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# Progress of the Inertial Mass Measurement Project at NIM *

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*An experiment is proposed to precisely measure the Planck constant. In this experiment, the Planck constant is measured based on the inertial mass measurement rather than the gravitational mass determinations in some other well-known experiments, e.g., the Kibble balance and counting atoms. We link the mechanical force to a quantum-traceable electrostatic force by a beam balance oscillator. After a 5-year continuous effort, the principle of the proposal is verified by a preliminary measurement with a relative uncertainty of* $5.4 \times 10^{-5}$. *The proposal has the potential to achieve much higher measurement accuracy with further improvements.*

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The Planck constant $h$ was first introduced in 1900 to describe Planck's hypothesis that the energy of the electromagnetic wave is quantized and one quantum of energy $E$ can be expressed as $E = h\nu$, where $\nu$ denotes the frequency.[1] Later the basic role of $h$ has also been reflected in many other aspects in quantum mechanics, e.g., the uncertainty principle, the Schrödinger equation, the Josephson effect and the quantum Hall effect. Precisely measuring $h$ has a fundamental importance in modern physics.[2] Currently, two precision experiments, i.e., the counting atoms and the Kibble balance, are being carried out to measure $h$ with a relative uncertainty with a few parts in $10^8$.[3–7]

The counting atoms method measures the Avogadro constant $N_{\rm A}$ by counting atoms in a purified silicon sphere, and then the Planck constant is determined based on the product of $N_{\rm A}h$, whose value is experimentally determined with a very high accuracy.[8] Specifically, $N_{\rm A}h$ can be expressed as

$$N_{\rm A}h = \frac{c_0\alpha^2 A_{\rm r}(e) M_{\rm u}}{2R_\infty}, \tag{1}$$

where $A_{\rm r}(e)$ is the electron relative atomic mass, $M_{\rm u}$ is the molar mass constant, $R_\infty$ is the Rydberg constant, $c_0$ is the speed of light in a vacuum, and $\alpha$ is the fine structure constant. With a long-term international collaboration of different national metrology institutes, the latest $h$ measurement of this project is in an uncertainty of $2.0 \times 10^{-8}$.[9]

The Kibble balance project was proposed by Kibble in 1975.[10] The measurement process of the Kibble balance is divided into two parts, i.e., the weighing mode and the velocity mode. In the weighing mode, an electrified coil is put in a magnetic field, and the magnetic force on the coil can balance the gravitational force of a test mass $m$, i.e., $BLI = mg$, where $B$ is the magnetic flux density, $I$ is the current through the coil, $L$ is the total length of the coil, and $g$ is the local gravitational acceleration. In the velocity mode, the coil is controlled to move with a velocity $v$ through the same magnetic field, and the induced voltage $U = BLv$ is measured. Here $BL$ can be eliminated by combining two modes, yielding a power balancing equation,

$$mgv = UI, \tag{2}$$

where $U$ is measured by linking it to the Josephson effect as

$$U = \frac{f_1 h}{2e}, \tag{3}$$

with $f_1$ being a known frequency and $e$ the electron charge. In Eq. (2), $I$ is measured by the Josephson effect in conjunction with the quantum Hall effect as

$$I = \frac{U_1}{R} = \frac{f_2 h}{2e}\frac{ne^2}{h} = \frac{f_2 ne}{2}, \tag{4}$$

where $U_1$ is the voltage drop on a resistor $R$ in series with the coil, $f_2$ is a known frequency, and $n$ is an integer number. Combining Eqs. (2)–(4), $h$ can be expressed as

$$h = \frac{4mgv}{f_1 f_2 n}. \tag{5}$$

All physical quantities in Eq. (5) can be determined at the $10^{-9}$ level, and hence $h$ in principle can be measured with an uncertainty of a few parts in $10^8$. The most accurate $h$ measurement results by the Kibble balance experiment are published by the National Research Council (NRC, Canada) and the National Institute of Standards and Technology (NIST, USA), with relative uncertainties of $1.9 \times 10^{-8}$[11] and $3.4 \times 10^{-8}$.[12] Other national metrology institutes still work hard to achieve the same degree of measurement uncertainty.[13–16]

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Note that the mass $m$ represented by counting atoms and the Kibble balances is the gravitational mass, which is linked to $h$ by the gravity of the test mass. In physics, there is another concept of mass, i.e., the inertial mass, which reflects the magnitude of inertia following Newton's second law $F = ma$. The inertial mass can be also employed to independently measure the Planck constant because the equivalence of the two types of mass has been verified by precision scientific experiments.[17]

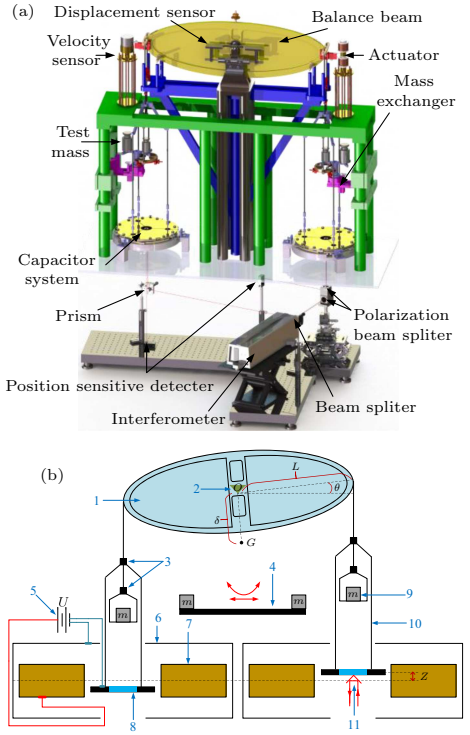


**Fig. 1.** (a) Experimental setup and (b) schematic diagram of the inertial mass measurement experiment. 1 balance beam; 2 knife edge; 3 flexure; 4 mass exchanger; 5 high voltage source; 6 guard electrode; 7 external electrode; 8 internal electrode; 9 test mass; 10 supporting frame; and 11 interferometer.

In 2012, a new $h$ determination principle was proposed based on the inertial mass measurement at National Institute of Metrology (NIM).[18] As shown in Fig. 1, an electromechanical oscillator has been employed in this experiment. The equal-arm balance has a low center of gravity and behaves as a pendulum, resulting in a known relation between the oscillation period $T$ and the test mass $m$.[19] Similar to the Kibble balance project, to determine $h$, the oscillation has to be linked to quantum electrical standards. In this approach, an electrostatic elastic force linked to the Josephson voltage standard and the calculable capacitor is generated: a kV level dc voltage $U$ is applied to an up–down symmetrical capacitor system, as shown in Fig. 1(b). The electrostatic force can also change the oscillation period of the balance. By measuring the oscillation period $T$ as functions of the test mass $m$ and the electrostatic elastic force, $h$ can be obtained. Note that $T$ is used to relate the mass $m$ to the voltage $U$, and it can be linked with a high accuracy time standard.[20]

Analysis in Ref. [18] shows that the motion equation of the balance can be written as

$$[J_0+2(m_0+m)L^2]\frac{d^2\theta}{dt^2}+M_0g\delta\sin\theta+\frac{LU^2}{2}\frac{\partial C(z)}{\partial z}=0, \tag{6}$$

where $\theta$ is the pivot angle, $J_0$ is the rotational inertia of the beam, $m$ is the test mass, $m_0$ is the mass of the suspension (not including for the test mass), $L$ is the beam length, $M_0$ is the mass of the balance beam, $\delta$ is the distance from the rotational center to the gravitational center of the beam, $U$ is the voltage applied to the capacitor, $C(z)$ is the capacitance as a function of the vertical displacement $z$, and $C(z)$ can be expanded as $C(z) = k_0 + k_1 z + k_2 z^2 + \ldots = k_2 z^2 + \Delta C$. Since $z = L\sin\theta$, Eq. (6) turns into

$$\frac{d^2\theta}{dt^2}+\frac{\beta_2+k_2U^2}{\beta_1+2m}\sin\theta+\frac{U^2}{2L(\beta_1+2m)}\frac{\partial\Delta C}{\partial z}=0, \tag{7}$$

where $\beta_1 = J_0/L^2 + 2m_0$ and $\beta_2 = M_0 g\delta/L^2$. Equation (7) can be transformed to a normalized oscillation equation, i.e.,

$$\begin{aligned}&\frac{d^2\theta}{dt^2}+{\omega_0}^2\theta=F(\theta),\\ &F(\theta)=-{\omega_0}^2(\sin\theta-\theta)-\frac{U^2}{2L(\beta_1+2m)}\frac{\partial\Delta C}{\partial z},\end{aligned} \tag{8}$$

where $\omega_0 = \sqrt{(\beta_2 + k_2U^2)/(\beta_1 + 2m)}$, and $F(\theta)$ represents the nonlinear component. Since $\theta$ is a small angle ($< 1.75$ mrad in this design), Eq. (8) is a weak nonlinear differential equation, whose approximate solution can be obtained by an averaging method[21] as

$$\begin{aligned}\theta&=a\cos(\omega_0 t+\psi),\\ \frac{da}{dt}&=-\frac{1}{2\pi\omega_0}\int_0^{2\pi}F(a\cos\phi)\sin\phi d\phi,\\ \frac{d\psi}{dt}&=-\frac{1}{2\pi a\omega_0}\int_0^{2\pi}F(a\cos\phi)\cos\phi d\phi.\end{aligned} \tag{9}$$

Note that $\cos\phi$ and $\sin\phi$ are orthogonal functions in the integral interval from 0 to $2\pi$ and hence $da/dt = 0$, giving a constant $a = \theta_0$. As a result, $d\psi/dt$ is a constant and $\psi$ can be expressed linearly as $\psi = c_1 t + c_0$. Note that all even terms in $\partial(\Delta C)/\partial z$ are eliminated in $d\psi/dt$ and only the terms with $k_4, k_6, k_8, \ldots$ are contained in $c_1$. Accordingly, the expression of the oscillation period $T$ can be written as

$$\begin{aligned}T&=\frac{2\pi}{\omega_0+c_1}=\sum_{p=0}^{\infty}\lambda_{2p}\theta_0^{2p}\\ &\approx\frac{2\pi}{\omega_0}+\frac{2\pi}{\omega_0}\left(\frac{1}{16}-\frac{3k_4U^2L^2}{4{\omega_0}^2(2m+\beta_1)}\right){\theta_0}^2.\end{aligned} \tag{10}$$

With $\theta_0 \to 0$, we can obtain a period solution, i.e.,

$$T=2\pi\sqrt{\frac{\beta_1+2m}{\beta_2+k_2U^2}}. \tag{11}$$

It is emphasized that we do not show the damping term in the derivation because a negative damping system is adopted to compensate for the damping force of air and mechanical frictions to a negligible level. In Eq. (11), $\beta_1$ and $\beta_2$ can be obtained by measuring the different periods $T_m$ under different test masses $m$ when $U = 0\,\mathrm{V}$ or measuring different periods $T_U$ under different voltages $U$ when $m = 0\,\mathrm{kg}$. In the simplest case, $(m_1,\ T_{m1})$, $(m_2,\ T_{m2})$, $(U_1,\ T_{U1})$ and $(U_2,\ T_{U2})$ are measured to eliminate $\beta_1$ and $\beta_2$, and we obtain

$$\frac{2(T_{m2}^2 m_1 - T_{m1}^2 m_2)}{T_{m1}^2 - T_{m2}^2} = \frac{k_2(U_1^2 - U_2^2)T_{U1}^2 T_{U2}^2}{4\pi^2(T_{U2}^2 - T_{U1}^2)}. \quad (12)$$

As shown in Eq. (3), the applied voltages $U_1$ and $U_2$ can be traced to the Josephson voltage standard. Here $h/e^2$ can be expressed by the fine structure constant $\alpha$, the vacuum permeability $\mu_0$ and the speed of light in vacuum $c_0$,[22] i.e.,

$$\frac{h}{e^2} = \frac{c_0\mu_0}{2\alpha}. \quad (13)$$

Equations (3) and (13) determine that $U_1^2 - U_2^2$ can be written as

$$U_1^2 - U_2^2 = \frac{c_0\mu_0(f_1^2 - f_2^2)h}{8\alpha}. \quad (14)$$

The combination of Eqs. (12) and (14) gives a determination of the Planck constant $h$, i.e.,

$$h = \frac{64\pi^2\alpha(T_{m2}^2 m_1 - T_{m1}^2 m_2)}{c_0\mu_0 k_2(f_1^2 - f_2^2)(T_{m1}^2 - T_{m2}^2)}\left(\frac{1}{T_{U1}^2} - \frac{1}{T_{U2}^2}\right). \quad (15)$$

Note that $k_2$ is determined by a measurement of the $C(z)$ curve, linked to calculable capacitor and frequency-based length measurement, which can be obtained as

$$k_2 = \frac{1}{2}\frac{\partial^2 C(z)}{\partial z^2}\Big|_{z=0}. \quad (16)$$

The condition $z = 0$ corresponds to Eq. (11), i.e., $\theta_0 \to 0$.

Obviously, the mass $m_i (i = 1, 2)$ in Eq. (15) is the inertial mass and the experimental measurement does not involve the gravitational acceleration $g$ as it does in the Kibble balance.[23] Moreover, the presented approach brings significant advantages that the air buoyancy in theory will be canceled out in the symmetry, and hence it is not required to build a complicated vacuum device.

This approach has another essential difference from the Kibble balance experiment, i.e., the Kibble balance system is described by a linear Eq. (2) while the presented system is determined by a nonlinear differential Eq. (6). The nonlinearity reflects in the correlation between $T$ and $\theta_0^2$, as shown in Eq. (10). If the nonlinearity is large, a calibration of $T - \theta_0^2$ with different oscillation amplitudes has to be carried out, which may create a considerable uncertainty due to the extrapolation at $\theta_0 = 0$. The nonlinear effect can be a critical problem, and how to minimize its influence should be well considered.

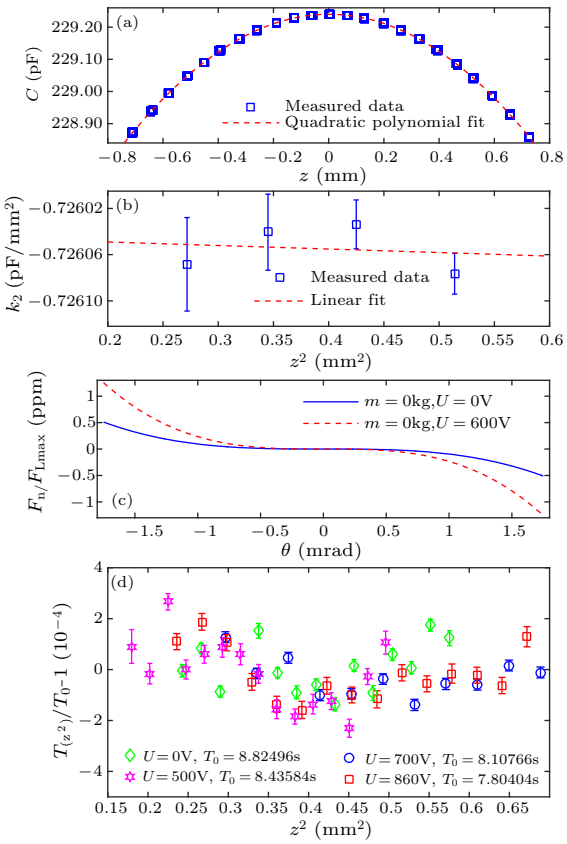


**Fig. 2.** Measurement results of (a) $C(z)$ curve, (b) $k_2$–$z^2$ line, (c) relative nonlinear components when $U = 0\,\mathrm{V}$ and $U = 600\,\mathrm{V}$ (d) residual $T$–$z_0^2$ profiles ($T_0$ is the mean value of $T$).

First, the nonlinear effect of the mechanical force is very small. When $U = 0$, Eq. (10) can be written as

$$T \approx \frac{2\pi}{\omega_0}\left(1 + \frac{\theta_0^2}{16}\right). \quad (17)$$

Since $\theta_0 < 1.75\,\mathrm{mrad}$ in the practical measurement, the $T$ correction due to the oscillation amplitude change is $\theta_0^2/16 \approx 2 \times 10^{-7}$. The nonlinearity can be much larger in the generation of the electrostatic force. In this approach, the capacitor system is specially designed to suppress the nonlinearity of the electrostatic force. Experimental measurements show that the nonlinearity or $k_4$ is related to the thickness of external electrodes. A mean of 4.9 mm thickness of both external electrodes is found as an optimization and adopted in this experimental setup.

The corresponding $C(z)$ curve is measured, as shown in Fig. 2(a). The discrete $C(z)$ is fitted by 4-order polynomials, i.e., $C(z) = k_0 + k_2 z^2 + k_4 z^4$. Based on Eq. (16), $k_2(z) = k_2 + 6k_4 z^2$. Since $k_2(z)$ is proportional to $z^2$, $k_2$ can be calculated by a linear extrapolation of $k_2(z)$ and $z^2$. The measurement result has been shown in Fig. 2(b). The slope of the fitting in Fig. 2(b), with an uncertainty of

$4.8 \times 10^{-5}\,\mathrm{pF/mm^4}$, is used to evaluate $k_4$. Here $k_4$ is calculated as $-5.07\times10^{-6}\,\mathrm{pF/mm^4}$, which yields a relative $T$ correction less than $7\times10^{-7}$ with $U = 860\,\mathrm{V}$ and $\theta_0 = 1.75\,\mathrm{mrad}$. The mean effect ($U = 600\,\mathrm{V}$) is comparable with that of the nonlinearity of the mechanical force.

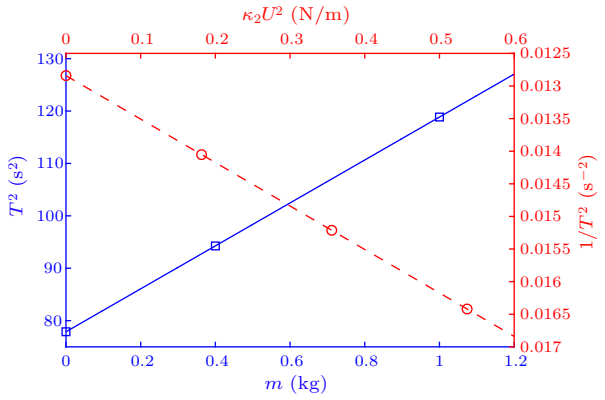


**Fig. 3.** The measurement results of the oscillation period.

The nonlinear component of the electrostatic force when $U = 0\,\mathrm{V}$ and $U = 600\,\mathrm{V}$ is also compared, as shown in Fig. 2(c). The shown curve is the differential signal of the total resorting force and its linear component. Note that this nonlinear force $F_n$ is converted into a relative one, i.e., divided by the maximum value of the linear force $F_{\mathrm{Lmax}}$. It can be seen that the relative value of the nonlinear force for $U = 600\,\mathrm{V}$ is about 2.5 times the value for $U = 0\,\mathrm{V}$.

**Table 1.** The uncertainty budget.

| Uncertainty component | Type | Contribution (ppm) |
|---|---|---|
| Statistic | A | 41 |
| Period measurement | A, B | 10 |
| Length measurement | A, B | 23 |
| Capacitance measurement | A, B | 12 |
| Voltage measurement | A, B | <2 |
| Mass measurement | A, B | <1 |
| Mechanical deformation | A, B | 20 |
| Alignment | A, B | 6 |
| Total ($k = 1$) | A, B | 54 |

As mentioned above, the nonlinearity effect of the electrical force in the present stage is very small, hence the oscillation period $T$ can be calculated by simply averaging the measured data. Figure 2(d) shows the relative residual $T$–$z^2$ profile with different applied voltages $U$. No considerable nonlinearity has been observed when the applied voltage is in a range of $U < 860\,\mathrm{V}$.

Based on Eq. (11), two fitted curves, $T^2$–$m$ (when $U = 0\,\mathrm{V}$) and $T^{-2}$–$k_2U^2$ (when $m = 0\,\mathrm{kg}$), are obtained by the measurement, as shown in Fig. 3. Each $T$ measurement point in Fig. 3 has a relative uncertainty lower than $1 \times 10^{-5}$. A linear fit of two curves yields a preliminary determination of the Planck constant $6.62595 \times 10^{-34}\,\mathrm{J\cdot s}$ with a relative uncertainty about $5.4 \times 10^{-5}$. The uncertainty budget of this preliminary measurement has been listed in Table 1. The large uncertainty sources, i.e., the statistic error, the mechanical deformation and the length measurement error, will be decreased by further experimental improvements, i.e., suppressing the influence of the ground vibration, adopting a new balance beam whose mass center is very close to the central pivot and improving alignment accuracy, and a smaller uncertainty is expected in the near future.

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