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Energy and Thermodynamics of the Quantum-Corrected Schwarzschild Black Hole

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Energy and thermodynamics are investigated in the Schwarzschild black hole spacetime when considering corrections due to quantum vacuum fluctuations. The Einstein and Møller prescriptions are used to derive the expressions of the energy in the background. The temperature and heat capacity are also derived. The results show that due to the quantum fluctuations in the background of the Schwarzschild black hole, all the energies increase and the Einstein energy differs from Møller's one. Moreover, when increasing the quantum correction factor a, the difference between Einstein and Møller energies, the Unruh–Verlinde temperature as well as the heat capacity of the black hole increases while the Hawking temperature remains unchanged.

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Through the years, many efforts were made to develop truly invariant general prescription for gravitational field energy.[1–7] Using the various prescriptions, many researchers derived energy and momentum expressions in various spacetimes.[8–24] The obtained results show that for the general Kerr–Schild class of spacetimes, the expressions of the energy coincide for all of the prescriptions except for Møller's prescription. This divergence guides such investigation leading to the choice of an appropriate result for such spacetime.[25] Theoretical studies on black holes have attracted substantial attention since the advent of general relativity. The investigation of energy distribution in spacetimes of black holes is one of the hot topics in modern physics.

Black holes are thermal systems, radiating as black bodies with characteristic temperatures and entropies. During the past 40 years, research in the theory of black holes in general relativity has brought to light strong hints of a very deep and fundamental relationship among gravitation, thermodynamics, and quantum theory. The cornerstone of this relationship is black hole thermodynamics, where it appears that certain laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole.[26] Over the last decade, there have been several outstanding approaches toward a statistical mechanical computation of the Bekenstein–Hawking (BH) entropy.[27–32] Black holes as thermodynamical systems are widely found in the literature.[27–43]

We know that the vacuum undergoes quantum fluctuations. This phenomenon is the appearance of energetic particles out of nothing, as allowed by the uncertainty principle. Vacuum fluctuations have observable consequences like the Casimir force between two plates in vacuum. Due to quantum fluctuations, the evolution of slices in the black hole geometry will lead to the creation of particle pairs, which facilitates black holes' radiation.[44] It is commonly believed that a successful quantization of gravity will provide us with modifications to the theory, which are necessary to avoid the prediction of geodesically incomplete space-time manifolds. Quantum corrections may completely change the gravitational equations and the corresponding space-time geometry at the Planck scale. The quantum-corrected Schwarzschild black hole could be generated due to backreaction of dilaton coupled matter in the early universe, which is the solution to quantum corrected equations of motion.[45]

Recently, Wontae and Yongwan[46] investigated the phase transition of the quantum-corrected Schwarzschild black hole and concluded that there appears a type of Grass–Perry–Yaffe phase transition due to the quantum vacuum fluctuations and this held even for the very small size black hole. More recently, we investigated quasinormal modes of a quantum-corrected Schwarzschild black hole and showed that the scalar field damps more slowly and oscillates more slowly due to the quantum fluctuations.[47,48]

In this Letter, the energy and thermodynamics of a quantum-corrected Schwarzschild black hole are investigated to highlight the energetic and thermodynamical behaviors of the black hole when the vacuum fluctuations are taken into account.

According to the work of Kazakov and Solodukhin on quantum deformation of the

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Schwarzschild solution,[49] the background metric of the Schwarzschild black hole is defined by

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where

$$f(r) = -\frac{2M}{r} + \frac{1}{r}\int^r U(\rho)d\rho. \quad (2)$$

For an empty space, $U(\rho) = 1$. Thus we obtain the Schwarzschild metric

$$ds^2 = \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

where M is the black hole mass. The event horizon of the black hole is localized at $r_{\rm EH} = 2M$.

Taking into account the quantum fluctuation of the vacuum, the quantity $U(\rho)$ transforms to[46]

$$U(\rho) = \frac{e^{-\rho}}{\sqrt{e^{-2\rho} - \frac{4}{\pi}G_{\rm R}}}, \quad (4)$$

where $G_{\rm R} = G_{\rm N}\ln(\mu/\mu_0)$, $G_{\rm N}$ is the Newton constant, μ is a scale parameter, ρ is analog to the logarithmic of an inverse radius ($\rho = \ln(\frac{1}{r})$) such that [49] $U = \frac{r}{\sqrt{r^2 - \frac{4}{\pi}G_{\rm R}}}$.

The background metric of the quantum-corrected Schwarzschild black hole can then be read

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (5)$$

where $f(r) = (-\frac{2M}{r} + \frac{\sqrt{r^2-a^2}}{r})$ and $a^2 = 4G_{\rm R}/\pi$. The event horizon of such a black hole is located at the radius $r_{\rm EH} = \sqrt{4M^2 + a^2}$. It is clear that the area of the event horizon increases with the quantum-correction parameter a.

The energy-momentum complex from Møller's prescription can be derived as [11,14]

$$\tau_\nu^\mu = \frac{1}{8\pi}\varepsilon_{\nu,\lambda}^{\mu\lambda}, \quad (6)$$

where the superpotentials are

$$\varepsilon_\nu^{\mu\lambda} = \sqrt{-g}\Big(\frac{\partial g_{\nu\sigma}}{\partial x^k} - \frac{\partial g_{\nu k}}{\partial x^\sigma}\Big)g^{\mu k}g^{\lambda\sigma}. \quad (7)$$

The energy of the physical system in a four-dimensional background is given by

$$E_{\rm M} = \int\int\int \tau_0^0 dx^1dx^2dx^3. \quad (8)$$

Substituting Eq. (5) into Eq. (7), Møller's energy for the quantum-corrected Schwarzschild black hole can be written as

$$E_{\rm M} = M + \frac{a^2}{2\sqrt{r^2 - a^2}}. \quad (9)$$

The energy from Einstein's energy–momentum complex is [1,14]

$$\theta_\nu^\mu = \frac{1}{16\pi}h_{\nu,l}^{\mu l}, \quad (10)$$

where

$$h_\nu^{\mu l} = -h_\nu^{l\mu} = \frac{g_{\nu n}}{\sqrt{-g}}[-g(g^{\mu n}g^{lm} - g^{ln}g^{\mu m})]_m. \quad (11)$$

Since Einstein's energy–momentum complex is restricted to quasi-Cartesian coordinates, we can express the above metric (5) in quasi-Cartesian coordinates defined as

$$\begin{aligned} T &= t + r - \int f(r)^{-1}dr, \\ x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta. \end{aligned} \quad (12)$$

The metric can then be expressed as

$$ds^2 = -dT^2 + dx^2 + dy^2 + dz^2 + (1 - f(r))\Big[dT - \frac{xdx + ydy + zdz}{r}\Big]^2. \quad (13)$$

The energy of the physical system is, once again, given by the formula

$$E_{\rm E} = \int\int\int \theta_0^0 dx^1dx^2dx^3, \quad (14)$$

where θ_0^0 is evaluated using Eqs. (10), (11) and (13) and substituted into Eq. (14) to obtain the energy

$$E_{\rm E} = M - \frac{1}{2}\sqrt{r^2 - a^2} + \frac{r}{2}. \quad (15)$$

The behavior of these energies is represented in Figs. 1 and 2.

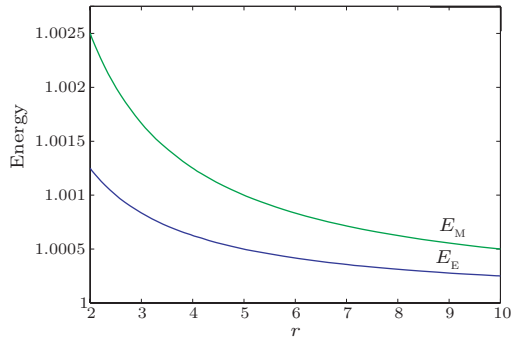

Fig. 1. Behavior of the Einstein and Møller energies versus the radial position r for $M = 1$.

These figures show that the energies decrease when increasing r, and for fixed r the Einstein energy is greater than the Møller energy. At the event horizon, $r = r_{\rm EH} = \sqrt{4M^2 + a^2}$, the two energies coincide when $a = 0$, and for $a \neq 0$ the gap between the two energies increases when increasing a.

The difference between the Einstein and Møller energies is

$$\Delta E = E_{\rm E} - E_{\rm M} = \frac{r}{2}\Big[1 - \sqrt{1 - \frac{a^2}{r^2}} - \frac{a^2}{r^2}\frac{1}{\sqrt{1-\frac{a^2}{r^2}}}\Big]. \quad (16)$$

For large $r \gg a$, the difference between the Einstein and Møller energies becomes

$$\Delta E \simeq \frac{r}{2}\Big[-\frac{a^2}{2r^2} - \frac{a^4}{2r^4}\Big] \simeq -\frac{a^2}{4r} = \frac{Q^2}{2r}, \quad (17)$$

with $Q = \pm i\frac{a}{\sqrt{2}}$. This expression is exactly the one obtained by Yang and Radinschi[50] for the Reissner–Nordström black hole.

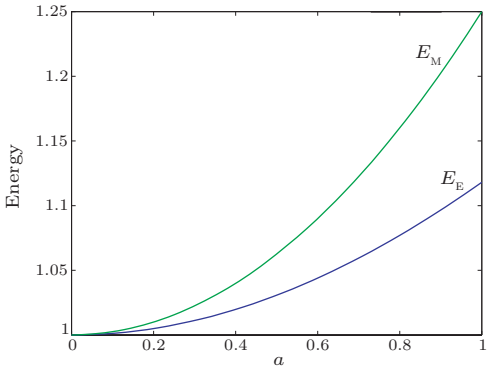

Fig. 2. Behavior of Einstein and Møller energies versus the quantum correction parameter a for $M = 1$ and $r = r_{\rm EH}$.

On the other hand, for $r \gg a$, the metric (5) coincides with the Reissner–Nordström metric but with an imaginary electric charge $Q = \pm i\frac{a}{\sqrt{2}}$.[47,48] Thus the relationship[50]

$$\Delta E = T_0^0 \times (kr^3) \quad (18)$$

would also be applied to the quantum-corrected Schwarzschild black hole.

The temperature of the black hole at a given radius r is given by the Unruh–Verlinde matching.[51,52] The Unruh–Verlinde temperature is given by [53–55]

$$T = \frac{\hbar}{2\pi} e^{\phi} n^{\alpha} \nabla_{\alpha}\phi, \quad (19)$$

where n_α is a unit vector, which is normal to the holographic screen, e^ϕ is the red-shift factor, and ϕ is the generalized form of the Newtonian potential given by

$$\phi = \frac{1}{2}\log(-g^{\mu\nu}\xi_\mu\xi_\nu), \quad (20)$$

with $g^{\mu\nu}$ being the background metric, and ξ_μ being the Killing time-like vector. The Killing vector for our spherically symmetric spacetime is

$$\xi_\mu = (-f(r), 0, 0, 0). \quad (21)$$

Using Eq. (5) in Eq. (21) and substituting into Eq. (20), we can obtain the following expression for the potential

$$\phi = \frac{1}{2}\log\Big(-\frac{2M}{r} + \frac{\sqrt{r^2-a^2}}{r}\Big), \quad (22)$$

and the expression of the temperature is

$$T = \frac{\hbar}{4\pi r^2}\Big(2M + \frac{a^2}{\sqrt{r^2-a^2}}\Big). \quad (23)$$

Its behavior is represented in Fig. 3.

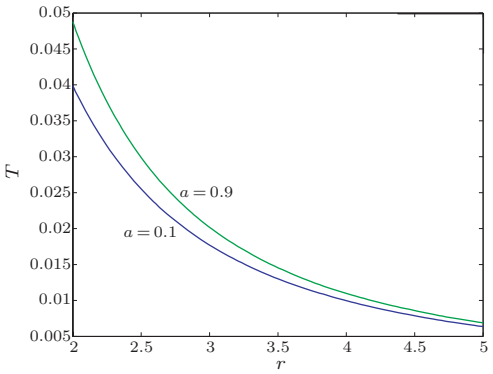

Fig. 3. Variation of the temperature versus the radial position r for different values of the quantum-correction parameter a.

Through this figure, we can see that the temperature decreases when increasing r. Moreover, when increasing the quantum-correction parameter a, the temperature increases.

The horizon radius $r_{\rm H}$ of the quantum-corrected Schwarzschild black hole is given by

$$r_{\rm H} = \sqrt{4M^2 + a^2}. \quad (24)$$

The entropy of the black hole is given by

$$S = \frac{A}{4} = \pi r_{\rm H}^2 = 4\pi M^2 + \pi a^2. \quad (25)$$

The first law of thermodynamics for the black hole can be written as

$$E = E_0 + \int_{S_0}^{S} T dS. \quad (26)$$

Substituting Eqs. (23) and (25) into (26), we obtain

$$E = E_0 + \frac{\hbar}{2r^2}\Big[\frac{8M^3}{3} + \frac{a^2 r_{\rm H}^2}{\sqrt{r^2-a^2}}\Big]. \quad (27)$$

For $a = 0$, we have $E = M$. Thus the energy yields

$$E = M + \frac{\hbar a^2 r_{\rm H}^2}{2r^2\sqrt{r^2-a^2}}, \quad (28)$$

which corresponds to the Møller energy when setting $\hbar = 1$ and $r_{\rm H}/r \sim 1$.

The Hawking temperature is the temperature at the event horizon.

Using this expression, the Hawking temperature reads

$$T_{\rm H} = T|_{r=r_{\rm H}} = \frac{\hbar}{8\pi M}, \tag{29}$$

which corresponds to the Hawking temperature of the Schwarzschild black hole free from any kind of correction.

Substituting Eq. (25) into Eq. (29) yields

$$T_{\rm H} = \frac{\hbar}{4\pi}\left(\frac{S}{\pi} - a^2\right)^{-\frac{1}{2}}. \tag{30}$$

The heat capacity of the black hole is

$$C = T_{\rm H}\left(\frac{\partial S}{\partial T_{\rm H}}\right) = -2S + 2\pi a^2. \tag{31}$$

When $a = 0$, the heat capacity becomes $C = -2S$, which corresponds to the expression for the Schwarzschild black hole found in the literature. That is not the case for the expression obtained in Ref. [46]. We can clearly see that the quantum vacuum fluctuations contribute to the heat capacity of the black hole by a positive quantity $C_{\rm a} = 2\pi a^2$.

In summary, we have studied energy and thermodynamics for the Schwarzschild black hole when considering quantum corrections due to the quantum vacuum fluctuations. Concerning energy, we derive the energy from the Einstein and Møller prescriptions. Their behaviors show that when $a = 0$, the two energies coincide with ($E_{\rm E} = E_{\rm M} = M$). When introducing the quantum correction to the black hole, the Møller energy becomes greater than the Einstein energy, and the difference increases when increasing the quantum-correction parameter a (see Figs. 1 and 2). The event horizon radius of the black hole as well as its Unruh–Verlinde temperature increases with the quantum-correction parameter. The Hawking temperature stays unchanged although the horizon changes. The heat capacity increases with the quantum-correction parameter. Thus we can conclude that quantum vacuum fluctuations generate positive energy increasing the energy of the black hole and help thermodynamically stabilize the black hole. At the event horizon, quantum vacuum fluctuations increase the energies (see Fig. 2) without changing the Hawking temperature, which remains proportional to the inverse of the black hole mass.

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