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Renormalization of Tripartite Entanglement in Spin Systems with Dzyaloshinskii–Moriya Interaction *

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Quantum entanglement represents a fundamental feature of quantum many-body systems. We combine tripartite entanglement with quantum renormalization group theory to study the quantum critical phenomena. The Ising model and the Heisenberg XXZ model in the presence of the Dzyaloshinskii–Moriya interaction are adopted as the research objects. We identify that the tripartite entanglement can signal the critical point. The derivative of tripartite entanglement shows singularity as the spin chain size increases. Furthermore, the intuitive scaling behavior of the system selected is studied and the result allows us to precisely quantify the correlation exponent by utilizing the power law.

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Entanglement[1] is regarded as an indispensable resource in quantum information and quantum computation. The study on entanglement includes but is not limited to the followings subject: entanglement distillation,[2] preserving and recovering entanglement,[3,4] entanglement detection,[5] entanglement in nonlinear systems,[6,7] and the relationship between entanglement and some basic physics problems, such as quantum phase transitions (QPTs).[8] QPTs occur at absolute zero temperature. It signifies the sudden change in the ground state of a many-body system when a physical parameter such as magnetic field, chemical composition, or pressure is adjusted slightly. QPTs can be used to restore qubits and to generate quantum entanglement in quantum information processing.

As one of the important issues in the field of quantum information science, using entanglement to detect the occurrence of QPT has already received a great deal of attention. Osterloh *et al.*[9] found that entanglement shows scaling behavior in the vicinity of the quantum phase transition point. Osborne and Nielsen[10] pointed out that the next-nearest neighbor entanglement shows a maximum value at the transition point. Gu *et al.*[11] gave the scaling behavior of entanglement. Basu *et al.*[12] investigated the evolution of entanglement towards criticality in the XY model. We can find that not only is entanglement a more valuable resource, but also that it plays a crucial role in the signature of QPTs. These results greatly promote the understanding of QPTs.

In addition, renormalization group, as a powerful tool, can help researchers to study the critical phenomena in multipartite systems. Wilson[13] introduces it from quantum field theory and has great success in explaining the critical phenomena. In recent years, combining the quantum renormalization group (QRG) method and the quantum entanglement theory to study QPTs has attracted a great deal of interest.[14–16] Ma *et al.*[17,18] have found that the first derivative of the entanglement exhibits nonanalytic behavior at the critical point. Kargarian *et al.*[19–21] showed that the nonanalytic behavior of entanglement is mirrored from the correlation length exponent.

The above-mentioned results are very important and will promote the advance of quantum information theory. However, we also notice that these studies through QRG theory mainly concentrate on bipartite entanglement. Researchers generally investigate the properties of the reduced density matrix. Some immediate questions thus arise. Is multipartite entanglement suitable for detecting QPTs when QRG theory is adopted? Could multipartite entanglement reflect the quantum criticality of the spin model in the thermodynamic limit? We will use tripartite entanglement to investigate the spin model with the Dzyaloshinskii–Moriya (DM) interaction.[22,23] The DM interaction can be used to describe Cs_2CuCl_4, Cu benzoate, and Kagom antiferromagnet. Many physical systems have been simulated by this kind of interaction. Since DM interaction connects magnetization to polarizations, it is also a key element in multiferroics.[24,25] However, most studies on DM interaction mainly concentrate on its effect on entanglement and are not the detection of QPTs. In this Letter, we employ QRG to deal with QPT problems in the Ising and XXZ models with DM interaction. The Ising model is one of the simplest models that contain many novel and interesting physical connotations. It can be used to describe many physical phenomena, such as order-disorder transformation in alloys.[26] The XXZ model can be used to construct interaction-free subspaces and can achieve universal quantum computation.[27] Thus it is meaningful to study the characteristics of these models in

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the QPT.

For a given quantum state ρ, the negativity (Ne) is[28,29]

$$Ne(\rho) = \frac{\|\rho^{T_i}\|_1 - 1}{2}, \tag{1}$$

where $\|\rho^{T_i}\|$ indicates the trace norm of ρ^{T_i}, and T_i denotes the partial transpose of density matrix ρ with respect to party i. It is well known that the partial transpose of a density matrix does not change the trace of a density matrix, thus we can easily find that the negativity of ρ equals the sum of the absolute values of the negative eigenvalues of ρ^{T_i}.

The tripartite negativity of a state ρ reads[30]

$$N_3(\rho) = (Ne_{\mathrm{A-BC}} Ne_{\mathrm{B-AC}} Ne_{\mathrm{C-AB}})^{1/3}, \tag{2}$$

where the bipartite negatives are defined by Eq. (1), and $Ne_{\mathrm{I-JK}} = -\sum_i \lambda_i$, with λ_i being the negative eigenvalues of ρ^{T_i}.

The Hamiltonian of the Ising model with z-direction DM interaction is[14]

$$H = \frac{J}{4}\sum_i^N [\sigma_i^z \sigma_{i+1}^z + D(\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x)], \tag{3}$$

where J is the exchange interaction, D is the DM interaction, and $\sigma^{x,y,z}$ are the usual Pauli operators. Based on the Kadanoffs scheme, the lattice is divided into blocks and then the Hamiltonian is separated as interblock and intrablock. We take three-site as one block in this work. The Hamiltonian now can be divided as the block Hamiltonian

$$H_{\mathrm{B}} = \frac{J}{4}\sum_i^{N/3} [\sigma_{1,I}^z \sigma_{2,I}^z + \sigma_{2,I}^z \sigma_{3,I}^z + D(\sigma_{1,I}^x \sigma_{2,I}^y - \sigma_{1,I}^y \sigma_{2,I}^x + \sigma_{2,I}^x \sigma_{3,I}^y - \sigma_{2,I}^y \sigma_{3,I}^x)]$$

and the interacting Hamiltonian

$$H_{\mathrm{BB}} = \frac{J}{4}\sum_i^{N/3} [\sigma_{3,I}^z \sigma_{1,I+1}^z + D(\sigma_{3,I}^x \sigma_{1,I+1}^y - \sigma_{3,I}^y \sigma_{1,I+1}^x)].$$

To preserve only the ground-state parts we built a projection operator $P = \prod_{i=1}^{N/3} P_0^L = \prod_{i=1}^{N/3} (|\Uparrow\rangle_L\langle\varphi_0| + |\Downarrow\rangle_L\langle\varphi_0|)$. The relation between the original Hamiltonian and the effective Hamiltonian is associated by the projection operator, which is constructed by the two lowest eigenstates of three-site interblock Hamiltonian H_{B},

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2q(1+q)}}[2D|\downarrow\uparrow\uparrow\rangle + i(1+q)|\uparrow\downarrow\uparrow\rangle \\ &\quad - 2D|\uparrow\uparrow\downarrow\rangle] \\ |\psi_0'\rangle &= frac1\sqrt{2q(1+q)}[2D|\downarrow\downarrow\uparrow\rangle + i(1+q)|\downarrow\uparrow\downarrow\rangle \\ &\quad - 2D|\uparrow\downarrow\downarrow\rangle], \end{aligned} \tag{4}$$

where $q = \sqrt{1+8D^2}$. The full Hamiltonian is projected onto these eigenstates, which gives the effective renormalized Hamiltonian[14]

$$H^{\mathrm{eff}} = H_0^{\mathrm{eff}} + H_1^{\mathrm{eff}} = P^\dagger H_{\mathrm{B}} P + P^\dagger H_{\mathrm{BB}} P. \tag{5}$$

The effective Hamiltonian structure is similar to Eq. (3),

$$H^{\mathrm{eff}} = \frac{J'}{4}[\sum_i^{N/3} \sigma_i^z \sigma_{i+1}^z + D'(\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x)], \tag{6}$$

where the renormalized parameters J' and D' are

$$\begin{aligned} J' &= J(\frac{1+q}{2q})^2, \\ D' &= \frac{16D^3}{(1+q)^2}, \end{aligned} \tag{7}$$

with $q = \sqrt{1+8D^2}$.

The density matrix of Eq. (6) is defined as

$$\rho = |\psi_0\rangle\langle\psi_0| = \frac{1}{2q(1+q)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4D^2 & 2Di(q+1) & 0 & -4D^2 & 0 & 0 & 0 \\ 0 & -2Di(q+1) & (q+1)^2 & 0 & 2Di(q+1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4D^2 & -2Di(q+1) & 0 & 4D^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

After some algebra, we can derive the tripartite entanglement of Eq. (8). Due to the tediousness of the expressions, the results are illustrated through numerical calculations. Figure 1 shows the evolution of tripartite entanglement N_3 under different QRG steps, and also its derivative. We observe that the tripartite entanglements will cross each other at the critical point $D_{\mathrm{c}} = 1$, corresponding to the DM interaction, which is the same as the exchange interaction. After enough times of iterations N_3 develops two saturated values, one is 0 and the other is 0.908, which represent the gapped phase and the saturated chiral phase, respectively. At the critical point it can switch from 0 to 0.908. This indicates that the tripartite entanglement can effectively detect the occurrence of QPTs. In Fig. 1(b), the derivative of tripartite entanglement exhibits singular behavior when the system size increases. This is the result of discontinuous change

of N_3 at $D_c = 1$.[19] It is worth remarking that the plot for $0 \leq D < 1$ becomes more pronounced as the size increases while the relevant change in the region $D > 1$ is rather small. Hence, it is important to point out that the system undergoes qualitative change in the gapped phase.

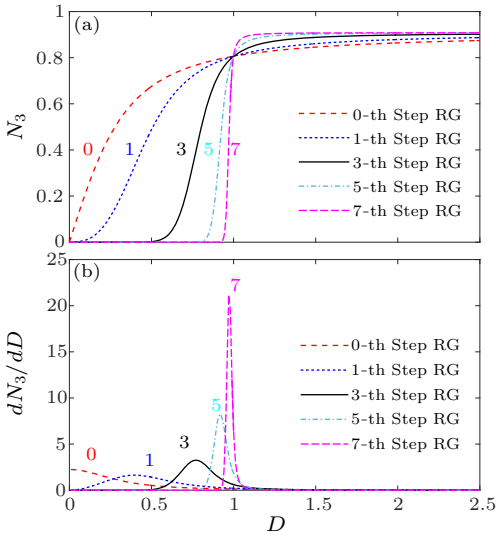

Fig. 1. (a) The change of tripartite entanglement of the Ising model versus D under different steps of QRG iterations. (b) The first derivative of tripartite entanglement of the Ising model versus D for different QRG iterations. All the parameters are dimensionless.

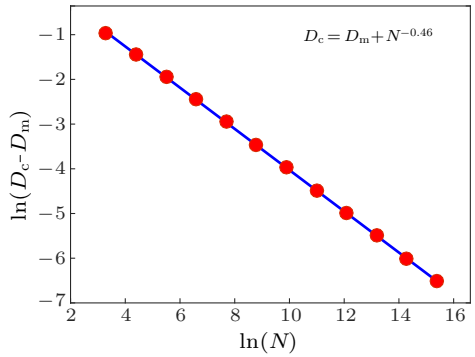

Fig. 2. The scaling behavior of $\ln(D_c - D_m)$ in terms of system size $\ln(N)$ of the Ising model.

Figure 2 displays the scaling behavior of $\ln(D_c - D_m)$ with respect to $\ln(N)$, where D_m is the position of the maximum values of $\frac{dN_3}{dD}$. The linear behavior is shown in this figure. Numerical results show that the tripartite entanglement exponent relation is $D_c = D_m + N^{-0.46}$. Thus the critical behavior of the Ising model can be characterized by the tripartite entanglement in the vicinity of the critical point.

The Hamiltonian of the XXZ model with DM interaction in the z-direction reads[19]

$$H = \frac{J}{4}\sum_{i}^{N}[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z + D(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)], \quad (9)$$

where J is the exchange interaction, Δ is the anisotropy parameter, D is the DM interaction, and $\sigma_i^\tau\{\tau = (x, y, z)\}$ are Pauli operators at site i. After adopting the similar QRG procedure, the effective Hamiltonian structure is similar to Eq. (9),

$$H^{\text{eff}} = \frac{J'}{4}\sum_{i}^{N/3}[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta'\sigma_i^z\sigma_{i+1}^z + D'(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)], \quad (10)$$

where J', Δ' and D' read the renormalized equations

$$J' = \frac{4J(1+D^2)}{q^2},$$
$$\Delta' = \frac{\Delta}{1+D^2}\left(\frac{\Delta+q^2}{4}\right)^2,$$
$$D' = D. \quad (11)$$

with $q = \sqrt{\Delta^2 + 8(1+D^2)}$.

The two degenerate ground states are

$$|\varphi_0\rangle = \frac{1}{\sqrt{2q(\Delta+q)(1+D^2)}}[2(D^2+1)|\downarrow\downarrow\uparrow\rangle - (1-iD)(q+\Delta)|\downarrow\uparrow\downarrow\rangle - 2[2iD+(D^2-1)]|\uparrow\downarrow\downarrow\rangle],$$
$$|\varphi_0'\rangle = \frac{1}{\sqrt{2q(\Delta+q)(1+D^2)}}[2(D^2+1)|\downarrow\uparrow\uparrow\rangle - (1-iD)(q+\Delta)|\uparrow\downarrow\uparrow\rangle - 2[2iD+(D^2-1)]|\uparrow\uparrow\downarrow\rangle]. \quad (12)$$

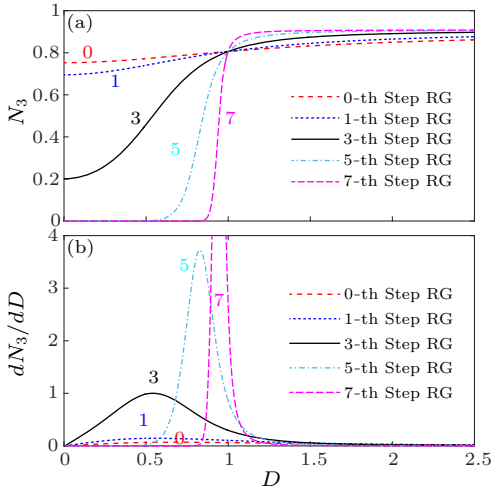

Fig. 3. (a) The change of tripartite entanglement of the XXZ model versus D in terms of QRG iterations when $\Delta = 2$. (b) The first derivative of tripartite entanglement of the XXZ model versus D in terms of QRG iterations when $\Delta = 2$. All the parameters are dimensionless.

Similarly, we can also obtain the density matrix of Eq. (12), and then calculate its tripartite entanglement. However, we do not show it here for the tedious expressions. In Fig. 3, we show the performances of the tripartite entanglement of the XXZ model. The tripartite entanglement develops two saturated values: one is zero for $0 \leq D < 1$ and the other is 0.9077 for $D > 1$. The tripartite entanglement is suppressed in the Ising phase, while in the gapless phase the entanglement is enhanced by the DM interaction. The

singularity is detected in Fig. 3(b) when the derivative is related to D.

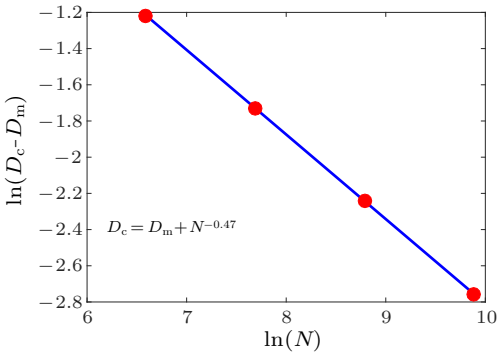

Fig. 4. The scaling behavior of $\ln(D_{\rm c} - D_{\rm m})$ in terms of system size $\ln(N)$ for the XXZ model.

The size dependence of $\ln(D_{\rm c} - D_{\rm m})$ is shown in Fig. 4, which demonstrates a scaling behavior. This model follows a power law $D_{\rm c} = D_{\rm m} + N^{-0.47}$ with entanglement exponent $\theta = -0.47$. Since the entanglement exponent is related to the correlation length exponent by $\theta = 1/\nu$,[17] the singular property and the scaling behavior of the model can be reflected by it.

In conclusion, we have investigated the effect of DM interaction on quantum critical phenomena in two kinds of spin systems. Tripartite entanglement is adopted as a quantity to measure the total entanglement in a three-site block state. On the one hand, we show that the critical behavior of the system can be described by the tripartite entanglement. On the other hand, our results clearly demonstrate that the derivative of the tripartite entanglement is accompanied by the singular characteristic in the vicinity of the critical point. Our finding reveals the ability of the tripartite entanglement to accurately detect the critical point, and more importantly, provides the power law by which the correlation length exponent of a spin system is totally determined.

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