non-slip self-similar boundary layer flow over a moving sheet with power-law velocity for $M=0$,[8] the mass transfer velocity is proportional to $x^{(n-1)/2}$. The mass transfer of the slip MHD flow has distinctive features as compared to self-similar non-slip flow. The magnetic parameter M has an important impact on the direction of the mass transfer at the wall of the moving sheet. Changing M can change the critical distance from the slot, $X_{\rm Cr}$, with the mass transfer changing from suction to injection. The critical distance from the slot $X_{\rm Cr}$ can be obtained as

$$X_{\rm Cr}=\Big(\frac{(1+\delta\beta)(\beta^2-M)}{\lambda n}\Big)^{\frac{1}{n-1}}, \tag{31}$$

which requires that $\beta^2-M>0$ for a positive value of λn and $\beta^2-M<0$ for a negative value of λn. The increases in slip parameter δ increase $X_{\rm Cr}$. However, the increase of M can reduce $X_{\rm Cr}$ for a positive value of λn, but can increase $X_{\rm Cr}$ for a negative value of λn.

For the exponential velocity case $U_{\rm w}(X)=e^{\rm kX}$, where k is a real number. The solutions become

$$U(X,Y)=\Big(\frac{1}{1+\delta\beta}\Big)\lambda e^{\rm kX}e^{-\beta Y}, \tag{32}$$

$$V(X,Y)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}ke^{\rm kX}e^{-\beta Y}. \tag{33}$$

The wall mass transfer velocity becomes

$$V_{\rm w}(X)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}ke^{\rm kX}. \tag{34}$$

Mass suction or injection at the wall can exist, depending on the combined effect of the magnetic parameter M, the slip parameter, value of k, and the movement (λ) of the sheet. Equation (34) can be used to derive the mass transfer velocity for various flows at the moving sheet. A similar mass transfer feature to the power-law velocity profiles can be found in Eq. (34). Due to the variation of the distance X, the variation of mass transfer depends on the combination of λ and k. The critical distance $X_{\rm Cr}$ can be obtained as

$$X_{\rm Cr}=\frac{1}{k}\ln\Big(\frac{(1+\delta\beta)(\beta^2-M)}{\lambda k}\Big). \tag{35}$$

For the periodic wall velocity case $U_{\rm w}(X)=sin(kX)$, where k is a real number. The solutions are

$$U(X,Y)=\Big(\frac{1}{1+\delta\beta}\Big)\lambda\sin(kX)e^{-\beta Y}, \tag{36}$$

$$V(X,Y)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}k\cos(kX)e^{-\beta Y}. \tag{37}$$

The wall mass transpiration rate is

$$V_{\rm w}(X)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}k\cos(kX). \tag{38}$$

It oscillates around $(-\beta+\frac{M}{\beta})$ with a magnitude of $\frac{|\lambda k|}{(1+\delta\beta)\beta}$. Either suction or injection at the moving sheet can occur, depending on the combined effect of magnetic parameter M, slip parameter δ, and the stretching or shrinking of the sheet. If $>\beta^2+\frac{|\lambda k|}{(1+\delta\beta)}$, then mass injection will always happen at the wall. Otherwise, either mass injection or mass suction at the wall is possible. The peak suction rate along the sheet is $(\frac{|\lambda k|}{(1+\delta\beta)\beta}+\beta-\frac{M}{\beta})$ and the peak mass injection rate is $(\frac{|\lambda k|}{(1+\delta\beta)\beta}-\beta+\frac{M}{\beta})$.

For the bilinear wall velocity case $U_{\rm w}(X)=\frac{X}{1+kX}$, where k is a real number. The solutions become

$$U(X,Y)=\Big(\frac{1}{1+\delta\beta}\Big)\lambda\frac{X}{1+kX}e^{-\beta Y}, \tag{39}$$

$$V(X,Y)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}\frac{1}{(1+kX)^2}e^{-\beta Y}. \tag{40}$$

The wall mass transpiration rate is

$$V_{\rm w}(X)=\Big(-\beta+\frac{M}{\beta}\Big)+\frac{\lambda}{(1+\delta\beta)\beta}\frac{1}{(1+kX)^2}. \tag{41}$$

For the bilinear sheet velocity, either mass suction or mass injection can occur, and the solutions exist for an arbitrary value of k. For a shrinking sheet, if $>\beta^2+\frac{|\lambda|}{(1+\delta\beta)}$, then mass injection will always occur at the wall. For a stretching sheet, if $M<\beta^2-\frac{|\lambda|}{(1+\delta\beta)}$,

then mass suction will always happen at the wall. The wall mass transfer velocity peaks at $X = 0$ and decreases monotonically to $(-\beta + \frac{M}{\beta})$ as $X \to \infty$. When $X \to \infty$, the flow approaches to the asymptotic suction flow profiles.

In summary, seven examples of different surface moving velocities are discussed and analyzed. Some of these velocity distributions are directly related to practical applications. For example, a constant wall velocity often occurs in the steel sheet forming and hot rolling process.[1−3,5] Linear and power-law velocity distributions come from the polymer or plastic sheet stretching near the slot in the production process depending on the rheology behavior of the material.[30−32] The applications of periodically stretching wall motions have been discussed and studied in Ref. [33] and these can be used to analyze the flow over living systems with stretchable and flexible walls. The rest velocity distributions are of great theoretical interest to reveal new flow behavior under different flow configurations. As mentioned above, flows under slip regimes have wide applications in micro and nano scale problems[17] and flows with particulates.[18] The MHD effects find many applications in controlling the fluid motion of electrically conducting hydrodynamic fluids used in industries and research laboratories.[34] Last but not least, the current results can provide various analytical solutions for many different flow configurations as benchmark problems for numerical code validation.

We have reported a general solution for the slip MHD flow over a stretching or shrinking sheet with an arbitrary velocity. The analytical solution exists when a prescribed mass transfer velocity at the wall is applied, which is determined by the moving velocity of the stretching or shrinking sheet. The flow is in general not self-similar and has features different from non-slip flow over a moving sheet. Various velocity profiles of the moving sheet including constant, linear, quadratic, power-law, exponential, periodic, and bilinear velocity distributions are employed as examples of such slip MHD flows over a moving sheet. The equations governing the velocity distribution of the flow are obtained and discussed in detail. The results show that the combined impact of the magnetic field, the slip parameter, and the movement of the sheet determines whether the analytical solution exists and also affects the mass suction or injection at the wall. Interesting flow behaviors are found for constant and linear velocity profiles at the sheet, where solutions can exist for an impermeable moving sheet.

References

[1] Fisher E G 1976 *Extrusion of Plastics* (New York: Wiley)
[2] Altan T, Oh S and Gegel H 1979 *Metal Forming: Fundamentals and Applications* (Metals Park, OH: American Society of Metals)
[3] Jaluria Y 2018 *Continuous Materials Processing. In Advanced Materials Processing and Manufacturing* (Cham: Springer) p 107
[4] Sakiadis B C 1961 *AIChE J.* **7** 26
[5] Tsou F K, Sparrow E M and Goldstain R J 1967 *Int. J. Heat Mass Transfer* **10** 219
[6] Crane L J 1970 *Z. Angew. Math. Phys.* **21** 645
[7] Chen C K and Char M I 1988 *J. Math. Anal. Appl.* **135** 568
[8] Fang T 2008 *Int. J. Heat Mass Transfer* **51** 2199
[9] Magyari E and Keller B 2000 *Eur. J. Mech.* B **19** 109
[10] Liao S J 2005 *Int. J. Heat Mass Transfer* **48** 2529
[11] Wang C Y 1991 *Annu. Rev. Fluid Mech.* **23** 159
[12] Fang T, Zhang J and Yao S 2009 *Chin. Phys. Lett.* **26** 014703
[13] Asghar S, Afzal S and Ahmad A 2017 *Chin. Phys.* B **26** 014704
[14] Nadeem S and Hayat T 2016 *Chin. Phys.* B **25** 114701
[15] Bhattacharyya K, Hayat T and Alsaedi A 2014 *Chin. Phys.* B **23** 124701
[16] Ahmad A, Asghar S and Alsaedi A 2014 *Chin. Phys.* B **23** 074401
[17] Gal-el-Hak M 1999 *J. Fluids Eng. Trans. ASME* **121** 5
[18] Yoshimura A and Prudhomme R K 1988 *J. Rheol.* **32** 53
[19] Andersson H I 1995 *Acta Mech.* **113** 241
[20] Pop I and Na T Y 1998 *Mech. Res. Commun.* **25** 263
[21] Liao S J 2003 *J. Fluid Mech.* **488** 189
[22] Sajid M, Hayat T and Javed T 2007 *Nonlinear Dyn.* **51** 259
[23] Fang T G, Zhang J and Yao S S 2009 *Commun. Nonlinear Sci. Numer. Simul.* **14** 3731
[24] Fang T G, Zhang J and Yao S S 2010 *Chin. Phys. Lett.* **27** 124702
[25] Fang T G and Aziz A 2010 *Z. Naturforsch.* A **65** 1087
[26] Weidman P D and Magyari E 2010 *Acta Mechanica* **209** 353
[27] Fang T G ang Zhong Y F 2010 *Commun. Nonlinear Sci. Numer. Simul.* **15** 3768
[28] Ahmad A and Asghar S 2011 *Appl. Math. Lett.* **24** 1905
[29] Magyari E and Kumaran V 2010 *Commun. Nonlinear Sci. Numer. Simul.* **15** 3237
[30] Vleggaar J 1977 *Chem. Eng. Sci.* **32** 1517
[31] Ali M E 1995 *Int. J. Heat Fluid Flow* **16** 280
[32] Dean M M 1980 *Annu. Rev. Fluid Mech.* **12** 365
[33] Lage J L and Bejan A 1990 *ASME J. Heat Transfer* **112** 92
[34] Galtier S 2016 *Introduction to Modern Magnetohydrodynamics* (Cambridge: Cambridge University Press)