Modulational Instability of Dipolar Bose–Einstein Condensates in Optical Lattices with Three-Body Interactions *

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Motivated by the recent experiment [Nature 530 (2016) 194] in which a stable droplet in a dipolar quantum gas has been created by the interaction-induced instability, we focus on the modulation instability of an optically-trapped dipolar Bose–Einstein condensate with three-body interaction. Within the mean-field level, we analytically solve the discrete cubic-quintic Gross–Pitaevskii equation with dipole–dipole interaction loaded into a deep optical lattice and show how combined effects of the three-body interaction and dipole–dipole interaction on the condition of modulational instability. Our results show that the interplay of the three-body interaction and dipole–dipole interaction can dramatically change the modulation instability condition compared with the ordinary Gross–Pitaevskii equation. We believe that the predicted results in this work can be useful for the future possible experiment of loading a Bose–Einstein condensate of ^{164}Dy atoms with strong magnetic dipole–dipole interaction into an optical lattice.

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Recent experiments with ultracold dysprosium[1] have reported a surprising observation: a Bose–Einstein condensate (BEC) of dysprosium atoms can spontaneously form a periodic array of droplets, as one tunes the interplay between the s-wave and dipole–dipole interactions (DDIs) with the Feshbach resonance. The stable formation of such droplet crystal eludes the traditional theories based on the competition between the short-ranged s-wave interatomic interaction and the long-ranged DDI. In particular, its explanation suggests inclusion of new nonlinear terms which represent a high order dependence on density in the mean-field description represented by the Gross–Pitaevskii (GP) equation. So far two such terms have been proposed, one from the three-body interaction (TBI),[2,3] and the other stems from the Lee–Huang–Yang effect.[4–6]

Motivated by the above theoretical and experimental interests, we instead consider a quantum gas with ultracold dysprosium trapped into a deep optical lattice, which contains both the two-body and three-body interactions as well as DDIs. Similar to the key scenario in Refs. [1,7,8], we use the Feshbach resonance technology to tune the two-body interaction to zero. We consider a scenario where the TBI is tuned from positive to negative value, thus inducing modulational instability (MI). Notice that MI is a general feature in discrete as well as continuum nonlinear wave equations, which is referred to a process of a plane wave becoming nonlinear growth due to a perturbation and is believed to be one of the most prevalent instabilities in nature. The MI has been found in diverse nonlinear systems ranging from deep water[9] and nonlinear optics to the ultracold atomic gases.[10–12] Meanwhile, the MI has been recognized as responsible for dephasing and localization phenomena in BECs in the presence of an optical lattice.[13–15] Owing to its generality, it is of natural interest to investigate MI in this novel nonlinear system.

In this Letter, we have studied the MI of a BEC of ^{164}Dy atoms with strong magnetic DDI and TBI trapped in an optical lattice. Within the mean-field level, we model such a system with the discrete cubic-quintic GP equation with DDI, where the cubic and quintic nonlinear terms represent the two-body and three-body interactions, respectively. In general, the process of the perturbations spontaneously seeding the nonlinear growth of a plane wave solution can be captured by MI. Using the linear stability analysis, we derive the MI criterion of our model system. Our results show that the interplay of the TBI and DDI can dramatically change the modulation instability condition compared with the ordinary GP equation.

The dynamics of dipolar BEC with TBI can be described by the Hamiltonian as follows:[16,17]

$$\begin{aligned} H = & \int d\boldsymbol{r}\Big[\frac{\hbar^2}{2m}|\nabla\Psi(\boldsymbol{r},t)|^2 + V_{\rm opt}(\boldsymbol{r})|\Psi(\boldsymbol{r},t)|^2 \\ & + \frac{g}{2}|\Psi(\boldsymbol{r},t)|^4 + \frac{g_3}{3}|\Psi(\boldsymbol{r},t)|^6\Big] \\ & + \int d\boldsymbol{r}d\boldsymbol{r}' V_{\rm dd}(\boldsymbol{r}-\boldsymbol{r}')|\Psi(\boldsymbol{r},t)|^2|\Psi(\boldsymbol{r}',t)|^2, \quad (1) \end{aligned}$$

where $\Psi(\boldsymbol{r},t)$ labels the condensate wave function, which is normalized to the total number of the condensate atoms N. The external potential reads $V_{\rm opt}(\boldsymbol{r}) = V_x\cos^2(K_x) + V_\perp(\cos^2(K_y) + \cos^2(K_z))$ with K being the wave vector of the lasers creating the optical lattice. We consider the chemical potential of the model system $\mu \ll V_\perp$ and $V_x \ll V_\perp$, under which the transverse degrees of freedom are frozen by the tight confinement. In such a way, our model

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can be treated as an effective one-dimensional geometry. Here $g = \frac{4\pi\hbar^2 a_s}{m}$ represents the two-body interatomic interaction with a_s being the s-wave scattering length. The dipolar interaction is in the form of $V_{dd}(\boldsymbol{r}) = N\frac{\mu_0\mu_d^2}{4\pi}\frac{1-3\cos^2\theta}{|r|^3}$, where μ_0 and $\mu_d = 9.93\mu_B$ are the vacuum permeability and the magnetic dipole moment of a ^{164}Dy atom respectively with μ_B being the Bohr magneton, and θ labels the angle between $\boldsymbol{r}$ and μ_d.

The term of the form $\frac{g_3}{3}|\Psi(\boldsymbol{r},t)|^6$ in Eq. (1) represents the TBI arising from triple collisions and is characterized by the three-body coupling constant within the context of pseudopotentials. We intend to remark a long history of research of the theoretical determination of the three-body coupling constant in a dilute BEC: (i) In 1959, Wu[18] first predicted a general form of the three-body coupling constant with $g_3 = 16\pi\hbar^2 a_s^4(4\pi - 3\sqrt{3})\log(C\sqrt{na_s^3})/m$ for a Bose gas of hard spheres with the constant C being only determined recently by Braaten and Nieto[19] using effective field theory. (ii) Very recently, many efforts have been made to identify experimentally accessible systems that have three-body or higher-order interactions. For example, Ref. [20] reported that a Bose–Hubbard model with two- and three-body interactions between nearest neighbors can be realized by polar molecules interacting via dipolar interactions driven by microwave fields.

In this work, we consider the situation where the optical lattice along the x direction is sufficiently large and the chemical potential is small compared with the interband gap. In such a way, the BEC is confined to the lowest Bloch band and the tight-binding approximation becomes valid. Thus we can rewrite the condensate wave function in the form of

$$\Psi(\boldsymbol{r},t) = \sqrt{N}\sum_{n=1}^{M} u_n(t)\,\Phi(\boldsymbol{r}), \tag{2}$$

where N is the total number of atoms, and $\Phi(\boldsymbol{r})$ is the condensate wave function localized in the site n with $\int \Phi_n\Phi_{n\pm1} = 0$ and $\int|\Phi_n|^2 = 1$. If we only consider nearest-neighbor sites of n, the dynamic of quasi-1D dipolar BEC in a deep optical lattice can be described by 1D GPE,

$$\begin{aligned} i\frac{d}{dt}u_n(t) = &-J(u_{n+1}+u_{n-1}) + U_0|u_n|^2u_n \\ &+ U_3|u_n|^4u_n + U_{dd}(|u_{n+1}|^2+|u_{n-1}|^2)u_n, \end{aligned} \tag{3}$$

which is characterized by four parameters: the tunneling rate J, the on-site s-wave scattering interaction U_0, the effective three-body interaction U_3 arising from triple collisions, and the effective dipole–dipole interaction at different relative distances U_{dd}. We remark that Eq. (3) has extended Eq. (2) in Ref. [21] to account for the presence of dipole–dipole and three-body interactions.

The tunneling parameter J in Eq. (3) can be calculated from

$$J = -\int d\boldsymbol{r}\Big(\frac{\hbar^2}{2m}\nabla\Phi_n\nabla\Phi_{n+1} + V_{opt}\nabla\Phi_n\nabla\Phi_{n+1}\Big), \tag{4}$$

where n and $n+1$ are indices of the nearest neighboring sites.

The on-site interaction term U_0 can be written as

$$U_0 = gN\int d\boldsymbol{r}\,\Phi_n^4 + N\int d\boldsymbol{r}d\boldsymbol{r}'V_{dd}(\boldsymbol{r}-\boldsymbol{r}')\Phi_n^2(\boldsymbol{r}')\Phi_n^2(\boldsymbol{r}), \tag{5}$$

where the first term comes from the two-body scattering interaction while the last term stems from the dipole–dipole-induced on-site interaction.

The effective three-body interaction U_3 in Eq. (3) is given by

$$\begin{aligned} U_3 = N^2\frac{16\pi\hbar^2}{m}a_s^4(4\pi-3\sqrt{3})\log(C\sqrt{na_s^3})\int d\boldsymbol{r}\,\Phi_n^6 \\ + N\int d\boldsymbol{r}'V_{dd}(\boldsymbol{r}-\boldsymbol{r}')\Phi_n^2(\boldsymbol{r}')\Phi_n^2(\boldsymbol{r}), \end{aligned} \tag{6}$$

where the first term arises from triple collisions, and the last term stems from the dipole–dipole-induced interaction. We point out that the three-body scattering is treated here as a perturbation to the usual two-body pseudopotential. However, as pointed out in Ref. [20], by loading the polar molecules into an optical lattice and adding an external microwave field, one can achieve a situation where only observable TBI exists. Hence, it is necessary to investigate the unique role played by TBIs.

The DDI, represented by the last term in Eq. (3), is long-range and anisotropic in nature, which brings new features to the model system. In particular, the contribution of DDI to Eq. (3) can be decomposed into an on-site component and a long-range component. The long-range part of the DDI can be simplified into the interaction U_{dd} between two dipoles localized at sites n and $n\pm1$, respectively,

$$\begin{aligned} U_{dd} = N\int d\boldsymbol{r}d\boldsymbol{r}'V_d(\boldsymbol{r}-\boldsymbol{r}')[\Phi_n^2(\boldsymbol{r}')\Phi_{n+1}^2(\boldsymbol{r}) \\ + \Phi_n^2(\boldsymbol{r}')\Phi_{n-1}^2(\boldsymbol{r})]. \end{aligned} \tag{7}$$

By rescaling the time and energy with $\hbar/J$ and J, respectively, we can rewrite Eq. (3) into the dimensionless form

$$\begin{aligned} i\frac{d}{dt}u_n(t) = &-(u_{n+1}+u_{n-1}) + q|u_n|^2u_n \\ &+ \lambda|u_n|^4u_n + d(|u_{n+1}|^2+|u_{n-1}|^2)u_n, \end{aligned} \tag{8}$$

with $q = U_0/J$, $\lambda = U_3/J$ and $d = U_{dd}/J$. Note that q can be adjusted by the Feshbach resonance technique and d can be controlled by changing the orientation of dipoles.

The stationary solution to Eq. (8) is the plane wave solution in the form of

$$u_n = u_0\exp[i(kn-\omega t)], \tag{9}$$

where k is the carrier wave vector, and ω is the chemical potential of condensate atoms

$$\omega = 2\cos k + (q+2d)u_0^2 + \lambda u_0^4. \tag{10}$$

To compute the modulation stability of this plane-wave solution, one introduces the perturbation ansatz

$$u_n(t) = [u_0 + \delta u_n(t)] \exp(ikn + \omega t). \quad (11)$$

Plugging this ansatz into Eq. (8) and linearizing $u_n(t)$, we obtain the evolution equation for the perturbation

$$\begin{aligned} i\frac{d}{dt}\delta u_n = &[2(q+d)u_0^2 + 3\lambda u_0^4 - \omega(k)]\delta u_n + (qu_0^2 \\ &+ 2\lambda u_0^4)\delta u_n^* + (\delta u_{n+1}e^{ik} + \delta u_{n-1}e^{-ik}) \\ &+ du_0^2(\delta u_{n+1} + \delta u_{n-1} + \delta u_{n+1}^* + \delta u_{n-1}^*). \end{aligned} \quad (12)$$

We pose δu_n in the form

$$\delta u_n(t) = u_1 e^{i(Qn+\Omega t)} + u_2^* e^{-i(Qn+\Omega^* t)}, \quad (13)$$

where Q is the modulational wave vector and the perturbation frequency Ω can be determined by the following linear system

$$\begin{pmatrix} -\Omega - \omega(k) + a^+ & b \\ b & \Omega - \omega(k) + a^- \end{pmatrix} \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = 0, \quad (14)$$

where $b \equiv qu_0^2 + 2\lambda u_0^4 + 2du_0^2 \cos Q$ and $a^{\pm} \equiv 2(q+d)u_0^2 + 3\lambda u_0^4 + 2\cos(Q \pm k) + 2du_0^2 \cos Q$. We can obtain the eigenfrequency

$$\Omega = \frac{1}{2}[M \pm \sqrt{d^2 - 4b^2 + 4(\omega(k) - a^+)(\omega(k) - a^-)}], \quad (15)$$

where $M \equiv a^+ - a^-$. The carrier plane wave of the dipolar condensate is modulationally unstable when the eigenfrequencies Ω become imaginary, thus we define the instability gain $G = \mathrm{Im}[\Omega]$. Let us discuss the stability of uniform plane waves ($k = 0$). If $G \neq 0$, the nonlinear plane wave experiences MI. In the following, based on Eq. (15), we study how the important ingredient of TBI determines the MI condition of a dipolar BEC in deep optical lattice. Hereafter, we assume that the two-body interaction parameter $q = 0$, which is reasonable for the Dy experiment and can be tuned to zero through the Feshbach resonances method. We mainly discuss the interplay between DDI and TBI on the MI.

In the first scenario where the DDI is absent ($d = 0$), the system reduces to a discrete nonlinear Schrödinger where the only nonlinearity comes from the TBI, and the instability gain $G = 2\mathrm{Im}[\sqrt{(\cos Q - 1)(\cos Q - 1 + 2\lambda u_0^4)}]$. It is clear that the MI may only occur in the case of $\lambda > 0$. Specifically, for $\lambda u_0^4 > 1$ we have MI for all Q values. Therefore, the plane matter waves will always be unstable. For $0 < \lambda u_0^4 < 1$, there is MI for $Q < 2\arcsin(\sqrt{\lambda u_0^4})$. Figure 1 presents the 3D plots as functions of u_0 and Q for different values of TBI strength λ. It is clearly shown that for a positive λ, there always exists MI of the plane matter wave, whereas the amplitude of MI gain G and the region of MI are dramatically influenced by the strength of TBI.

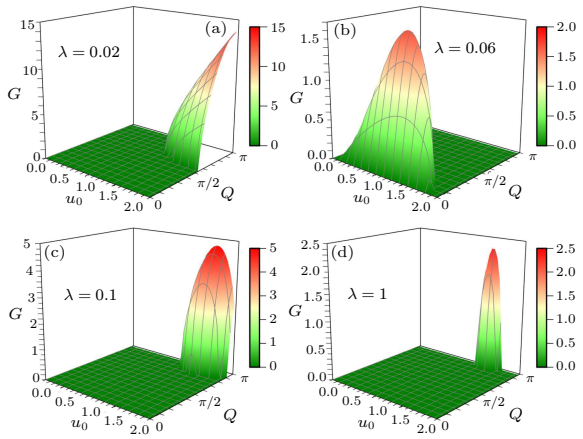

Fig. 1. Modulational instability gain G for DDI parameter $d = 0$ and TBI strength parameter for $\lambda = 0.02$ (a), $\lambda = 0.06$ (b), $\lambda = 0.1$ (c), and $\lambda = 1$ (d), respectively.

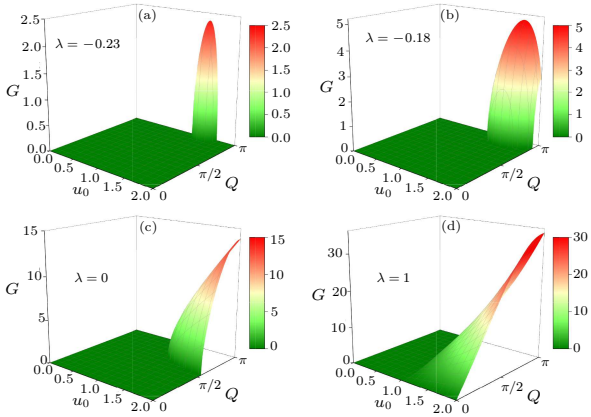

Fig. 2. Modulational instability gain G for DDI parameter $d = -1$ and TBI strength parameter for (a) $\lambda = -0.23$, (b) $\lambda = -0.18$, (c) $\lambda = 0$, (d) $\lambda = 1$.

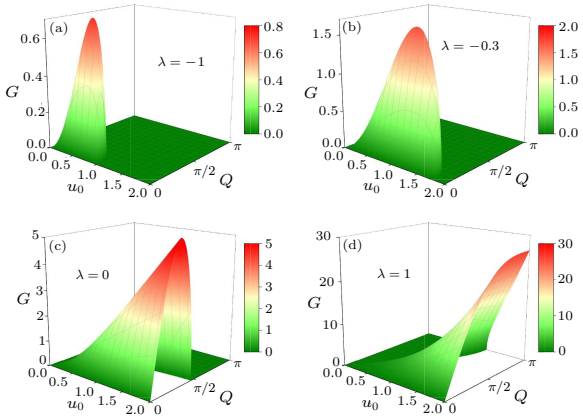

Fig. 3. Modulational instability gain G for DDI parameter $d = 1$ and TBI strength parameter for (a) $\lambda = -1$, (b) $\lambda = -0.3$, (c) $\lambda = 0$, (d) $\lambda = 1$.

In the second scenario, the DDI is turned on ($d \neq 0$). In this case, the instability gain G is expressed as $G = 2(\mathrm{Im}[\sqrt{(\cos Q - 1)(\cos Q - 1 + 2\lambda u^4 + 2du^2 \cos Q)}]$. To observe the MI, the term $(\cos Q - 1 + 2\lambda u^4 + 2du^2 \cos Q)$ must be positive. Thus it is clearly shown that both the parameters λ and d determine the MI condition. From Fig. 2, considering the attractive DDI where $d = -1$, we find that there exists a critical TBI

value $\lambda_{\rm cr} = -0.23$. In the interval $\lambda < \lambda_{\rm cr}$, the plane wave is always stable, but in the regime of $\lambda > \lambda_{\rm cr}$, the MI will occur, as shown clearly that with the increase of λ, the region of MI in the u_0 and Q plane and the amplitude of G both are increasing. These reflect that with an attractive DDI, a positive TBI can help the dipolar BEC to formulate the localized states. In Fig. 3, for an exclusive DDI where $d = 1$, we find that MI always exists in the whole regime of λ, and the TBI only changes the MI region and the amplitude of MI gain parameter G. This means that for an exclusive DDI, the localized states can be easily excited in dipolar BEC, but the TBI can change the region where the localized state occurs. For experimental scenarios, we mention that to observe our theoretical predictions, one can choose ^{64}Dy as an example, with $N = 15 \times 10^3$,[1,5] and using the Feshbach resonances method and letting the effective two-body interaction $q = 0$, the TBI can also be tuned at will.[22] The deep optical lattice can be achieved with a far off-resonant standing light wave, and the depth of the potential is proportional to the intensity of the light wave as shown in Ref. [23].

In summary, motivated by the recent experimental achievement of using the dipolar BEC to form a stable droplet, we have investigated the modulation instability of a dipolar BEC with the TBI loaded into a deep optical lattice. Our results show that the combined effects of DDI and TBI can significantly change the MI condition compared with the case of the non-dipolar one.

References

[1] Kaudau H et al 2016 *Nature* **530** 194
[2] Bisset R N and Blakie P B 2015 *Phys. Rev.* A **92** 061603(R)
[3] Xi K T and Saito H 2016 *Phys. Rev.* A **93** 011604(R)
[4] Bisset R N et al 2016 *Phys. Rev.* A **94** 033619
[5] Wächtler F et al 2016 *Phys. Rev.* A **94** 043618
[6] Edler D et al 2017 *Phys. Rev. Lett.* **119** 050403
[7] Blakie P B 2016 *Phys. Rev.* A **93** 033644
[8] Adhikari S K 2017 *Phys. Rev.* A **95** 023606
[9] Benjamin T B and Feir J E 1967 *J. Fluid Mech.* **27** 417
[10] Bhat I A et al 2015 *Phys. Rev.* A **92** 063606
[11] Nguyen J H V et al 2017 *Science* **356** 422
[12] Wu L and Zhang J F 2007 *Chin. Phys. Lett.* **24** 1471
[13] Wu B and Niu Q 2001 *Phys. Rev.* A **64** 061603(R)
[14] Bai X D and Xue J K 2013 *Phys. Rev.* E **88** 062916
[15] Zheng G P et al 2015 *Commun. Theor. Phys.* **63** 565
[16] Rojas-Rojas S et al 2011 *Phys. Rev.* A **84** 033621
[17] Chen H et al 2010 *Int. J. Mod. Phys.* B **24** 5695
[18] Wu T T 1959 *Phys. Rev.* **115** 1390
[19] Braaten E and Nieto A 1999 *Eur. Phys. J.* B **11** 143
[20] Büchler H P et al 2007 *Nat. Phys.* **3** 726
[21] Smerizi A et al 2002 *Phys. Rev. Lett.* **89** 170402
[22] Inouye S et al 1998 *Nature* **392** 151
[23] Anker T et al 2005 *Phys. Rev. Lett.* **94** 020403