A Realization of the $W_{1+\infty}$ Algebra and Its n-Algebra *

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We consider a realization of the $W_{1+\infty}$ algebra and investigate its n-algebra, which is different from the n-algebra of Zhang et al. [2016 arXiv:1606.07570v2] It is found that the generators W_m^s with any fixed superindex $s \geqslant 1$ yield the null sub-2s-algebra. The nontrivial sub-4-algebra and Virasoro–Witt 3-algebra are presented. Moreover, we extend the generators to the multi-variables case. These generators also yield the $W_{1+\infty}$ algebra and null n-algebras.

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The $W_{1+\infty}$ algebra is an important infinite-dimensional algebra which is the higher-spin extensions of the Virasoro algebra.[1–3] It has attracted much interest from physical and mathematical points of view. In the context of integrable systems, the $W_{1+\infty}$ algebra was found to be associated with the first Hamiltonian structure of the KP hierarchy.[4,5] It is well-known that Calogero–Moser–Sutherland (CMS) models are the one-dimensional many-body integrable systems, which have received lots of attention.[6–8] The $SU(\nu)$ Calogero spin system is the generalization of the quantum Calegero model. In this integrable system, Hikami and Wadati found that there exists a higher symmetry, i.e., $W_{1+\infty}$ algebra.[9] This W algebraic structure unifies the Calogero type and the Sutherland type (trigonometric potential).

Recently great attention has been paid to n-algebra, especially 3-algebra due to a world-volume description of multiple $M2$-branes proposed by Bagger and Lambert,[10] and Gustavsson.[11] One also found the applications of n-algebra in the integrable systems[12,13] and condensed matter physics.[14–17] We have mentioned above that the $W_{1+\infty}$ algebra has a deep intrinsic connection to the KP hierarchy. Chen *et al.* presented a $W_{1+\infty}$ 3-algebra and investigated its connection with the integrable systems.[13] In this Letter, we consider a realization of the $W_{1+\infty}$ algebra and study its n-algebra.

Let us introduce the operators W_m^s in the following way,

$$W_m^s = \frac{1}{2(m+s)}[x^2, W_{m+2}^{s-1}], \quad s = 2, 3, \ldots, \tag{1}$$

and we set

$$W_m^1 = -\partial^m, \quad \text{for} \quad m \geqslant 1. \tag{2}$$

Substituting the generators (2) into the recursive relation (1), we obtain

$$W_m^s = (-1)^s \sum_{r=0}^{s-1} \frac{C_{s-1}^r A_{m+s-1}^r}{2^r} x^{s-1-r} \partial^{m+s-1-r}, \quad s = 2, 3, \ldots. \tag{3}$$

The generators (3) form the so-called $W_{1+\infty}$ algebra[9]

$$[W_n^s, W_m^r] = (n(r-1) - m(s-1))W_{n+m}^{s+r-2} + \ldots, \tag{4}$$

where $(\ldots)$ can be understood as indicating the lower-order terms corresponding to the quantum effect.

The first few commutators of Eq. (4) are

$$\begin{aligned}
&[W_n^1, W_m^1] = 0, \\
&[W_n^2, W_m^1] = -mW_{n+m}^1, \\
&[W_n^2, W_m^2] = (n-m)W_{n+m}^2, \\
&[W_n^3, W_m^1] = -2mW_{n+m}^2, \\
&[W_n^3, W_m^2] = (n-2m)W_{n+m}^3 \\
&\qquad + \frac{1}{4}(n+2)(m+1)mW_{n+m}^1, \\
&[W_n^3, W_m^3] = 2(n-m)W_{n+m}^4 - \frac{1}{2}(n-m) \\
&\qquad \cdot (n+2)(m+2)W_{n+m}^2.
\end{aligned} \tag{5}$$

As the case of the well-known generators $L_m = -x^{m+1}\partial$, we note that the generators $W_m^2 = \frac{1}{2(m+2)}[x^2, W_{m+2}^1] = x\partial^{m+1} + \frac{m+1}{2}\partial^m$ also yield the Virasoro–Witt algebra. To explore the difference of these two realizations of Virasoro–Witt algebra, let us consider the case of 3-algebra.

The operator Nambu 3-bracket is defined by[18,19]

$$\begin{aligned}
[A, B, C] &= A[B, C] + B[C, A] + C[A, B] \\
&= [B, C]A + [C, A]B + [A, B]C,
\end{aligned} \tag{6}$$

where $[A, B] = AB - BA$.

By direct calculation, the Nambu 3-bracket $[L_k, L_m, L_n]$ yields the null 3-algebra[20]

$$[L_k, L_m, L_n] = 0. \tag{7}$$

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Substituting W_m^2 and W_m^1 into the Nambu 3-bracket (6), we obtain the 3-algebra

$$[W_{m_1}^2, W_{m_2}^2, W_{m_3}^2] = \frac{1}{4}(m_3 - m_2)(m_3 - m_1) \cdot (m_2 - m_1) W_{m_1+m_2+m_3}^1, \tag{8a}$$

$$[W_{m_1}^2, W_{m_2}^2, W_{m_3}^1] = (m_2 - m_1) W_{m_1+m_2+m_3}^2, \tag{8b}$$

$$[W_{m_1}^2, W_{m_2}^1, W_{m_3}^1] = (m_2 - m_3) W_{m_1+m_2+m_3}^1, \tag{8c}$$

$$[W_{m_1}^1, W_{m_2}^1, W_{m_3}^1] = 0. \tag{8d}$$

It satisfies the so-called Bremner identity (BI)[21,22]

$$\begin{aligned}&\epsilon^{i_1 i_2 \dots i_6}[[A, [B_{i_1}, B_{i_2}, B_{i_3}], B_{i_4}], B_{i_5}, B_{i_6}] \\ &= \epsilon^{i_1 i_2 \dots i_6}[[A, B_{i_1}, B_{i_2}], [B_{i_3}, B_{i_4}, B_{i_5}], B_{i_6}],\end{aligned} \tag{9}$$

where $i_1, \dots, i_6$ are implicitly summed from 1 to 6. However, the Filippov condition or fundamental identity (FI),[23]

$$\begin{aligned}&[A, B, [C, D, E]] = [[A, B, C], D, E] \\ &+ [C, [A, B, D], E] + [C, D, [A, B, E]],\end{aligned} \tag{10}$$

does not hold for it. Thus the 3-algebra (8) is not a Nambu 3-algebra. From Eqs. (7) and (8), we note that for the two different realizations of Virasoro–Witt algebra, their Nambu 3-brackets lead to the different 3-algebras.

Let us consider the operators

$$Q_m = W_m^2 + \lambda m W_m^1, \quad R_m = W_m^1. \tag{11}$$

These operators yield the Virasoro–Witt algebra

$$\begin{aligned}&[Q_n, Q_m] = (n - m) Q_{n+m}, \\ &[Q_n, R_m] = -mR_{n+m}, \ [R_n, R_m] = 0,\end{aligned} \tag{12}$$

and 3-algebra

$$\begin{aligned}[Q_k, Q_m, Q_n] &= (\frac{1}{4} - \lambda^2)(n - m) \cdot (n - k)(m - k) R_{k+m+n}, \\ [Q_k, Q_m, R_n] &= (m - k)(Q_{k+m+n} - 2\lambda n R_{k+m+n}), \\ [Q_k, R_m, R_n] &= (m - n) R_{k+m+n}, \\ [R_k, R_m, R_n] &= 0.\end{aligned} \tag{13}$$

Let us take $\hat{Q}_m = -(\frac{1}{4} - \lambda^2)^{-\frac{1}{4}} Q_m$, $\hat{R}_m = (\frac{1}{4} - \lambda^2)^{\frac{1}{4}} R_m$ and $z = \frac{2\lambda}{(\frac{1}{4} - \lambda^2)^{1/2}}$, where $\lambda \neq \pm\frac{1}{2}$, then the 3-algebra (13) becomes

$$\begin{aligned}[\hat{Q}_k, \hat{Q}_m, \hat{Q}_n] &= (n - m)(n - k)(k - m) \hat{R}_{k+m+n}, \\ [\hat{Q}_k, \hat{Q}_m, \hat{R}_n] &= (k - m)(\hat{Q}_{k+m+n} + zn\hat{R}_{k+m+n}), \\ [\hat{Q}_k, \hat{R}_m, \hat{R}_n] &= (n - m)\hat{R}_{k+m+n}, \\ [\hat{R}_k, \hat{R}_m, \hat{R}_n] &= 0.\end{aligned} \tag{14}$$

We immediately recognize that the 3-algebra (14) is nothing but the so-called Virasoro–Witt 3-algebra derived in Ref. [24]. When $z = \pm 2i$ with $i = \sqrt{-1}$, it satisfies the FI (10).

The n-bracket is defined by[25,26]

$$[B_{i_1}, B_{i_2}, \dots, B_{i_n}] = \epsilon_{12\dots n}^{\sigma_1 \sigma_2 \dots \sigma_n} B_{i_{\sigma_1}} B_{i_{\sigma_2}} \dots B_{i_{\sigma_n}}, \tag{15}$$

where $\epsilon_{12\dots n}^{\sigma_1 \sigma_2 \dots \sigma_n}$ is the Lévi–Cività symbol

$$\epsilon_{j_1 \dots j_p}^{i_1 \dots i_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \dots & \delta_{j_p}^{i_p} \end{pmatrix}. \tag{16}$$

Let us pause here to look at the operators

$$\bar{W}_m^s = x^{m+s-1} (\frac{\partial}{\partial x})^{s-1}, \ s \in \mathbb{Z}_+, \ m \in \mathbb{N}. \tag{17}$$

They yield the infinite-dimensional algebra

$$\begin{aligned}[\bar{W}_{m_1}^{s_1}, \bar{W}_{m_2}^{s_2}] &= (\sum_{\alpha=0}^{s_1-1} C_{s_1-1}^{\alpha} A_{m_2+s_2-1}^{\alpha} \\ &- \sum_{\alpha=0}^{s_2-1} C_{s_2-1}^{\alpha} A_{m_1+s_1-1}^{\alpha}) \bar{W}_{m_1+m_2}^{s_1+s_2-1-\alpha},\end{aligned} \tag{18}$$

where $A_m^{\alpha} = m(m-1)\dots(m-\alpha+1)$ for $\alpha \leqslant m$, $A_m^{\alpha} = 0$ for $\alpha > m$, and $C_m^{\alpha} = \frac{m(m-1)\dots(m-\alpha+1)}{\alpha!}$. The algebra (18) is also called the $W_{1+\infty}$ algebra in the literature.[27,28]

In Ref. [29], Zhang *et al.* derived the $W_{1+\infty}$ n-algebra and found that

$$[\bar{W}_{m_1}^{s+1}, \bar{W}_{m_2}^{s+1}, \dots, \bar{W}_{m_{2s}}^{s+1}] = V_{2s} \bar{W}_{m_1+m_2+\dots+m_{2s}}^{s+1}, \tag{19}$$

and

$$[\bar{W}_{m_1}^{s+1}, \bar{W}_{m_2}^{s+1}, \dots, \bar{W}_{m_{2s+1}}^{s+1}] = 0, \tag{20}$$

where V_{2s} is the Vandermonde determinant

$$\begin{aligned}V_{2s} &= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ m_1 & m_2 & \cdots & m_{2s-1} & m_{2s} \\ m_1^2 & m_2^2 & \cdots & m_{2s-1}^2 & m_{2s}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1^{2s-2} & m_2^{2s-2} & \cdots & m_{2s-1}^{2s-2} & m_{2s}^{2s-2} \\ m_1^{2s-1} & m_2^{2s-1} & \cdots & m_{2s-1}^{2s-1} & m_{2s}^{2s-1} \end{vmatrix} \\ &= \prod_{1 \leqslant j < k \leqslant 2s} (m_k - m_j).\end{aligned} \tag{21}$$

Since the $W_{1+\infty}$ sub-$2s$-algebra (19) satisfies the generalized Jacobi identity (GJI)[26]

$$\epsilon_{12\dots 2n-1}^{i_1 i_2 \dots i_{2n-1}} [[B_{i_1}, B_{i_2} \dots, B_{i_n}], B_{i_{n+1}}, \dots, B_{i_{2n-1}}] = 0, \tag{22}$$

it is a generalized Lie algebra or higher order Lie algebra.

Let us now focus on the case of the operators (3). By direct calculation, we note that the form of n-algebra appears to become more complicated. We present a conjecture as follows:

$$\begin{aligned}
&[W^{s_1}_{m_1}, W^{s_2}_{m_2}, \ldots, W^{s_{2n}}_{m_{2n}}]\\
&= f(m_1, \ldots, m_{2n}, s_1, \ldots, s_{2n}) W^{s_1+\ldots s_{2n}-(5n-3)}_{m_1+\ldots m_{2n}} + \ldots,\\
&[W^{s_1}_{m_1}, W^{s_2}_{m_2}, \ldots, W^{s_{2n+1}}_{m_{2n+1}}]\\
&= g(m_1, \ldots, m_{2n+1}, s_1, \ldots, s_{2n+1}) W^{s_1+\ldots s_{2n+1}-(5n-2)}_{m_1+\ldots m_{2n+1}}\\
&\quad + \ldots, \qquad (23)
\end{aligned}$$

where $(\ldots)$ stands for the lower-order terms corresponding to the quantum effect.

We check the cases of n-bracket with $n \leqslant 7$ and confirm that the conjecture (23) holds for these cases. Let us list the several n-algebras as follows:

$$\begin{aligned}
&[W^{s_1}_{m_1}, W^{s_2}_{m_2}, W^{s_3}_{m_3}]\\
&= -[m_1(s_2-s_3) + m_2(s_3-s_1)\\
&\quad + m_3(s_1-s_2)] W^{s_1+s_2+s_3-3}_{m_1+m_2+m_3} + \ldots,\\
&[W^{s_1}_{m_1}, W^{s_2}_{m_2}, W^{s_3}_{m_3}, W^{s_4}_{m_4}]\\
&= f(m_1, \ldots, m_4, s_1, \ldots, s_4) W^{s_1+s_2+s_3+s_4-7}_{m_1+m_2+m_3+m_4} + \ldots,\\
&[W^{s_1}_{m_1}, W^{s_2}_{m_2} \ldots, W^{s_5}_{m_5}]\\
&= g(m_1, \ldots, m_5, s_1, \ldots, s_5) W^{s_1+\ldots+s_5-8}_{m_1+\ldots+m_5} + \ldots, \qquad (24)
\end{aligned}$$

where

$$\begin{aligned}
&f(m_1, \ldots, m_4, s_1, \ldots, s_4)\\
&= -\sum_{\substack{r_1+\ldots+r_4+\alpha_1+\alpha_2+\alpha_3=4,\\ r_1,\ldots,r_4\geqslant 0,\\ \alpha_1,\alpha_2,\alpha_3\geqslant 0}} \Big\{ \frac{C^{r_1}_{s_1-1}\cdots C^{r_4}_{s_4-1}}{2^{r_1+\ldots+r_4}} \epsilon^{i_1\ldots i_4}_{1\ldots 4}\\
&\cdot C^{\alpha_3}_{s_{i_4}-1-r_{i_4}} C^{\alpha_2}_{s_{i_3}+s_{i_4}-2-r_{i_4}-r_{i_3}-\alpha_3}\\
&\cdot C^{\alpha_1}_{s_{i_2}+s_{i_3}+s_{i_4}-3-r_{i_4}-r_{i_3}-r_{i_2}-\alpha_3-\alpha_2}\\
&\cdot A^{r_{i_1}+\alpha_1}_{m_{i_1}+s_{i_1}-1} A^{r_{i_2}+\alpha_2}_{m_{i_2}+s_{i_2}-1} A^{r_{i_3}+\alpha_3}_{m_{i_3}+s_{i_3}-1} A^{r_{i_4}}_{m_{i_4}+s_{i_4}-1} \Big\}, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
&g(m_1, \ldots, m_5, s_1, \ldots, s_5)\\
&= \sum_{\substack{r_1+\ldots+r_5+\alpha_1+\ldots+\alpha_4=4,\\ r_1,\ldots,r_5\geqslant 0,\\ \alpha_1,\ldots,\alpha_4\geqslant 0}} \Big\{ \frac{C^{r_1}_{s_1-1}\cdots C^{r_5}_{s_5-1}}{2^{r_1+\ldots+r_5}} \epsilon^{i_1\ldots i_5}_{1\ldots 5}\\
&\cdot C^{\alpha_4}_{s_{i_5}-1-r_{i_5}} C^{\alpha_3}_{s_{i_4}+s_{i_5}-2-r_{i_4}-r_{i_5}-\alpha_4}\\
&\cdot C^{\alpha_2}_{s_{i_3}+s_{i_4}+s_{i_5}-3-r_{i_3}-r_{i_4}-r_{i_5}-\alpha_4-\alpha_3}\\
&\cdot C^{\alpha_1}_{s_{i_2}+s_{i_3}+s_{i_4}+s_{i_5}-4-r_{i_2}-r_{i_3}-r_{i_4}-r_{i_5}-\alpha_4-\alpha_3-\alpha_2}\\
&\cdot A^{r_{i_1}+\alpha_1}_{m_{i_1}+s_{i_1}-1} A^{r_{i_2}+\alpha_2}_{m_{i_2}+s_{i_2}-1} A^{r_{i_3}+\alpha_3}_{m_{i_3}+s_{i_3}-1}\\
&\cdot A^{r_{i_4}+\alpha_4}_{m_{i_4}+s_{i_4}-1} A^{r_{i_5}}_{m_{i_5}+s_{i_5}-1} \Big\}. \qquad (26)
\end{aligned}$$

Although it is hard to give the explicit expression of $W_{1+\infty}$ n-algebra, straightforward calculation gives the following multibracket for the generators W^s_m with any fixed superindex $s \geqslant 1$,

$$\begin{aligned}
&[W^s_{m_1}, W^s_{m_2}, \ldots, W^s_{m_{2s-1}}]\\
&= \epsilon^{i_1\ldots i_{2s-1}}_{1\ldots(2s-1)} W^s_{m_{i_1}} \ldots W^s_{m_{i_{2s-1}}}\\
&= (-1)^{(2s-1)s} \sum_{r_1=0}^{s-1} \cdots \sum_{r_{2s-1}=0}^{s-1} \sum_{\alpha_{2s-2}=0}^{b_{2s-2}} \cdots\\
&\quad \sum_{\alpha_1=0}^{b_1} \frac{(\prod_{j=1}^{2s-2} C^{r_j}_{s-1} C^{\alpha_j}_{b_j}) C^{r_{2s-1}}_{s-1}}{2^{r_1+\ldots+r_{2s-1}}}\\
&\quad \cdot \epsilon^{i_1\ldots i_{2s-1}}_{1\ldots(2s-1)} A^{r_1+\alpha_1}_{m_{i_1}+s-1} \ldots A^{r_{2s-2}+\alpha_{2s-2}}_{m_{i_{2s-2}}+s-1}\\
&\quad A^{r_{2s-1}}_{m_{i_{2s-1}}+s-1} x^{b_0} \partial^{m_1+\ldots+m_{2s-1}+b_0}\\
&= (-1)^{s+1} \sum_{r_1=0}^{s-1} \cdots \sum_{r_{2s-1}=0}^{s-1} \sum_{\alpha_{2s-2}=0}^{b_{2s-2}} \cdots\\
&\quad \sum_{\alpha_1=0}^{b_1} (-1)^{\sigma} \frac{(\prod_{j=1}^{2s-2} C^{r_j}_{s-1} C^{\alpha_j}_{b_j}) C^{r_{2s-1}}_{s-1}}{2^{r_1+\ldots+r_{2s-1}}}\\
&\quad \cdot V_{2s-1} W^1_{m_1+\ldots+m_{2s-1}}, \qquad (27)
\end{aligned}$$

where $b_j = (2s-1-j)(s-1) - r_{2s-1} - \ldots - r_{j+1} - \alpha_{2s-1} - \ldots - \alpha_{j+1}$, $j = 0, 1, \ldots, 2s-2$, we denote the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & 2s-3 & 2s-2\\ \alpha_1+r_1 & \alpha_2+r_2 & \cdots & \alpha_{2s-2}+r_{2s-2} & r_{2s-1} \end{pmatrix}$$

and $\alpha_{2s-1} = 0$.

Let us take the scaled generators $W^s_m \to Q_s^{-\frac{1}{2s-1}} W^s_m$, where the scaling coefficient Q_s is given by

$$\begin{aligned}
Q_s &= (-1)^{s+1} \sum_{r_1=0}^{s-1} \cdots \sum_{r_{2s-1}=0}^{s-1} \sum_{\alpha_{2s-2}=0}^{b_{2s-2}} \cdots\\
&\quad \sum_{\alpha_1=0}^{b_1} (-1)^{\sigma} \frac{1}{2^{r_1+\ldots+r_{2s-1}}} (\prod_{j=1}^{2s-2} C^{r_j}_{s-1} C^{\alpha_j}_{b_j}) C^{r_{2s-1}}_{s-1}.
\end{aligned}$$

Then Eq. (27) can be rewritten as

$$[W^s_{m_1}, W^s_{m_2}, \ldots, W^s_{m_{2s-1}}] = V_{2s-1} W^1_{m_1+\ldots+m_{2s-1}}. \qquad (28)$$

Taking $s = 2$ in Eq. (28), it gives the 3-bracket (8a). Note that the operators (3) do not yield the nontrivial sub-2s-algebra (19). However, we find that there are the closed 3-algebra (8) and 4-algebra

$$\begin{aligned}
&[W^3_{m_1}, W^3_{m_2}, W^3_{m_3}, W^3_{m_4}]\\
&= -\frac{9}{2} V_4 W^3_{m_1+m_2+m_3+m_4}\\
&\quad + \frac{1}{8} V_4 [3 \sum_{1\leqslant j<k\leqslant 4} m_j m_k\\
&\quad + \sum_{j=1}^{4} (6m_j + m_j^2) + 9] W^1_{m_1+m_2+m_3+m_4},
\end{aligned}$$

$$
\begin{aligned}
&[W^3_{m_1}, W^3_{m_2}, W^3_{m_3}, W^1_{m_4}]\\
&= -10m_4V_3W^3_{m_1+m_2+m_3+m_4}\\
&\quad+\frac{1}{2}m_4V_3\Big[\sum_{1\leqslant j<k\leqslant 3} m_jm_k+(4-m_4)(m_1+m_2+m_3)\\
&\quad+6m_4-3m_4^2+4\Big]W^1_{m_1+m_2+m_3+m_4},\\
&[W^3_{m_1}, W^3_{m_2}, W^1_{m_3}, W^1_{m_4}]\\
&=2m_3m_4(m_1-m_2)(m_3-m_4)W^1_{m_1+m_2+m_3+m_4},\\
&[W^3_{m_1}, W^1_{m_2}, W^1_{m_3}, W^1_{m_4}]=0,\\
&[W^1_{m_1}, W^1_{m_2}, W^1_{m_3}, W^1_{m_4}]=0.
\end{aligned}
\tag{29}
$$

Since the 4-algebra (29) satisfies the GJI (22), it is a generalized Lie algebra.

According to the definition of n-bracket and using Eq. (28), we obtain

$$
\begin{aligned}
&[W^s_{m_1}, W^s_{m_2}, \ldots, W^s_{m_{2s}}]\\
&= -Q_s^{-\frac{1}{2s-1}}(-1)^s\sum_{r=0}^{s-1}\frac{C^r_{s-1}}{2^r}\\
&\quad\cdot\begin{vmatrix} A^r_{m_1+s-1} & \cdots & A^r_{m_{2s}+s-1}\\ 1 & \cdots & 1\\ m_1 & \cdots & m_{2s}\\ \vdots & \vdots & \vdots\\ m_1^{2s-2} & \cdots & m_{2s}^{2s-2}\end{vmatrix}\\
&\quad\cdot x^{s-1-r}\partial^{m_1+\ldots+m_{2s}+s-1-r}\\
&=0.
\end{aligned}
\tag{30}
$$

As examples, in the following we list the several null $2s$-algebras

$$
\begin{aligned}
&[W^2_{m_1}, W^2_{m_2}, W^2_{m_3}, W^2_{m_4}]=0,\\
&[W^3_{m_1}, W^3_{m_2}, \ldots, W^3_{m_6}]=0,\\
&[W^4_{m_1}, W^4_{m_2}, \ldots, W^4_{m_8}]=0.
\end{aligned}
\tag{31}
$$

Let us consider the following multi-variables extension of Eq. (1),

$$
\hat{W}^s_m=\frac{1}{2(m+s)}\Big[\sum_{i=1}^N x_i^2, \hat{W}^{s-1}_{m+2}\Big],\quad s=2,3,\ldots, \tag{32}
$$

and

$$
\hat{W}^1_m=-\sum_{i=1}^N\partial_i^m,\ \text{for } m\geqslant 1. \tag{33}
$$

In the $SU(\nu)$ Calogero spin system, Hikami and Wadati[9] constructed a series of higher spin operators $\tilde{W}^l_m$ in terms of Eq. (32), where $\tilde{W}^1_m=\sum_{j,k=1}^N(L^m)_{jk}$, $m\geqslant 1$ and the Lax operator L is given by

$$
L_{jk}=\delta_{jk}p_j+(1-\delta_{jk})ia\frac{P_{jk}}{x_j-x_k}, \tag{34}
$$

in which $p_j=-i\frac{\partial}{\partial x_j}$ denotes the momentum operator of the jth particle, the real parameter a controls the strength of the interaction, and P_{jk} is a permutation operator which exchanges the spin state of the jth and the kth particles. When there is no interaction term, i.e., $a=0$, we note that the operators $\tilde{W}^l_m$ reduce to Eq. (32). That is to say, the operators $\hat{W}^1_m$ are the special case of $\tilde{W}^l_m$, which indeed corresponds to the case of N free particles.

Substituting the operators (33) into the recursive relation (32), we obtain

$$
\begin{aligned}
&\hat{W}^s_m=(-1)^s\sum_{i=1}^N\sum_{r=0}^{s-1}\frac{C^r_{s-1}A^r_{m+s-1}}{2^r}x_i^{s-1-r}\partial_i^{m+s-1-r},\\
&\quad s=2,3,\ldots.
\end{aligned}
\tag{35}
$$

It is not difficult to verify that the operators (35) also form the $W_{1+\infty}$ algebra (4) as the case of single variable. However, the same algebraic structure is invalid for n-bracket.

Let us focus on the generators $\hat{W}^s_m$ with any fixed superindex. Performing lengthy but straightforward calculations, we find that the multi-bracket $[\hat{W}^s_{m_1}, \hat{W}^s_{m_2}\ldots, \hat{W}^s_{m_{(2s-1)N}}]$ does not form the closed $(2s-1)N$-algebra. However, there is the null $(2sN-N+1)$-algebra

$$
[\hat{W}^s_{m_1}, \hat{W}^s_{m_2}, \ldots, \hat{W}^s_{m_{(2s-1)N+1}}]=0. \tag{36}
$$

Note that the null algebra (36) is determined by the superindex and number of variables.

In summary, based on a realization of the $W_{1+\infty}$ algebra, we have presented the $W_{1+\infty}$ n-algebra and found that there exist two nontrivial sub-n-algebras, i.e., the generalized Lie algebra (29) and the Virasoro–Witt 3-algebra (8). Moreover, a remarkable property is that the generators W^s_m with any fixed superindex $s\geqslant 1$ yield the null sub-$2s$-algebra. We have also investigated the case of the generators with multi-variables. Although they yield the $W_{1+\infty}$ algebra, we find that the quantum multibrackets of these operators do not yield any nontrivial n-algebra. It is interesting to note that there are the null multibrackets for the generators with any fixed superindex. Recently, the relationship between the infinite-dimensional 3-algebras and the integrable systems has been studied.[12,13] For the future investigations, it is of interest to discuss the application of $W_{1+\infty}$ n-algebra in the integrable systems.

It should be pointed out that the general realization of the $W_{1+\infty}$ algebra has been presented in the $SU(\nu)$ Calogero spin system by Hikami and Wadati.[9] Since there is no interaction term, the multi-variables realization of $W_{1+\infty}$ algebra presented in this study can be regarded as the special case of the general realization of Hikami and Wadati. In the $SU(\nu)$ Calogero

spin system, the conserved operators yield the $W_{1+\infty}$ algebra. It remains an open question whether the quantum multibrackets of these conserved operators yield the null n-algebras.

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