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Effects of Homogeneous Plasma on Strong Gravitational Lensing of Kerr Black Holes *

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Considering the Kerr black hole surrounded by a homogeneous unmagnetized plasma medium, we study the strong gravitational lensing on the equatorial plane of the Kerr black hole. It is found that the presence of the uniform plasma can increase the photon-sphere radius $r_{\rm ps}$, the coefficients $\bar{a}$ and $\bar{b}$, the angular position of the relativistic images (θ_∞), the deflection angle $\alpha(\theta)$ and the angular separation s. However, the relative magnitude $r_{\rm m}$ decreases in the presence of the uniform plasma medium. It is also shown that the impact of the uniform plasma on the effect of strong gravitational lensing becomes smaller as the spin of the Kerr black hole increases in the prograde orbit ($a > 0$). In particular, for the extreme black hole ($a = 0.5$), the effect of strong gravitational lensing in the homogeneous plasma medium is the same as the case in vacuum for the prograde orbit.

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It is of interest to study gravitational lensing in a plasma[1–4] since most lenses are surrounded by interstellar or intergalactic media. A general theory of geometrical optics in a curved spacetime with an isotropic dispersive medium was proposed in the textbook of Synge.[5] In the book of Perlick,[6] the general formulae for the exact light deflection angle in the Schwarzschild and Kerr metric, in the presence of plasma, are obtained in the form of integrals. Several similar approaches above the geometric optics approximation through the magnetized plasma in the vicinity of the compact object have been presented.[1,7] Based on the general approach of the geometric optics, the gravitational lensing in inhomogeneous and homogeneous plasma around black holes has been studied[8–15] recently as an extension of vacuum studies. However, this research work is restricted to the static spacetime and slowly rotating compacted object in the plasma medium.

The main purpose of this work is to study the strong gravitational lensing[16–31] by the Kerr black hole in a homogeneous unmagnetized plasma medium, to extend the results[9] to the case of rotating gravitational lens, and to see the impacts of homogeneous plasma on photon sphere radius, the deflection angle, the coefficients and the observable quantities of strong gravitational lensing.

Considering a rotating Kerr black hole surrounded by plasma, we can give the Kerr metric in the standard Boyer–Lindquist coordinates as follows:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\phi \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2+a^2 \right. \\ &\left. +\frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{1}$$

with

$$\Delta \equiv r^2 - 2Mr + a^2, \quad \Sigma \equiv r^2 + a^2\cos^2\theta. \tag{2}$$

We assume that the spacetime is filled with a non-magnetized cold plasma, whose electron plasma frequency $w_{\rm p}$ is a function of the radius coordinate only,[5,10]

$$\omega_{\rm p}(r)^2 = \frac{4\pi e^2}{m}N(r), \tag{3}$$

where e is the charge of the electron, m is the electron mass, and $N(r)$ is the number density of the electrons in the plasma. When $\omega_{\rm p}$ is a constant, plasma is homogeneous. In this study we only consider homogeneous plasma. The refraction index n of this plasma depends on the plasma frequency $\omega_{\rm p}$ and the frequency ω of the photon as it is measured by a static observer,[1,2,5,10]

$$n^2 = 1 - \frac{\omega_{\rm p}^2}{\omega^2}. \tag{4}$$

Let us now study the strong gravitational lensing of the rotating Kerr black hole surrounded by plasma. As in Refs. [20–30] we just consider that both the observer and the source lie in the equatorial plane of the black hole and the whole trajectory of the photon is limited on the same plane. Using the condition $\theta = \pi/2$ and taking $2M = 1$, Eq. (1) is reduced to

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\phi^2 - 2D(r)dtd\phi, \tag{5}$$

with

$$A(r) = 1 - \frac{1}{r}, \quad B(r) = \frac{r^2}{a^2 - r + r^2}, \tag{6}$$

$$C(r) = a^2 + \frac{a^2}{r} + r^2, \quad D(r) = \frac{a}{r}. \tag{7}$$

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The general relativistic geometrical optics on the background of the curved spacetime, in a refractive and dispersive plasma medium, was developed by Synge.[5] Based on the Hamiltonian approach for the description of the geometrical optics, the Hamiltonian for the photon around the Kerr black hole surrounded by plasma has the form [7]

$$H(x^i, p_i) = \frac{1}{2}[g^{ik}p_i p_k + \hbar^2\omega_{\rm p}^2] = 0. \tag{8}$$

It is interesting to notice that the expression of the Hamiltonian for the photon around the Kerr black hole surrounded by homogeneous plasma is similar to the Hamiltonian of the massive particle in vacuum. Using the Hamiltonian for the photon around the Kerr black hole, we can obtain the Hamiltonian differential equations

$$\frac{dx^i}{d\lambda} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\lambda} = -\frac{\partial H}{\partial x^i}, \tag{9}$$

then we obtain two constants of motions which are the energy and the angular momentum of the particle,

$$E = -p_{\rm t} = \hbar\omega, \quad L = P_\phi. \tag{10}$$

Let us consider a homogeneous plasma with $\omega_{\rm p} =$ const. We introduce notations of $\hat{E}$ and $\hat{L}$,

$$\frac{-p_{\rm t}}{\hbar\omega_{\rm p}} = \frac{\omega}{\omega_{\rm p}} = \hat{E} > 1, \quad \frac{L}{\hbar\omega_{\rm p}} = \hat{L} > 0. \tag{11}$$

From this equation, we find an expression for $\dot{t}, \dot{r}, \dot{\phi}$ in terms of $\hat{E}$ and $\hat{L}$,

$$\begin{aligned}
\frac{dt}{d\lambda} &= \frac{\hbar\omega_{\rm p}(\hat{E}C(r) - \hat{L}D(r))}{D(r)^2 + A(r)C(r)}, \\
\frac{d\phi}{d\lambda} &= \frac{\hbar\omega_{\rm p}(\hat{E}D(r) + \hat{L}A(r))}{D(r)^2 + A(r)C(r)}, \\
\left(\frac{dr}{d\lambda}\right)^2 &= [\hbar^2\omega_{\rm p}^2(\hat{E}^2C(r) - (A(r)C(r) + D(r)^2) \\
&\quad - 2\hat{E}\hat{L}D(r) - \hat{L}^2A(r))] \\
&\quad \cdot \left\{(B(r)[D(r)^2 + A(r)C(r)]\right\}^{-1},
\end{aligned} \tag{12, 13}$$

where λ is an affine parameter along the geodesics. Under the condition $\frac{dr}{d\lambda}|_{r=r_0} = 0$, we can obtain the angular momentum

$$\hat{L}(r_0) = \frac{-\hat{E}D(r_0) + \sqrt{A(r_0)PC(r_0) + \hat{E}^2D^2(r_0)}}{A(r_0)}, \tag{14}$$

with

$$PC(r_0) = \hat{E}^2C(r_0) - (A(r_0)C(r_0) + D(r_0)^2). \tag{15}$$

For the photon moving in the plasma with the effective mass[7,9] $m_{\rm eff} = \hbar\omega_{\rm p}$, the impact parameter can be expressed as

$$u = \frac{L}{\sqrt{E^2 - m_{\rm eff}^2}} = \frac{\hat{L}}{\sqrt{\hat{E}^2 - 1}}. \tag{16}$$

Now let us turn to the photon sphere, and the definition of the photon sphere can be found in Refs. [17,20]. The feature of the radius of the photon sphere is that the deflection angle in a strong field limit is divergent, when the photon reaches the radius of the photon sphere ($r = r_{\rm ps}$) in an unstable circular orbit. If a photon falls inside this sphere, it is destined to be absorbed by the black hole. Of course, we will have different photon spheres for photons winding in the same sense of the rotation of the black hole (prograde photons) and for photons winding in the opposite sense (retrograde photons). Here $r_{\rm ps}$ satisfies the condition $\dot{r} = 0$ and $\ddot{r} = 0$. Under this condition, the equation for the photon sphere reads

$$\begin{aligned}
&A(r)PC'(r) - A'(r)PC(r) \\
&+ 2\hat{L}\hat{E}[A'(r)D(r) - A(r)D'(r)] = 0.
\end{aligned} \tag{17}$$

The largest real root external to the horizon of this equation is defined as the radius of the photon sphere $r_{\rm ps}$. In the case of the Kerr black hole surrounded by the plasma, the analytical expression of marginally circular photon orbits takes the form

$$\begin{aligned}
&a^4 + 2a^2(2 - a^2)r + (4 + 8a^2 + a^4 - 4\hat{E}^2a^2 \\
&- 4\hat{E}^4a^4 + 2a^2(4 - 3\hat{E}^2) - 4a^2\hat{E}^2(\hat{E}^2 - 1))r^2 \\
&+ (-8 - 4a^2 + 4a^2\hat{E}^2 - 4(4 - 3\hat{E}^2)(1 + a^2 \\
&- a^2\hat{E}^2) + 4a^2(\hat{E}^4 - 1))r^3 \\
&+ (4 + (4 - 3\hat{E}^2)(12 + 2a^2 - 7a^2\hat{E}^2) \\
&+ 4(\hat{E}^2 - 1)(a^2\hat{E}^2 - 2a^2 - 2))r^4 \\
&+ ((4 - 3\hat{E}^2)(6\hat{E}^2 - 16) + 4(\hat{E}^2 - 1)(8 + a^2 \\
&- 2a^2\hat{E}^2 - 3\hat{E}^2))r^5 + ((4 - 3\hat{E}^2)^2 \\
&+ 4(\hat{E}^2 - 1)(5\hat{E}^2 - 11))r^6 + 4(\hat{E}^2 - 1)(6 \\
&- 5\hat{E}^2)r^7 + 4(\hat{E}^2 - 1)^2r^8 = 0.
\end{aligned} \tag{18}$$

Obviously, circular photon orbits depend on the plasma frequency, in particular the photon radius in a static Schwarzschild black hole surrounded by homogeneous plasma has the analytical expression from Eq. (17),

$$r_{\rm ps} = \frac{3\hat{E}^2 - 4 + \hat{E}\sqrt{-8 + 9\hat{E}^2}}{4(\hat{E}^2 - 1)}. \tag{19}$$

This is just the result obtained in Ref. [9]. As $\hat{E} \to \infty$, the photon radius $r_{\rm ps} \to \frac{3}{2}$ corresponds to the photons of the static Schwarzschild black hole in a vacuum. In Fig. 1, we present the variation of the photon-sphere radius $r_{\rm ps}$ with the rotational parameter a in different plasma media and vacua. As is expected, compared to the case in a vacuum, the presence of plasma can increase the photon-sphere radius $r_{\rm ps}$. It is also shown that the growth of the photon-sphere radius in the prograde orbit ($a > 0$) is less than the case in the retrograde orbit ($a < 0$). In particular, for the extreme black hole ($a = 0.5$), the photon sphere radius in the

plasma medium is the same as in a vacuum for the prograde orbit.

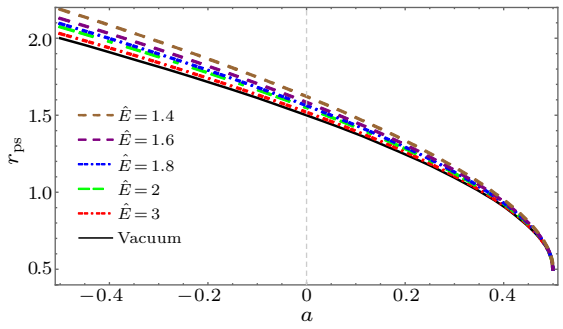

Fig. 1. Variation of the radius of the photon sphere $r_{\rm ps}$ with the parameter a in different plasma media and vacua of the Kerr black hole.

The deflection angle for the photon coming from infinity in a stationary Kerr black hole in the plasma medium can be given as

$$\alpha(r_0) = I(r_0) - \pi, \tag{20}$$

with

$$\begin{aligned} I(r_0) = 2\int_{r_0}^{\infty} &\{\sqrt{B(r)|A(r_0)|}[\hat{E}D(r) + \hat{L}A(r)]dr\} \\ &\cdot \Big\{\sqrt{D^2(r) + A(r)C(r)}\{\mathrm{sgn}(A(r_0)) \\ &\cdot [A(r_0)PC(r) - A(r)PC(r_0) \\ &+ 2\hat{E}\hat{L}[A(r)D(r_0) - A(r_0)D(r)]]\}^{1/2}\Big\}^{-1}, \end{aligned} \tag{21}$$

where $\mathrm{sgn}(X)$ gives the sign of X.

It is obvious that the deflection angle increases as the parameter r_0 decreases. For a certain value of r_0, the deflection angle becomes 2π, thus the light ray makes a complete loop around the lens before reaching the observer. If r_0 is equal to the radius of the photon sphere $r_{\rm ps}$, we can find that the deflection angle diverges and the photon is captured by the compact object.

To find the behavior of the deflection angle when the photon is close to the photon sphere, we use the evaluation method proposed by Bozza.[20] The divergent integral in Eq. (21) is first split into the divergent part $I_D(r_0)$ and the regular one $I_{\rm R}(r_0)$, and then both of them are expanded around $r_0 = r_{\rm ps}$ with sufficient accuracy. This technique has been widely used in the study of the strong gravitational lensing for various black holes.[20–30] Let us now define a variable

$$z = 1 - \frac{r_0}{r}, \tag{22}$$

and rewrite Eq. (21) as

$$I(r_0) = \int_0^1 R(z, r_0) f(z, r_0) dz, \tag{23}$$

with

$$\begin{aligned} R(z, r_0) = &\frac{2r_0}{\sqrt{PC(z)}(1-z)^2} \\ &\cdot \frac{\sqrt{B(z)|A(r_0)|}[\hat{E}D(z) + \hat{L}A(z)]}{\sqrt{D^2(z) + A(z)C(z)}}, \end{aligned} \tag{24}$$

$$\begin{aligned} f(z, r_0) = &\Big\{\mathrm{sgn}(A(r_0))\Big[A(r_0) - A(z)\frac{PC(r_0)}{PC(z)} \\ &+ \frac{2\hat{E}\hat{L}}{PC(z)}(A(z)D(r_0) - A(r_0)D(z))\Big]\Big\}^{-1/2}. \end{aligned} \tag{25}$$

Obviously, the function $R(z, r_0)$ is regular for all values of z and r_0. However, the function $f(z, r_0)$ diverges as z tends to zero, i.e., as the photon approaches the photon sphere. Thus the integral (23) can be separated into two parts $I_D(r_0)$ and $I_{\rm R}(r_0)$,

$$\begin{aligned} I_D(r_0) &= \int_0^1 R(0, r_{\rm ps}) f_0(z, r_0) dz, \\ I_{\rm R}(r_0) &= \int_0^1 [R(z, r_0) f(z, r_0) - R(0, r_0) f_0(z, r_0)] dz. \end{aligned} \tag{26}$$

Expanding the argument of the square root in $f(z, r_0)$ to the second order in z, we have

$$f_0(z, r_0) = \frac{1}{\sqrt{p(r_0)z + q(r_0)z^2}}, \tag{27}$$

where

$$\begin{aligned} p(r_0) = &\frac{r_0}{PC(r_0)}\{A(r_0)PC'(r_0) - A'(r_0)PC(r_0) \\ &+ 2\hat{E}\hat{L}[A'(r_0)D(r_0) - A(r_0)D'(r_0)]\}, \\ q(r_0) = &\frac{r_0}{2PC^2(r_0)}\{2(PC(r_0) \\ &- r_0PC'(r_0))([A(r_0)PC'(r_0) \\ &- A'(r_0)PC(r_0)] + 2\hat{E}\hat{L}[A'(r_0)D(r_0) \\ &- A(r_0)D'(r_0)]) \\ &+ r_0PC(r_0)([A(r_0)PC''(r_0) \\ &- A''(r_0)PC(r_0)] + 2\hat{E}\hat{L}[A''(r_0)D(r_0) \\ &- A(r_0)D''(r_0)])\}. \end{aligned} \tag{28}$$

From Eq. (28), we can find that if r_0 approaches the radius of photon sphere $r_{\rm ps}$, the coefficient $p(r_0)$ vanishes and the leading term of the divergence in $f_0(z, r_0)$ is z^{-1}, which implies that the integral (23) diverges logarithmically. The coefficient $q(r_0)$ takes the form

$$\begin{aligned} q(r_{\rm ps}) = &\frac{\mathrm{sgn}(A(r_{\rm ps}))r_{\rm ps}^2}{2PC(r_{\rm ps})}\{A(r_{\rm ps})PC''(r_{\rm ps}) \\ &- A''(r_{\rm ps})PC(r_{\rm ps}) + 2\hat{E}\hat{L}[A''(r_{\rm ps})D(r_{\rm ps}) \\ &- A(r_{\rm ps})D''(r_{\rm ps})]\}. \end{aligned} \tag{29}$$

Therefore, the deflection angle in the strong field region can be expressed as[20]

$$\alpha(\theta) = -\bar{a}\log\Big(\frac{\theta D_{\rm OL}}{u_{\rm ps}} - 1\Big) + \bar{b} + \mathcal{O}(u - u_{\rm ps}), \tag{30}$$

with

$$\bar{a} = \frac{R(0, r_{\rm ps})}{2\sqrt{q(r_{\rm ps})}}, \quad b_{\rm R} = I_{\rm R}(r_{\rm ps}),$$

$$u_{\rm ps} = \frac{-\hat{E}D(r_{\rm ps}) + \sqrt{A(r_{\rm ps})PC(r_{\rm ps}) + \hat{E}^2 D^2(r_{\rm ps})}}{A(r_{\rm ps})\sqrt{\hat{E}^2 - 1}},$$

$$\begin{aligned}\bar{b} = & -\pi + b_{\rm R} \\ & + \bar{a}\log\Big\{[2q(r_{\rm ps})PC(r_{\rm ps})] \\ & \cdot \big\{u_{\rm ps}\sqrt{\hat{E}^2 - 1}|A(r_{\rm ps})|[\hat{E}D(r_{\rm ps}) \\ & + \hat{L}_{\rm ps}A(r_{\rm ps})]\big\}^{-1}\Big\},\end{aligned} \tag{31}$$

where the quantity $D_{\rm OL}$ is the distance between the observer and gravitational lens. Making use of Eqs. (30) and (31), we can study the properties of strong gravitational lensing in the rotating Kerr black hole in the presence of homogeneous plasma. In Fig. 2, we plot the changes of the coefficients $\bar{a}$ and $\bar{b}$ with a for a different ratio of photon frequency to plasma frequency $\omega_{\rm p}$. It is shown that the coefficients $\bar{a}$ and $\bar{b}$ in the strong field limit are functions of the parameters a and $\hat{E}$. Compared with the vacuum case, the presence of plasma can increase the coefficients $\bar{a}$ and $\bar{b}$. It is also shown that the growths of the coefficients $\bar{a}$ and $\bar{b}$ in the prograde orbit ($a > 0$) are less than those in the case with the retrograde orbit ($a < 0$). In particular, for the extreme black hole ($a = 0.5$), the coefficients $\bar{a}$ and $\bar{b}$ in the plasma medium are the same as those in a vacuum for the prograde orbit. With the help of the coefficients $\bar{a}$ and $\bar{b}$, we plot the change of the deflection angles evaluated at $u = u_{\rm ps} + 0.00326$ with the rotational parameter a for a different ratio of photon frequency to plasma frequency $\omega_{\rm p}$ in Fig. 3. It is shown that in the strong field limit the deflection angles $\alpha(\theta)$ have similar properties to the coefficient $\bar{a}$.

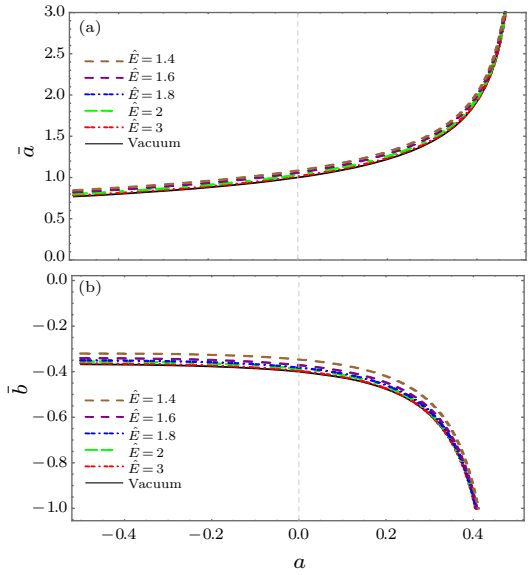

Fig. 2. Variation of the coefficients $\bar{a}$ and $\bar{b}$ for the strong gravitational lensing with the parameter a in different plasma media and vacua of the Kerr black hole.

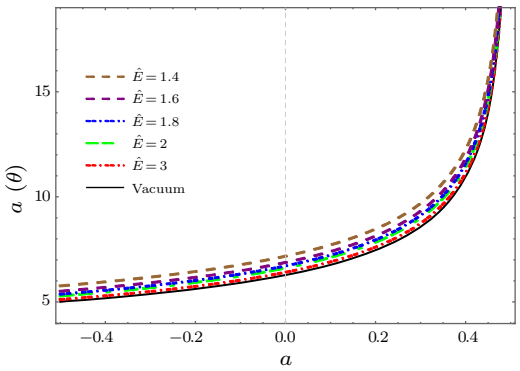

Fig. 3. Variation of deflection angle $\alpha(\theta)$ evaluated at $u = u_{\rm ps} + 0.00326$ with the parameter a in different plasma media and vacua of the Kerr black hole.

Let us now study the effect of the homogeneous plasma medium on the observable quantities of strong gravitational lensing. Here we consider only the case in which the source, lens and the observer are highly aligned, thus the lens equation in strong gravitational lensing can be approximated well as[21]

$$\gamma = \frac{D_{\rm OL} + D_{\rm LS}}{D_{\rm LS}}\theta - \alpha(\theta) \mathrm{mod} 2\pi, \tag{32}$$

where $D_{\rm LS}$ is the lens–source distance, $D_{\rm OL}$ is the observer–lens distance, γ is the angle between the directions of the source and the optical axis, and $\theta = u/D_{\rm OL}$ is the angular separation between the lens and the image. Following Ref. [21] we can find that the angular separation between the lens and the nth relativistic image is

$$\theta_n \simeq \theta_n^0\Big(1 - \frac{u_{\rm ps}e_n(D_{\rm OL} + D_{\rm LS})}{\bar{a}D_{\rm OL}D_{\rm LS}}\Big), \tag{33}$$

with

$$\theta_n^0 = \frac{u_{\rm ps}}{D_{\rm OL}}(1 + e_n), \quad e_n = e^{\frac{\bar{b} + |\gamma| - 2\pi n}{\bar{a}}}, \tag{34}$$

where the quantity θ_n^0 is the image positions corresponding to $\alpha = 2n\pi$, and n is an integer. According to the past oriented light ray which starts from the observer and finishes at the source the resulting images stand on the eastern side of the black hole for direct photons ($a > 0$) and are described by positive γ. Retrograde photons ($a < 0$) have images on the western side of the compact object and are described by negative values of γ. In the limit $n \to \infty$, we can find that $e_n \to 0$, which means that the relationship between the minimum impact parameter $u_{\rm ps}$ and the asymptotic position of a set of images θ_∞ can be simplified further to

$$u_{\rm ps} = D_{\rm OL}\theta_\infty. \tag{35}$$

To obtain the coefficients $\bar{a}$ and $\bar{b}$, we need to separate at least the outermost image from all the others. As shown in Refs. [20,21], we consider here the simplest case in which only the outermost image θ_1 is resolved as a single image and all the remaining ones

are packed together at θ_∞. Thus the angular separation between the first image and other ones can be expressed as[20,21]

$$s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{\bar{b}-2\pi}{\bar{a}}}. \quad (36)$$

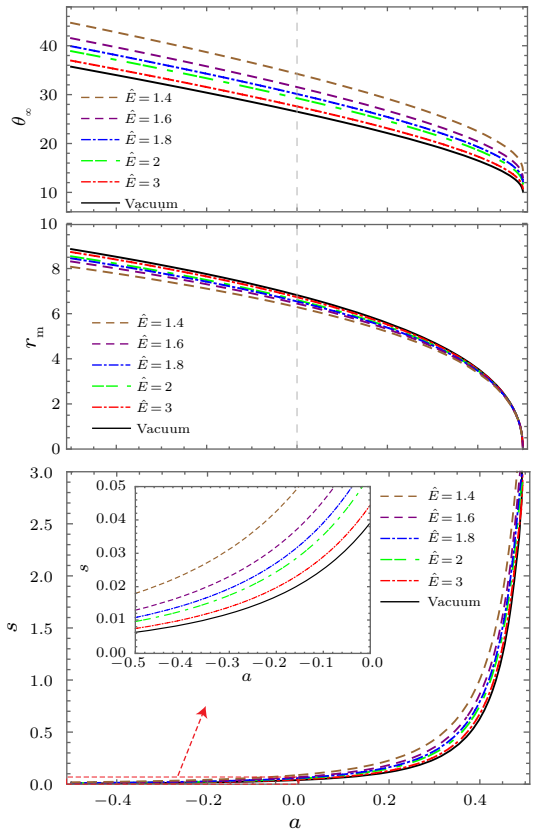

Fig. 4. Variation of the innermost relativistic image θ_∞, the relative magnitudes r_m and the angular separation s with the parameter a in different plasma media and vacua of the Kerr black hole.

By measuring s and θ_∞, we can obtain the strong deflection limit coefficients $\bar{a}$, $\bar{b}$ and the minimum impact parameter u_{ps}. Comparing their values with those predicted by the theoretical models, we can obtain the information of the Kerr black hole. The mass of the central object of our Galaxy is estimated recently to be $4.4\times10^6 M_\odot$[32] and its distance is around 8.5 kpc, thus the ratio of the mass to the distance $M/D_{OL} \approx 2.4734 \times 10^{-11}$. Making use of Eqs. (31), (35) and (36) we can estimate the values of the coefficients and observable quantities for gravitational lens in the strong field limit. The numerical value for the angular position of the relativistic images θ_∞, the angular separation s and the relative magnitudes r_m are plotted in Fig. 4. We find that the variation of the angular position of the relativistic images θ_∞ with the rotational parameter a in different plasma media and vacua is similar to that of the photon-sphere radius r_{ps}. However, the variation of the relative magnitudes r_m is contrary to the case of the photon-sphere radius r_{ps}. We also find that the variation of the angular separation s with the rotational parameter a in different plasma media and vacua is similar to that of the deflection angle $\alpha(\theta)$. Thus we can conclude that the effects of homogeneous plasma on strong gravitational lensing of the Kerr black hole in the prograde orbit is larger that in the retrograde orbit.

In summary, we have investigated the strong gravitational lensing of the Kerr black hole surrounded by a homogeneous plasma. We have derived the expression for the deflection angle of light in the Kerr black hole in the presence of homogeneous plasma and have numerically calculated the coefficient of the deflection angle. It is shown that the presence of the uniform plasma increases the photon-sphere radius r_{ps}, the coefficients $\bar{a}$ and $\bar{b}$, the angular position of the relativistic images θ_∞, the deflection angle $\alpha(\theta)$ and the angular separation s. However, the relative magnitudes r_m decrease in the presence of the uniform plasma medium. It is also shown that the impact of the uniform plasma on the effect of strong gravitational lensing becomes smaller as the spin of the Kerr black hole increases in the prograde orbit ($a > 0$). In particular, for the extreme black hole ($a = 0.5$), the effect of strong gravitational lensing in a homogeneous plasma medium is the same as the case in a vacuum for the prograde orbit.

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