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Sound Wave of Spin–Orbit Coupled Bose–Einstein Condensates in Optical Lattice *

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We study the phonon mode excitation of spin–orbit (SO) coupled Bose–Einstein condensates trapped in a one-dimensional optical lattice. The sound speed of the system is obtained analytically. Softening of the phonon mode, i.e., the vanishing of sound speed, in the optical lattice is revealed. When the lattice is absent, the softening of phonon mode occurs only at the phase transition point, which is not influenced by the atomic interaction and Raman coupling when the SO coupling is strong. However, when the lattice is present, the softening of phonon modes can take place in a regime near the phase transition point. Particularly, the regime is widened as lattice strength and SO coupling increase or atomic interaction decreases. The suppression of sound speed by the lattice strongly depends on atomic interaction, Raman coupling, and SO coupling. Furthermore, we find that the sound speed in plane wave phase regime and zero-momentum phase regime behaves with very different characteristics as Raman coupling and SO coupling change. In zero-momentum phase regime, sound speed monotonically increases/decreases with Raman coupling/SO coupling, while in plane wave phase regime, sound speed can either increase or decrease with Raman coupling and SO coupling, which depends on atomic interaction.

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In cold-atom systems, the elementary excitation and collective mode play important roles for describing the superfluidity.[1,2] In particular, the speed of the elementary excitation is a basic parameter describing how fast the excited mode propagates in the system and it is related to the superfluidity of the system. In recent years, artificial spin–orbit (SO) coupling has been successfully realized in ultracold bosonic[3,4] and fermonic[5,6] systems. Due to the SO coupling, the Bose–Einstein condensates (BECs) display many novel and interesting phases and stabilities in ground states.[7–9] Specifically, the elementary excitations and the sound wave in SO coupled BECs have been studied in free space.[10–14] The investigations have shown that the speed of sound along the direction of SO coupling vanishes and it is deeply quenched near the phase transition between the plane-wave phase and the zero momentum phase.[12,14] Particularly, the roton maxson and a softening of phonon mode have been successionally observed in experiments.[15]

On the other hand, adding an optical lattice on top of the SO coupled BECs provides a direct observation of the breaking of Galilean invariance.[16] Motivated by the experiments, an outstanding challenge along this line is to understand the competition between the SO coupling and the optical lattice in determining the properties of the ground states and the elementary excitations of the BECs. The combination of SO coupling with optical lattice results in the prediction and discovery of topological insulators.[17–20] Compared with the uniform case, the ground states and dynamics of the SO coupled BECs in an optical lattice become significantly different.[21–25] However, the analytical description of the low-energy excitation of the SO coupled BECs in the optical lattice is still missing.

In this Letter, we study the elementary excitations of SO coupled BECs trapped in an optical lattice. The sound speed of the system is analytically derived using the perturbation method.[26] Rich competitive relationship among SO coupling, Raman coupling, interatomic interaction and lattice potential for determining the phonon mode excitation is revealed explicitly. The lattice potential makes the phonon mode excitation in plane-wave phase, and zero-momentum phase regimes behave with very different characteristics. The presence of the optical lattice enhances the phonon mode softening.

Motivated by the recent experiments,[3,4,16] we consider SO coupled BECs residing in a one-dimensional (1D) optical lattice. Assuming the SO coupling with equal contributions of Rashba[27] and Dresselhaus,[28] the dimensionless mean-field ground state of the system can be described by

$$\Big[-\frac{1}{2}\partial_x^2 \pm ik_{\rm L}\partial_x + \frac{k_{\rm L}^2}{2} + V_{\rm opt}(x) + |\psi_\sigma|^2 + \beta|\psi_{\bar\sigma}|^2\Big]\psi_\sigma + \Omega\psi_{\bar\sigma} = \mu\psi_\sigma, \quad (1)$$

where μ is the chemical potential of the system, $\{\sigma,\bar\sigma\} = \{\uparrow,\downarrow\}$, $\psi_\sigma \to \psi_\uparrow$, $\psi_{\bar\sigma} \to \psi_\downarrow$. The wave functions $\psi_\uparrow$ and $\psi_\downarrow$ are related to the two pseudospin components of the BECs, $\beta = |\alpha_{12}/\alpha_{22}|$ is the ratio of the inter-species interaction and intra-species interaction, which shows the strength of the interaction between the different species, and $\alpha_{i,j}$ $(i,j = 1,2)$ is the s-wave scattering lengths. Ignoring the effects from the Zeeman splitting between the spin-up and spin-down components, we assume $\alpha_{11} = \alpha_{22}$. In Eq. (1), the

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length is measured in units of $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$, where $\omega_\perp$ is the external trapping frequency in the transverse direction, and m is the mass of the atom. Time is measured in units of $\omega_\perp^{-1}$, energy is in units of $\hbar\omega_\perp$, density is in units of $(2|\alpha_{11}|)^{-1}$, $k_{\rm L}$ is the strength associated with the SO coupling, Ω is the Raman coupling constant, and $V_{\rm opt}(x) = s\cos(bx)$ is the external 1D optical lattice potential, where $s = v_0 E_{\rm R}/(\hbar\omega_\perp)$, v_0 is the strength of the optical lattice, $E_{\rm R} = \hbar^2\pi^2/(2md^2)$ is the recoil energy of the optical lattice, $b = (2\pi/d)a_\perp$, and d is the period of the optical lattice. Here we use $k_{\rm L} \to k_{\rm L}a_\perp$ and $\Omega \to \Omega/(\hbar\omega_\perp)$. According to the experiments,[21–23] for ^{87}Rb, $\omega_\perp$ can be adjusted to 2000 Hz, d is about 7.5μm, and for weak optical lattice, $v_0 \sim E_{\rm R}$, then we can estimate $\beta \sim 5$, $s \sim 1$, $\Omega \sim 100$, and $k_{\rm L} \sim 20$.

For weak optical lattice and in plane-wave and zero-momentum phase regimes, the ground state of the SO coupled BECs in an optical lattice can be described by the Bloch wave[21,22] $\psi_{k\sigma}(x) = \sqrt{n_0}e^{ikx}\phi_{k\sigma}(x)$, where the Bloch wave vector k is associated with the sound speed of the condensates in the lattice, n_0 is the average density, and $\phi_{k\sigma}(x)$ is a periodic function with the same period of the optical potential satisfying the normalization condition. Substituting the Bloch-wave states into Eq. (1), we obtain the following equations for each Bloch wave state $\phi_{k\sigma}(x)$,

$$\Big[-\frac{1}{2}(\partial_x + ik)^2 \pm ik_{\rm L}\partial_x \mp k_{\rm L}k + V_{\rm opt}(x) + n_0|\phi_{k\sigma}(x)|^2 + \beta n_0|\phi_{k\bar\sigma}(x)|^2 + \frac{k_{\rm L}^2}{2}\Big]\phi_{k\sigma}(x) + \Omega\phi_{k\bar\sigma}(x) = \mu\phi_{k\sigma}(x). \tag{2}$$

The sound speed can be defined as[29]

$$c_{\rm s} = (\kappa M^*)^{-1/2}, \tag{3}$$

where $\kappa^{-1} = n_0\partial\mu/\partial n_0$, and $M^{*-1} = \partial^2\varepsilon/\partial k^2$. As discussed in Ref. [11], when $k_{\rm L}^2 \gg n_0$, the sound speed of SO coupled BECs can be well predicated by Eq. (3). In our work, we mainly discuss sound speed of the BECs in the plane-wave phase for $\Omega < k_{\rm L}^2$ regime and the zero-momentum phase for $\Omega > k_{\rm L}^2$ regime.

When $\Omega < k_{\rm L}^2$, the lower energy branch has a double-well structure, with two degenerate single-particle ground states at $k = \pm k_m = \pm k_{\rm L}\sqrt{1-\Omega^2/k_{\rm L}^4}$. Considering $\Omega \ll {k_{\rm L}}^2$, one obtains the equations for the Bloch states at $k = k_m$ up to the second order of Ω,

$$\Big[-\frac{1}{2}(\partial_x + ik_{\rm L})^2 \pm ik_{\rm L}\partial_x \mp kk_{\rm L} + \frac{k_{\rm L}^2}{2} + V_{\rm opt}(x) + n_0|\phi_{k_m\sigma}|^2 + \beta n_0|\phi_{k_m\bar\sigma}|^2\Big]\phi_{k_m\sigma} + \Omega\phi_{k_m\bar\sigma} = \mu\phi_{k_m\sigma}. \tag{4}$$

For weak optical lattice, the optical potential can be regarded as a perturbation, we can expand the wave functions as[26,30,31] $\phi_{k_m\sigma} = \phi^{(0)}_{k_m\sigma} + \phi^{(1)}_{k_m\sigma} + \phi^{(2)}_{k_m\sigma} + \ldots$, and the chemical potential as $\mu_{k_m} = \mu^{(0)}_{k_m} + \mu^{(1)}_{k_m} + \mu^{(2)}_{k_m} + \ldots$, where the superscript denotes the order in s. Substituting these into Eq. (4), we can obtain the equations in different orders in s. For the zeroth order in s, we obtain

$$(\mp kk_{\rm L} + k_{\rm L}^2)\phi^{(0)}_{k_m\sigma} + n_0\phi^{(0)3}_{k_m\sigma} + n_0\beta\phi^{(0)}_{k_m\sigma}\phi^{(0)2}_{k_m\bar\sigma} + \Omega\phi^{(0)}_{k_m\bar\sigma} = \mu^{(0)}_{k_m}\phi^{(0)}_{k_m\sigma}, \tag{5}$$

along with the normalization condition $V^{-1}\int_V dx(|\phi_{k_m\sigma}|^2 + |\phi_{k_m\bar\sigma}|^2) = 1$. For $\Omega \ll {k_{\rm L}}^2$, the Raman coupling term in Eq. (5) can be treated as a perturbation. By means of employing the perturbation treatment, one can find the ground state wave functions of Eq. (5),[11] $\phi^{(0)}_{k_m\uparrow} = 1 - \Omega^2/[2(2k_{\rm L}^2 + n_0\beta - n_0)^2]$, $\phi^{(0)}_{k_m\downarrow} = -\Omega/(2k_{\rm L}^2 + n_0\beta - n_0)$, and the chemical potential $\mu^{(0)}_{k_m} = n_0 - [2k_{\rm L}^2/(2k_{\rm L}^2 + n_0\beta - n_0)^2]\Omega^2$. To the first order in s, we obtain $\mu^{(1)}_{k_m} = 0$ and

$$\Big[-\frac{1}{2}(\partial_x + ik_{\rm L})^2 \pm ik_{\rm L}\partial_x \mp kk_{\rm L} + \frac{k_{\rm L}^2}{2}\Big]\phi^{(1)}_{k_m\sigma} + V_{\rm opt}\phi^{(0)}_{k_m\sigma} + 3n_0\phi^{(0)2}_{k_m\sigma}\phi^{(1)}_{k_m\sigma} + n_0\beta(\phi^{(1)}_{k_m\sigma}\phi^{(0)2}_{k_m\bar\sigma} + 2\phi^{(0)}_{k_m\sigma}\phi^{(0)}_{k_m\bar\sigma}\phi^{(1)}_{k_m\sigma}) + \Omega\phi^{(1)}_{k_m\bar\sigma} = \mu^{(0)}_{k_m}\phi^{(1)}_{k_m\sigma}. \tag{6}$$

To the second order in s, we obtain

$$\begin{aligned}&\Big[-\frac{1}{2}(\partial_x + ik_{\rm L})^2 \pm ik_{\rm L}\partial_x \mp kk_{\rm L} + \frac{k_{\rm L}^2}{2}\Big]\phi^{(2)}_{k_m\sigma} + V_{\rm opt}(x)\phi^{(1)}_{k_m\sigma} + 3n_0(\phi^{(0)2}_{k_m\sigma}\phi^{(2)}_{k_m\sigma} + \phi^{(1)2}_{k_m\sigma}\phi^{(0)}_{k_m\sigma}) \\ &+ n_0\beta(\phi^{(0)2}_{k_m\bar\sigma}\phi^{(2)}_{k_m\sigma} + 2\phi^{(0)}_{k_m\sigma}\phi^{(0)}_{k_m\bar\sigma}\phi^{(2)}_{k_m\bar\sigma} + 2\phi^{(1)}_{k_m\sigma}\phi^{(1)}_{k_m\bar\sigma}\phi^{(0)}_{k_m\bar\sigma}) + \Omega\phi^{(2)}_{k_m\bar\sigma} \\ &= \mu^{(0)}_{k_m}\phi^{(2)}_{k_m\sigma} + \mu^{(2)}_{k_m}\phi^{(0)}_{k_m\sigma}.\end{aligned} \tag{7}$$

Based on the above equations, the chemical potential up to the second-order in s is

$$\mu_{k_m} = n_0 - \frac{2k_{\rm L}^2}{(2k_{\rm L}^2 + n_0\beta - n_0)^2}\Omega^2 + As^2, \tag{8}$$

where A is shown in the supplementary material. To calculate the effective mass, we also need to calculate the energy of the system at $k = k_m$, which can be obtained using the effective potential. The system can be treated as a noninteracting system in an effective potential,

$$V_{{\rm eff}\sigma}(x) = V_{\rm opt}(x) + n_0|\phi_{k_m\sigma}|^2 + \beta n_0|\phi_{k_m\bar\sigma}|^2 - \mu_{k_m}. \tag{9}$$

Since the correction to the system energy is of second order in the potential strength, it is sufficient to consider the first-order correction of the Bloch states. Substituting the wave functions up to the first-order into Eq. (9), the effective potential can be given as $V_{{\rm eff}\sigma}(x) = v_\sigma V_{\rm opt}(x)$, where v_σ is shown in the supplementary material. Then the energy equations for the system in the effective potential are

$$\Big[-\frac{1}{2}(\partial_x + ik_{\rm L})^2 \pm ik_{\rm L}\partial_x \mp kk_{\rm L} + \frac{k_{\rm L}^2}{2} + V_{{\rm eff}\sigma}\Big]\phi_{k_m\sigma} + \Omega\phi_{k_m\bar\sigma} = \epsilon\phi_{k_m\sigma}. \tag{10}$$

Similar to the calculation of the chemical potential, we can obtain the energy of the system from Eq. (10), and then M^* at $k = k_m$ can be calculated as

$$\frac{1}{M^*} = \frac{\partial^2 \epsilon}{\partial k^2}\Big|_{k=k_m} = 1 - \frac{\Omega^2}{k_{\rm L}^4} + Js^2, \tag{11}$$

where the coefficient J is shown in the supplementary material. Then, from Eq. (3), ignoring the influences of high order in Ω reasonably, we find that the expression of sound speed for the SO coupled BECs in $\Omega < k_{\rm L}^2$ regime is

$$\begin{aligned} c_{\rm s} = {} & \sqrt{n_0}\Big[1 - \frac{\Omega^2}{k_{\rm L}^4} + \frac{2k_{\rm L}^2(\beta-1)\Omega^2}{(2k_{\rm L}^2 + n_0\beta - n_0)^3}\Big] \\ & \cdot \Big\{1 + [4k_{\rm L}^6 J(\beta-1)\Omega^2 + (2k_{\rm L}^2 + n_0\beta \\ & - n_0)^3(k_{\rm L}^4(J+B) - B\Omega^2)]/[8k_{\rm L}^6(\beta-1)\Omega^2 \\ & + 2(2k_{\rm L}^2 + n_0\beta - n_0)^3(k_{\rm L}^4 - \Omega^2)]s^2\Big\}, \end{aligned} \tag{12}$$

where the coefficient B is shown in the supplementary material.

In the $\Omega > k_{\rm L}^2$ regime, the lower branch of the single-particle energy changes into a single-well structure, atoms condense at the $k = 0$ state, which is the so-called zero momentum phase. The equations for the Bloch states of Eq. (2) at $k = 0$ are described by

$$\begin{aligned} & \Big[-\frac{1}{2}\partial_x^2 \pm ik_{\rm L}\partial_x + \frac{k_{\rm L}^2}{2} + V_{\rm opt}(x) + n_0|\phi_{0\sigma}|^2 \\ & + \beta n_0|\phi_{0\bar{\sigma}}|^2\Big]\phi_{0\sigma} + \Omega\phi_{0\bar{\sigma}} = \mu_0\phi_{0\sigma}. \end{aligned} \tag{13}$$

Similar to the treatment for the small Raman coupling, taking the optical lattice as a perturbation, the equations for the Bloch states at the zeroth order in s are

$$\frac{k_{\rm L}^2}{2}\phi_{0\sigma}^{(0)} + n_0\phi_{0\sigma}^{(0)3} + n_0\beta\phi_{0\sigma}^{(0)}\phi_{0\bar{\sigma}}^{(0)2} + \Omega\phi_{0\bar{\sigma}}^{(0)} = \mu_0^{(0)}\phi_{0\sigma}^{(0)}, \tag{14}$$

along with the normalization condition $V^{-1}\int_V dx(|\phi_{0\sigma}|^2 + |\phi_{0\bar{\sigma}}|^2) = 1$. We obtain the wave functions $\phi_{0\uparrow,\downarrow}^{(0)} = \pm 1/\sqrt{2}$, where the spin-up and spin-down components have equal populations. The zeroth order is $\mu_0^{(0)} = \frac{1}{2}(k_{\rm L}^2 + n_0 + n_0\beta - 2\Omega)$.

To the first order in s, $\mu_0^{(1)} = 0$, we obtain

$$\begin{aligned} & \Big(-\frac{1}{2}\partial_x^2 \pm ik_{\rm L}\partial_x + n_0 + \Omega\Big)\phi_{0\sigma}^{(1)} + (\Omega - n_0\beta)\phi_{0\bar{\sigma}}^{(1)} \\ & + \frac{1}{\sqrt{2}}V_{\rm opt}(x) - \frac{1}{\sqrt{2}}\mu_0^{(1)} = 0. \end{aligned} \tag{15}$$

To the second order in s, we obtain

$$\begin{aligned} & \Big(-\frac{1}{2}\partial_{x^2} \pm ik_{\rm L}\partial_x\Big)\phi_{0\sigma}^{(2)} + V_{\rm opt}(x)\phi_{0\sigma}^{(1)} \\ & + \frac{n_0}{2}(\phi_{0\sigma}^{(2)} + \phi_{0\sigma}^{(2)*}) + \frac{3n_0}{\sqrt{2}}\phi_{0\sigma}^{(1)2} \pm \frac{\beta n_0}{\sqrt{2}}\phi_{0\bar{\sigma}}^{(1)2} \\ & - \frac{n_0\beta}{2}(\phi_{0\bar{\sigma}}^{(2)} + \phi_{0\bar{\sigma}}^{(2)*}) - \sqrt{2}\beta n_0\phi_{0\sigma}^{(1)}\phi_{0\bar{\sigma}}^{(1)} \\ & + \Omega(\phi_{0\sigma}^{(2)} + \phi_{0\bar{\sigma}}^{(2)}) = \pm\frac{1}{\sqrt{2}}\mu_{0\sigma}^{(2)}. \end{aligned} \tag{16}$$

Then, the chemical potential can be obtained as

$$\begin{aligned} \mu_0 = {} & \frac{1}{2}(k_{\rm L}^2 + n_0 + n_0\beta - 2\Omega) \\ & + s^2b^2[b^4 - 4b^2(k_{\rm L}^2 + n_0\beta - n_0 - 2\Omega) \\ & + 16k_{\rm L}^2(n_0\beta - n_0 - \Omega) \\ & + 4(n_0 - n_0\beta + 2\Omega)^2]/4[b^4 \\ & + 4b^2(-k_{\rm L}^2 + n_0 + \Omega) \\ & + 4n_0(n_0 - n_0\beta^2 + 2(1+\beta)\Omega)]^2. \end{aligned} \tag{17}$$

Based on Eq. (9), the effective potentials are

$$\begin{aligned} V_{{\rm eff}\uparrow}(x) = {} & \frac{V_{\rm opt}(x)}{C_1}[(b^2 - 2bk_{\rm L})(b^2 + 2bk_{\rm L} \\ & - 2n_0(\beta-1) + (2b^2 - 4bk_{\rm L} \\ & + 2n_0(\beta-1))\Omega - 8\Omega^2], \\ V_{{\rm eff}\downarrow}(x) = {} & \frac{V_{\rm opt}(x)}{C_1}[(b^4 + (4n_0(\beta-1) \\ & - 8\Omega)\Omega + 4bk_{\rm L}(n_0 - n_0\beta + \Omega) \\ & + 2b^2(-2k_{\rm L}^2 + n_0 - n_0\beta + \Omega)], \end{aligned} \tag{18}$$

where $C_1 = \sqrt{2}[b^4 + 4b^2(-k_{\rm L}^2 + n_0 + \Omega) + 4n_0(n_0 - n_0\beta^2 + 2\Omega(\beta+1))]$. At $k = 0$, M^* is

$$\frac{1}{M^*} = \frac{\partial^2 \epsilon}{\partial k^2}\Big|_{k=0} = 1 - \frac{k_{\rm L}^2}{\Omega} - Qs^2, \tag{19}$$

where Q is shown in the supplementary material. Then, according to Eq. (3), the expression of sound speed for the SO coupled BECs in $\Omega > k_{\rm L}^2$ regime is

$$\begin{aligned} c_{\rm s} = {} & \sqrt{\frac{n_0}{2}(1+\beta)\Big(1 - \frac{k_{\rm L}^2}{\Omega}\Big)} \\ & \cdot \Big[1 + \frac{2P(1 - \frac{k_{\rm L}^2}{\Omega}) - n_0Q(1+\beta)}{2n_0(1+\beta)(1 - \frac{k_{\rm L}^2}{\Omega})}s^2\Big], \end{aligned} \tag{20}$$

where the coefficient P is shown in the supplementary material.

Equations (12) and (20) give the sound speed of SO coupled BECs in the plane-wave phase and zero-momentum phase regimes, respectively. It is clear that, when the lattice is absent, i.e., $s = 0$, the sound speeds reduce to the case of free space, which agrees with the previous result.[11]

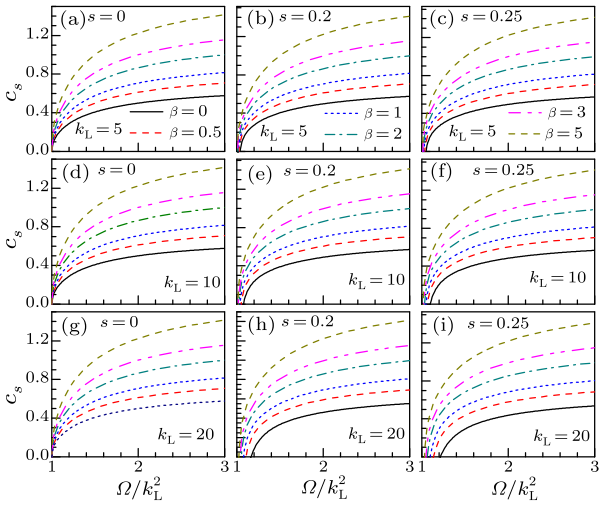

Fig. 1. Sound speed against $\Omega/k_{\rm L}^2$ for different β in zero-momentum phase regime.

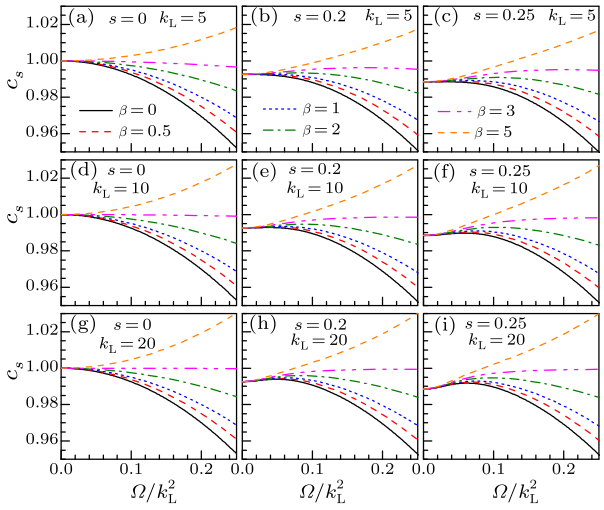

Fig. 2. Sound speed against $\Omega/k_{\rm L}^2$ for different β in plane-wave phase regime.

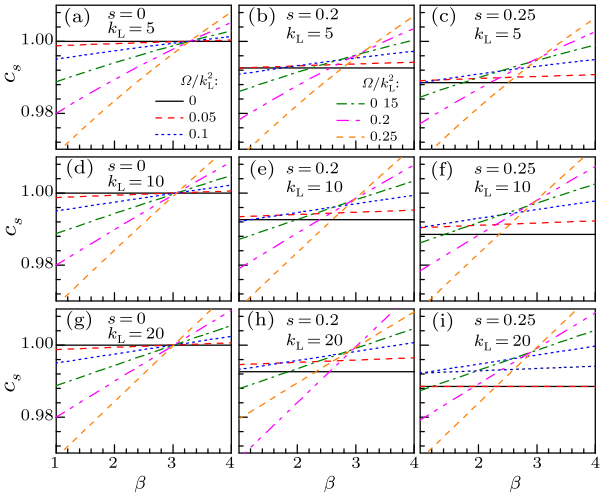

Fig. 3. Sound speed against β for different $\Omega/k_{\rm L}^2$ in $\Omega < k_{\rm L}^2$ regime.

To clearly show how the sound speed changes with Ω, $k_{\rm L}$ and β, and how it transits between the two phases, the sound speed of the two phase regimes given by Eqs. (12) and (20) are illustrated in Figs. 1 and 2, respectively. In all results, we set $b = 0.1$ and $n_0 = 1$. As demonstrated in Ref. [30], the sound speed for a weak optical lattice can be well predicted by the perturbation method when $s < 1$. Thus we set $s < 1$ in numerical results. We clearly see that the sound speed increases with β in the two regimes.

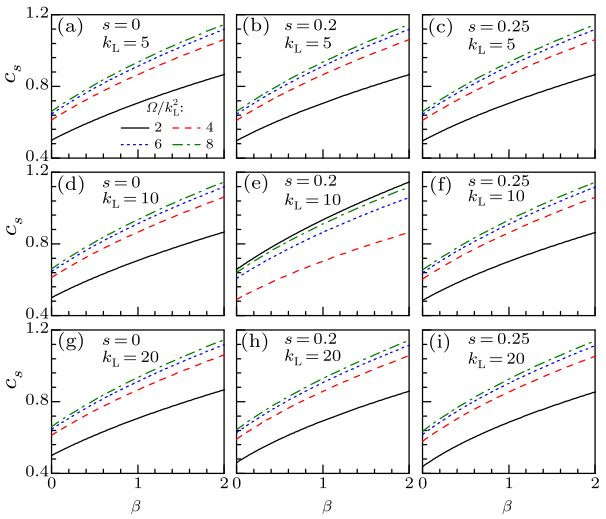

Fig. 4. Sound speed against β for different $\Omega/k_{\rm L}^2$ in $\Omega > k_{\rm L}^2$ regime.

Interestingly, we find from Fig. 1 that, in the zero-momentum phase regime, the sound speed always decreases rapidly and vanishes (i.e., phonon mode softening occurs) at $\Omega/k_{\rm L}^2 \sim 1$ or in a window near $\Omega/k_{\rm L}^2 \sim 1$, where the phase transition between the plane-wave phase and the zero-momentum phase occurs. The behaviors of sound speed displayed in Fig. 1 can be explained by the modification of the single-particle dispersion. Without the optical lattice (the first column in Fig. 1), it is clear from the expression of the effective mass given by Eq. (19) that the vanishing of sound speed in zero-momentum phase regime originates in the divergency of the effective mass at $\Omega/k_{\rm L}^2 = 1$, which marks the phase transition between the plane-wave phase and the zero-momentum and is consistent with the critical value[12] $\Omega/k_{\rm L}^2 = 1 + n_0(\beta-1)/(2k_{\rm L}^2) \approx 1$ for $k_{\rm L}^2 \gg n_0$. In particular, the critical value of Ω is exactly $\Omega/k_{\rm L}^2 = 1$ when $\beta = 1$. Considering the optical lattice, as is shown in the second and the third columns of Fig. 1, the situations are very different from the case without the lattice and the softening of the phonon mode in the zero-momentum phase becomes more obvious. As we can see in each row in Fig. 1, if $k_{\rm L}$ keeps the same while the optical lattice strength is enhanced, the critical value of $(\Omega/k_{\rm L}^2)_{\rm c}$, at which the sound speed vanishes for zero-phase mode, increases and depends on β. The smaller the β is, the larger the $(\Omega/k_{\rm L}^2)_{\rm c}$ shifts. Particularly, for a fixed β, the sound speed disappears in a regime of $1 < \Omega/k_{\rm L}^2 < (\Omega/k_{\rm L}^2)_{\rm c}$, and this regime is widened with the decrease of β or the increase of s. The shifting of $(\Omega/k_{\rm L}^2)_{\rm c}$ and disappearance of sound speed are more significant for larger $k_{\rm L}$ (see the third row in Fig. 1). That is, when the softening of phonon

mode occurs only at phase transition point $\Omega/k_{\rm L}^2 \approx 1$ for uniform space ($s = 0$), it can take place in a window of $\Omega/k_{\rm L}^2 \approx 1$ when the lattice presents, and β, $k_{\rm L}$, s have significant coupled influence on it. Physically, phonon mode softening indicates the appearance of dynamical instability and disappearance of superfluid in the system. The presence of optical lattice enhances this instability.

Figures 1 and 2 illustrate that the variation of sound speed against $\Omega/k_{\rm L}^2$ also behaves with different characteristics in plane wave phase and zero momentum phase regime. First, we discuss the situation in $\Omega < k_{\rm L}^2$ regime (see Fig. 2). In the free space, $s = 0$, there exists a critical value of $\beta_{\rm c}$, when $\beta < \beta_{\rm c}$, the sound speed decreases monotonically with Ω, when $\beta = \beta_{\rm c}$, the sound speed nearly keeps the same and it is independent of Ω, and when $\beta > \beta_{\rm c}$, the sound speed increases monotonically with Ω. Specifically, when Ω is small, β barely affects the sound speed, once Ω is large enough, β starts to influence the behaviors of sound speed. When the lattice presents, the situations are very different. The second and third columns of Fig. 2 for $\Omega/k_{\rm L}^2 < 1$ regime show that optical lattice suppresses the sound speed, especially for small Ω cases, that is, the suppressing of sound speed by optical lattice is more significant when Ω is weak. Interestingly, when $\beta > \beta_{\rm c}$, the sound speed increases monotonically with Ω, but when $\beta < \beta_{\rm c}$, the sound speed first increases with Ω and then decreases with Ω. This is different from the case of $s = 0$. The influence of β on sound speed is shifted to much larger value of Ω as the optical lattice strength increases. We obtain that due to the optical lattice, the behaviors of the sound speed become much richer than the case of the SO coupled BECs in free space.

The coupled effects of Ω, $k_{\rm L}$, β and s on sound speed are further illustrated in Fig. 3 for the $\Omega < k_{\rm L}^2$ case. We can see that the sound speed is proportional to β, interestingly, when $s = 0$, there exists a fixing point corresponding to $\beta = \beta_{\rm c}$, where the sound speed with different values of Ω degenerates. As $\beta < \beta_{\rm c}$, the sound speed decreases with Ω, and as $\beta > \beta_{\rm c}$, the sound speed increases with Ω. However, when the optical lattice exists, the situations are changed. As we can see in the second and third columns of Fig. 3, the fixing point $\beta_{\rm c}$ still exists except for the sound speed with small values of Ω, as is shown by the lines corresponding to $\Omega/k_{\rm L}^2 = 0$, 0.05 and 0.1 in Fig. 3, shifting downward and departing from the fixing point, and especially, when the optical lattice is deep and $k_{\rm L}$ is larger, the phenomena become much more obvious. These indicate that optical lattice suppresses sound speed, especially for small Ω. The smaller the Ω is, the larger the sound speed is suppressed. We also find that if Ω is small, β and $k_{\rm L}$ have very weak influence on sound speed and the sound speed strongly depends on s. These can be understood by Eq. (12), when $\Omega/k_{\rm L}^2 \to 0$, $c_{\rm s} \to \sqrt{n_0}(1-(b^6-36b^2n_0^2+48n_0^3)/(4(b^2+4n_0)^3)s^2)$. We can find that when the SO coupling is weak, the sound speed is only related to the optical lattice but not affected by β and $k_{\rm L}$. Figure 3 also indicates that the fixing point $\beta = \beta_{\rm c}$ changes with $k_{\rm L}$ and s. Except that, as we see in every row of Fig. 3, when $\beta = \beta_{\rm c}$, the sound speed nearly keeps the same, when $\beta > \beta_{\rm c}$, the sound speed decreases with $k_{\rm L}$. This can be understood as follows: without optical lattice, according to $c_{\rm s} = \sqrt{n_0}(1-\Omega^2/k_{\rm L}^4+2k_{\rm L}^2(\beta-1)\Omega^2/(2k_{\rm L}^2+n_0\beta-n_0)^3)$, there exists a critical $\beta_{\rm c} \approx 3.2$, when $\beta < \beta_{\rm c}$, the sound speed increases with $k_{\rm L}$, and when $\beta > \beta_{\rm c}$, the sound speed decreases with $k_{\rm L}$. Considering the optical lattice, $\beta_{\rm c}$ shifts with s and $k_{\rm L}$ but still exists.

In the $\Omega > k_{\rm L}^2$ regime, as we can see from Figs. 1 and 4, the sound speed monotonically increases with Ω and β. Specifically, when Ω is large enough, the sound speed will weakly depend on Ω. Also we find that the sound speed always decreases with $k_{\rm L}$ and s. This means that $k_{\rm L}$ and s always suppress the sound speed. The suppressing of sound speed by $k_{\rm L}$ and s is more obvious when Ω and β are small. According to the expressions of the effective mass and compressibility, we find that in this case $k_{\rm L}$ only changes the effective mass much but does not affect the compressibility, $k_{\rm L}$ always increases the effective mass, therefore the sound speed invariably decreases with $k_{\rm L}$.

In summary, we have investigated the sound wave of the SO coupled BECs trapped in a one-dimensional optical lattice. The dependence of the sound speed on Raman coupling strength, SO coupling strength, optical lattice strength and atomic interaction strengths is provided analytically. Our results may provide quantitative theoretical evidence for investigating the phonon mode excitation of the SO coupled BECs in optical lattice experimentally.

References

[1] Jin D S et al 1996 *Phys. Rev. Lett.* **77** 420
[2] Mewes M O et al 1996 *Phys. Rev. Lett.* **77** 988
[3] Lin Y J et al 2011 *Nature* **471** 83
[4] Zhang J Y et al 2012 *Phys. Rev. Lett.* **109** 115301
[5] Wang P J et al 2012 *Phys. Rev. Lett.* **109** 095301
[6] Cheuk L W et al 2012 *Phys. Rev. Lett.* **109** 095302
[7] Li Y et al 2012 *Phys. Rev. Lett.* **108** 225301
[8] Zhang Y C et al 2015 *Phys. Rev. Lett.* **115** 253902
[9] Sakaguchi H et al 2014 *Phys. Rev.* E **89** 032920
[10] Achilleos V et al 2013 *Phys. Rev. Lett.* **110** 264101
[11] Zheng W and Li Z B 2012 *Phys. Rev.* A **85** 053607
[12] Martone G I et al 2012 *Phys. Rev.* A **86** 063621
[13] Zhang Y C et al 2016 *Phys. Rev.* A **94** 033635
[14] Khamehchi M A et al 2014 *Phys. Rev.* A **90** 063624
[15] Ji S C et al 2015 *Phys. Rev. Lett.* **114** 105301
[16] Hamner C et al 2015 *Phys. Rev. Lett.* **114** 070401
[17] Kane C L and Mele E J 2005 *Phys. Rev. Lett.* **95** 226801
[18] Fu L et al 2007 *Phys. Rev. Lett.* **98** 106803
[19] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[20] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
[21] Chen Z and Liang Z X 2016 *Phys. Rev.* A **93** 013601
[22] Zhang Y P and Zhang C W 2013 *Phys. Rev.* A **87** 023611
[23] Toniolo D and Linder J 2014 *Phys. Rev.* A **89** 061605(R)
[24] Zhai H 2015 *Rep. Prog. Phys.* **78** 026001
[25] Sun Q et al 2013 *Phys. Rev.* A **88** 063637
[26] Taylor E and Zaremba E 2003 *Phys. Rev.* A **68** 053611
[27] Bychkov Y A and Rashba E I 1984 *J. Phys.* C **17** 6039
[28] Dresselhaus G 1955 *Phys. Rev.* **100** 580
[29] Menotti C et al 2004 *Phys. Rev.* A **70** 023609
[30] Zhang A X and Xue J K 2009 *Phys. Rev.* A **80** 043617
[31] Liang Z X et al 2008 *Phys. Rev.* A **78** 023622

Supplemental Material for “Parameters introduced to simplify expressions”

In this section we show parameters introduced to simplify expressions corresponding to the chemical potential, the sound speed, the effective potential, and the effective mass.

(1) In $\Omega < k_L^2$ regime. The coefficient A in the expression of the chemical potential Eq. (10) is:

$$A = \{2b^2 f^5(b^2 + f + 4bk_L)^2 + 4f^2(96 f k_L^2 n_0^2(\beta-1)^2 + 8bk_L(-f^3 + 2f^2k_L^2 + 4fk_L^4 + 2fn_0(f-2n_0)(\beta-1) -)n_0^3(\beta-5)(\beta-1)^2 + 2k_L^2 n_0^2(\beta-1)(7\beta-3)) + b^2[-2f^3 + 24fk_L^4 + 2n_0(8k_L^4(3-2\beta) + 2k_L^2 n_0(\beta-1)(\beta+7) - n_0^2(\beta-1)^2(3\beta-7)) + f^2(15f - 2n_0(3+4\beta))]\Omega^2\}/8f^2(b^2 + f + 4bk_L)(b^2 + 4n_0)(2f^3(b^2 + f + 4bk_L)(b^2 + 4n_0) + \langle 3b^4(f - 4n_0(\beta-1)) + 12b^3k_L(f - 4n_0(\beta-1)) + 16bk_L(2f^2 - fn_0 - 12n_0^2(\beta-1)) + 8fn_0(-25n_0(\beta-1) + 2k_L^2(8\beta+3)) + b^2(-24n_0(f-2n_0)(\beta-1) + f(44k_L^2 + 6n_0(\beta+1)))\rangle\Omega^2),$$

and the coefficient B in the expression of the sound speed Eq. (15) is:

$$B = 6b^2k_L^4n_0^2(8k_L^2 + 3n_0)(2\beta-1)^3/\{2k_L^4n_0(2\beta-1)(b^2 + n_0(2\beta-1))((b+2k_L)^2 + 5n_0) + [2k_L^4n_0(8k_L^4 + k_L^2n_0(38-4\beta) - n_0^2(\beta-1)(13+4\beta)) + (2k_L^2 + n_0 - n_0\beta)(24b^3k_Ln_0(2\beta-1)(2k_L^2 + 5n_0\beta))]\Omega^2\},$$

where $f = 2k_L^2 + n_0\beta - n_0$. The coefficients $v_\uparrow$, $v_\downarrow$ in the expression of the effective potential Eq. (12) are:

$$v_\uparrow = \{2f^2(b^2 + 2f + 4bk_L)(b^2 - 2n_0) + [-b^4 - 4b^3k_L + 8bk_L(4k_L^2 + n_0) + 2b^2(6k_L^2 + n_0(8-7\beta)) + 4(-2f^2 + 8k_L^4 - 6n_0(f + k_L^2) - 7n_0^2 + n_0\beta(4f + 4k_L^2 + 7n_0))\Omega^2]\}/[2f^2(b^2 + 2f + 4bk_L)(b^2 + 4n_0) + 8(-f^2 + 2k_L^2(b^2 + 2bk_L + 2k_L^2) - 2k_Ln_0(3b + 2k_L) + 9n_0^2 + 4n_0f\beta + n_0\beta(2bk_L + 4k_L^2 - 6n_0 - 3n_0\beta))],$$

$$v_\downarrow = [f(-(b+2k_L)^2 - 2n_0)(b^2 + 4n_0) + 4n_0\beta(f + (b+2k_L)^2 - 2n_0)\Omega]/[f^2(b^2 + 4n_0)((b+2k_L)^2 + 2n_0(\beta-1)) + 4(-f^2 + 2k_L^2(b^2 + 2bk_L + 2k_L^2) - 2k_Ln_0(3b + 2k_L) + 9n_0^2 + 4n_0f\beta + n_0\beta(2bk_L + 4k_L^2 - 6n_0 - 3n_0\beta))].$$

The coefficient J in the expression of the effective mass Eq. (14) is:

$$J = 2b\{-192k_L^6(b^2 + 2bk_L + 4k_L^2)^4v_\uparrow^2 + [3072b^2k_L^8v_\uparrow(42v_\uparrow - 29v_\downarrow) + 6912b^3k_L^7v_\uparrow(23v_\uparrow - 10v_\downarrow) + 6144bk_L^9v_\uparrow(7v_\uparrow - 6v_\downarrow) + 39b^{10}(v_\uparrow - v_\downarrow)(v_\uparrow + v_\downarrow) - 16384k_L^{10}v_\uparrow(2v_\uparrow + v_\downarrow) + 4b^6k_L^4(5851v_\uparrow^2 - 2160v_\uparrow v_\downarrow - 928v_\downarrow^2) + 2b^7k_L^3(3391v_\uparrow^2 - 912v_\uparrow v_\downarrow - 928v_\downarrow^2) + b^8k_L^2(1607v_\uparrow^2 - 256v_\uparrow v_\downarrow - 853v_\downarrow^2) + 64b^4k_L^6(1879v_\uparrow^2 - 796v_\uparrow v_\downarrow - 108v_\downarrow^2) + 96b^5k_L^5(633 - 258v_\uparrow v_\downarrow - 44v_\downarrow^2) + 12b^9k_L(22v_\uparrow^2 - 17v2^2)]\Omega^2\}/[k_L^2(b^2 + 2bk_L + 4k_L^2)^3(16bk_L^4(b^2 + 2bk_L + 4k_L^2) - 9b^3 - 24b^2k_L + 144bk_L^2 + 96k_L^3)\Omega^2].$$

(2) In $\Omega > k_L^2$ regime. The coefficient Q in the expression of the effective mass Eq. (24) is:

$$
\begin{aligned}
Q = \{&16b^2k_L^2(k_L^2 - \Omega)(16(3n_0(\beta - 1) - 4\Omega)(k_L^2 - \Omega) \\
&(n_0(\beta - 1) - \Omega)\Omega^2 + b^2[4k_L^4(n_0(\beta - 1) - 5\Omega)(n_0(\beta - 1) - \Omega) \\
&- 4k_L^2(n_0(\beta - 1) - 4\Omega)(n_0(\beta - 1) - \Omega)\Omega + \Omega^2(-27n_0^2(\beta - 1)^2 \\
&+ 64(n_0(\beta - 1)\Omega - 36\Omega^2))]\}/[\Omega(b^2 - 4k_L^2 + 4\Omega)^3(4b^2(-k_L^2 \\
&+ n_0 + \Omega) + 4n_0(n_0 - n_0\beta^2 + 2\Omega(\beta + 1)))^2].
\end{aligned}
$$

The coefficient P in the expression of the sound speed Eq. (25) is:

$$
\begin{aligned}
P = \{&2b^6(\beta - 1)(b^2 - 4k_L^2 - 2n_0(\beta - 1) + 4\Omega)(b^4 \\
&- k_L^2 2b^2 + n_0 2b^2 + 4n_0^2 - 4n_0^2\beta^2 + 8n_0(\beta + 1)\Omega + 2b^2\Omega] \\
&+ 2(b^2 + 2(n_0 - n_0\beta^2 + (\beta + 1)\Omega))(b^4 - 4b^2(k_L^2 + n_0(\beta - 1) \\
&- 2\Omega) + 16k_L^2(n_0(\beta - 1) - \Omega) + 4(n_0 - n_0\beta + 2\Omega)^2))]\}/[b^4 \\
&+ 4b^2(-k_L^2 + n_0 + \Omega) + 4n_0(n_0 - n_0\beta^2 + 2(\beta + 1)\Omega))]^3.
\end{aligned}
$$