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A Unified Approach to the Thermodynamics and Quantum Scaling Functions of One-Dimensional Strongly Attractive $SU(w)$ Fermi Gases *

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We present a unified derivation of the pressure equation of states, thermodynamics and scaling functions for the one-dimensional (1D) strongly attractive Fermi gases with $SU(w)$ symmetry. These physical quantities provide a rigorous understanding on a universality class of quantum criticality characterized by the critical exponents $z = 2$ and correlation length exponent $\nu = 1/2$. Such a universality class of quantum criticality can occur when the Fermi sea of one branch of charge bound states starts to fill or becomes gapped at zero temperature. The quantum critical cone can be determined by the double peaks in specific heat, which serve to mark two crossover temperatures fanning out from the critical point. Our method opens to further study on quantum phases and phase transitions in strongly interacting fermions with large $SU(w)$ and non-$SU(w)$ symmetries in one dimension.

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The experimental realization of the 1D quantum gases such as repulsive Bose gases,[1−6] Yang–Gaudin model,[7,8] multicomponent attractive Fermi gases[9] has provided a remarkable test ground for exactly solvable models. The mathematical theory of Bethe ansatz integrable models has become testable in ultracold atoms. The Bethe anssatz has also found success for other significant models such as the Kondo physics,[10] BCS pairing model,[11] strongly correlated electronic systems,[12−14] spin ladders[15,16] and quantum degenerate gases.[17,18]

Recent studies of the 1D Fermi gases with high spin symmetries[18−26] have given many theoretical predictions on the existence of bound states of multiparticles, quantum liquids and phase transitions. In this regard, exploring exactly solvable models of interacting fermions with high mathematical symmetries is highly desirable in order to understand new phases of matter. However, the thermodynamic properties of exactly solvable models with high symmetries at finite temperatures are notoriously difficult to extract and present a formidable challenge. Building on Yang–Yang thermodynamic Bethe ansatz equations, here we present a unified approach to the thermodynamics and quantum critical scalings in 1D strongly attractive Fermi gases with $SU(w)$ symmetry. Analytical results of the equation of states (EOS), dimensionless ratios and scaling functions of thermal and magnetic properties provide a rigorous understanding on a universality class of quantum criticality of free fermions. The quantum critical region can be determined by the double maxima in specific heat, which characterize the two crossover temperatures fanning out from the critical point. These results suggest to experimentally test the universal nature of multicomponent quantum liquids through the 1D strongly attractive Fermi gases of ultracold atoms.[9]

The model: The 1D $SU(w)$ Fermi gases with δ-function interaction confined to length L is described by the following Hamiltonian[27−30]

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}}\sum_{1\le i<j\le N}\delta(x_i - x_j) - E_z - \mu N, \tag{1}$$

and with the chemical potential μ and the effective Zeeman energy $E_z = \sum_{r=1}^{w}\frac{1}{2}r(w-r)n_r H_r$. Here N is the total particle number. There are w possible hyperfine states $|1\rangle, |2\rangle, \ldots, |w\rangle$ in which the fermions can occupy. Experimentally, $g_{1\mathrm{D}} = -2\hbar^2/ma_{1\mathrm{D}}$, with $a_{1\mathrm{D}}$ the effective scattering length in 1D,[31] can be tuned from a weak interaction to a strong coupling regime via Feshbach resonances. For our convenience, from now on, we choose our units such that $\hbar^2 = 2m = 1$ unless we particularly use the units. In this model, the two-body charge bound states involve the Bethe ansatz roots $\{\lambda_j \pm ic/2\}, j = 1, \ldots, M_2$ and the three-body bound states $\{\lambda_j \pm ic, \lambda_j\}, j = 1, \ldots, M_3$, and so on, where M_2 and M_3 are the numbers of charge bound states and three-body bound states, respectively.[23] The thermodynamics of the model is determined by the effective external fields H_r, chemical potential, interaction between different particles and spin wave fluctuations.

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The TBA equations: The thermodynamics of the Hamiltonian (1) is determined by the following TBA equations[13,23,30]

$$\epsilon^{(r)}(k) = rk^2 - r\mu - H_r - \frac{r(r^2-1)c^2}{12} - \sum_{q=1}^{w} \hat{a}_{rq} * F[\epsilon^{(q)}] + \sum_{q=1}^{\infty} a_q * F[\eta_{r,q}], \tag{2}$$

$$\eta_{r,l}(k) = l\cdot(2H_r - H_{r-1} - H_{r+1}) - a_l * F[\epsilon^{(r)}] - \sum_{q=1}^{\infty} U_{lq} * F[\eta_{r,q}] + \sum_{q=1}^{\infty} S_{lq} * F[\eta_{r-1,q}] + \sum_{q=1}^{\infty} S_{lq} * F[\eta_{r+1,q}], \tag{3}$$

where we denote

$$a_n(x) = \frac{1}{2\pi}\frac{n|c|}{(nc/2)^2 + x^2},\ \hat{a}_{lj}(x) = \sum_{q=1,2q\neq l+j}^{min(l,j)} a_{l+j-2q}(x),$$

$$F[\varepsilon] \triangleq -T\ln[1+\exp(-\frac{\varepsilon}{T})].$$

In the above equations, $*$ denotes the convolution $(f*g)(\lambda) = \int_{-\infty}^{\infty} f(\lambda-\lambda')g(\lambda')d\lambda'$ and the functions $U_{lj}(x)$ and $S_{lj}(x)$ are given in Ref. [23]. From the dressed energies $\epsilon^{(r)}(k)$ for bound states of r-atoms with $r = 1,\cdots,w$, one can obtain the pressure

$$p = \sum_{r=1}^{w} \frac{rT}{2\pi}\int_{-\infty}^{\infty} dk\ln(1+e^{-\epsilon^{(r)}/T}). \tag{4}$$

The summation of the pressures of all charge bound states services as the EOS, from which we can obtain full thermodynamics of the model at the temperatures ranging from zero to high. This form of the EOS gives rise to the additivity nature of quantum liquids in low temperatures.[19]

We are interested in the low temperature behavior of interacting fermions with high symmetries in 1D. We can see from the TBA equations (3) that the fermomagnetic ordering (the second term in (3)) drives the spin contributions $\eta_{r,l}(k)$ to the dressed energies of charge bound states exponentially small in strongly attractive regimes. Consequently, we can ignore all the string contributions in the above TBA equations when temperature is much less than the binding energies of the charge bound states $\varepsilon_r = \frac{1}{48}r(r^2-1)g_{1D}^2$. In the recent study,[19] it has been proved that the dimensionless Wislon ratios, i.e. either the ratio of the susceptibility χ to the specific heat c_V divided by the temperature

$$R_{\rm W}^{\rm s} = \frac{4}{3}\left(\frac{\pi k_{\rm B}}{\mu_{\rm B} g}\right)^2 \frac{\chi}{c_V/T}, \tag{5}$$

or the ratio of the compressibility κ to the specific heat c_V divided by the temperature

$$R_{\rm W}^{\rm c} = \frac{k_{\rm B}^2\pi^2}{3}\frac{\kappa}{c_V/T}, \tag{6}$$

essentially captures the quasiparticle nature of Fermi liquid,[32,33] as its value characterizes the interacting effect in the Fermi liquid. Here $k_{\rm B}$ is Boltzmann's constant, $\mu_{\rm B}$ is the Bohr magneton and g is the Lande factor. The two types of dimensionless ratios (5) and (6) characterize a competition between the fluctuations of two thermodynamic quantities. Thus a constant Wilson ratio implies that the two types of fluctuations are on an equal footing with respect to the temperature, regardless of the microscopic details of many-body systems.[19,34,35] In Fig. 1, we demonstrate the compressibility Wilson ratio elegantly maps out the full phase diagram of the $SU(2)$ Fermi gas[27,28] with a strong attraction at $T = 0.001\varepsilon_2/k_{\rm B}$, where ε_2 is the binding energy of a bound pair. This turns out that the low temperature $SU(2)$ TBA equations (2) provide rigorous results of quantum liquid behavior and quantum criticality, prompting us to explore universal thermodynamics and quantum scaling functions for the high symmetry Fermi gases with strong attractions through the $SU(w)$ TBA equations.

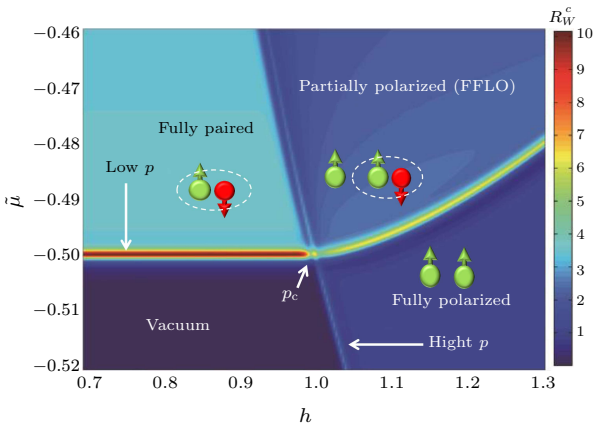

Fig. 1. Contour plot of the Wilson ratio $R_{\rm W}^{\rm c}$ in $\tilde{\mu}-h$ plane at temperature $T = 0.001\varepsilon_2/k_{\rm B}$. The calculation of the Wilson ratio (6) was carried out by numerically solving the TBA equations (2). Here $\tilde{\mu} = \frac{\mu}{|c|^2/2}$ and $h = \frac{H}{|c|^2/2}$ are the dimensionless chemical potential and magnetic field. $P_{\rm c}$ stands for the critical polarization. The $P = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow)$ denotes the particle number imbalance. The Wilson ratio gives different constant values which characterize three Luttinger liquid phases of fully paired state, partially-polarized Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) like state and the fully-polarized normal Fermi gas. The Wilson ratio remarkably displays two distinct plateaus of the integers 1 and 4 in strong coupling limit.

Without loss of accuracy at low temperature thermodynamics, we simplify the TBA equations (2) into the following form

$$\epsilon^{(r)}(k) = V_r - \sum_{q=1}^{w} \hat{a}_{rq} * F[\epsilon^{(q)}],\quad r = 1,2,\cdots,w, \tag{7}$$

with $V_r = rk^2 - r\mu - H_r - \frac{1}{12}r(r^2-1)c^2$, where the last effective field H_w is set to zero due to the spin

singlet charge bound states. For example, the $SU(3)$ case, i.e. $w=3$ in the Eq. (7) determines the low temperature properties of the 1D three-component strong interacting fermions.[36,37]

Equation of states: In the strong coupling region, i.e. $|c| \gg E_F$ (E_F is the Fermi energy) and at low temperatures, we have the following expansion form

$$\begin{aligned} -a_n * F[\epsilon^{(m)}](k) \approx & \frac{n|c|}{(nc/2)^2+k^2}\frac{p^{(m)}}{m} \\ & + \frac{1}{2\pi}\frac{32}{3m\sqrt{m}}\frac{1}{n^3|c|^3}\Gamma(\frac{5}{2})T^{\frac{5}{2}}\mathrm{Li}_{\frac{5}{2}}(-e^{A^{(m)}/T}), \end{aligned} \tag{8}$$

which is obtained by integration by parts. In Eq. (8) the $\mathrm{Li}(x)$ is the polylog function. Substituting this relation into the TBA equations (7), we have

$$\begin{aligned} \epsilon^{(r)}(k) = & V_r + \sum_{m=1}^{w}\sum_{\substack{q=1 \\ 2q<r+m}}^{\min(r,m)} \frac{4}{|c|}\frac{p^{(m)}}{m(r+m-2q)} \\ & - \sum_{m=1}^{w}\sum_{\substack{q=1 \\ 2q<r+m}}^{\min(r,m)} \frac{16}{|c|^3}\frac{p^{(m)}}{m(r+m-2q)^3}k^2 \\ & + \sum_{m}^{w} Z_{rm}\frac{T^{5/2}}{|c|^3}\mathrm{Li}_{\frac{5}{2}}(-e^{A^{(m)}/T}). \end{aligned} \tag{9}$$

In the above equation, we defined matrix,

$$Z_{rm} = \sum_{\substack{q=1 \\ 2q<r+m}}^{\min(r,m)} \frac{4}{\sqrt{\pi}}\frac{1}{(r+m-2q)^3 m^{3/2}} \tag{10}$$

and the $A^{(m)}$ collects the constant terms (terms are independent of the k) in the dressed energy $\epsilon^{(m)}(k)$. We would like to mention that the polylog functions involve different modes of generating functions of Fermi integrals. In contrast to the Somerfield expansion with respect to the powers of the temperature t, here the polylog functions contain enough thermal and quantum fluctuations that are acquired by the quantum criticality. Therefore, the polylog functions essentially characterise the singular behavior of the 1D strongly interacting fermions even near quantum phase transitions. If we consider the first three orders in the pressure, only the constant term and the quadratic term ($O(k^2)$ terms) contribute to the thermodynamic quantities, and we can safely drop high order terms in k. Explicitly, we express the dressed energy as

$$\epsilon^{(r)} = rD_r k^2 - A^{(r)}. \tag{11}$$

Integrating Eq. (4) by parts we have

$$p^{(m)} = -\frac{\sqrt{m}}{2\sqrt{\pi D_m}}T^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(-e^{A^{(m)}/T}). \tag{12}$$

The parameter D_r is a modification to the quadratic term in the dressed energy $\epsilon^{(m)}(k)$ and could be read off from Eq. (9),

$$D_r = 1 - \sum_{m=1}^{w}\sum_{\substack{q=1 \\ 2q<r+m}}^{\min(r,m)} \frac{16}{|c|^3}\frac{p^{(m)}}{rm(r+m-2q)^3}. \tag{13}$$

Here we only consider the first three orders in the dressed energy equations, thus $D_r \approx 1$, and the EOS becomes

$$p^{(r)} = -\frac{\sqrt{r}}{2\sqrt{\pi}}T^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(-e^{\frac{A^{(r)}}{T}}), \tag{14}$$

$$A^{(r)} = A_0^{(r)} - \sum_{m=1}^{N}\frac{D_{rm}p^{(m)}}{|c|} - \frac{T^{\frac{5}{2}}Z_{rm}\mathrm{Li}_{\frac{5}{2}}\left(-e^{\frac{A^{(m)}}{T}}\right)}{|c|^3},$$

where $A_0^{(r)}$ and D_{rm} are determined by Eq. (9),

$$\begin{aligned} A_0^{(r)} &= r\mu + H_r + \frac{1}{12}r(r^2-1)c^2, \\ D_{rm} &= \sum_{\substack{q=1 \\ 2q<r+m}}^{\min(r,m)} \frac{4}{m(r+m-2q)}. \end{aligned}$$

In order to simplify the EOS, we define the dimensionless quantities and parameters

$$\begin{aligned} \tilde{p}^{(r)} &= \frac{p^{(r)}}{|c|^3}, \quad \tilde{A}^{(r)} = \frac{A^{(r)}}{|c|^2}, \\ \tilde{\mu} &= \frac{\mu}{|c|^2}, \quad h_r = \frac{H_r}{|c|^2}, \quad t = \frac{T}{|c|^2}. \end{aligned} \tag{15}$$

Then the dimensionless EOS is given by

$$\tilde{p}^{(r)} = -\frac{\sqrt{r}}{2\sqrt{\pi}}t^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(-e^{\frac{\tilde{A}^{(r)}}{t}}), \tag{16}$$

$$\tilde{A}^{(r)} = \tilde{A}_0^{(r)} - \sum_{m=1}^{N} D_{rm}\tilde{p}^{(m)} - Z_{rm}t^{\frac{5}{2}}\mathrm{Li}_{\frac{5}{2}}(-e^{\frac{\tilde{A}^{(m)}}{t}}),$$

where $\tilde{A}_0^{(r)} = r\tilde{\mu} + h_r + \frac{1}{12}r(r^2-1)$. For simplifying our notations, we further define matrices

$$\begin{aligned} & (\ \mathbf{Li_s})_{rm} = \mathrm{Li}_s(-e^{\tilde{A}^{(r)}/t})\delta_{rm}, \quad (\boldsymbol{D})_{rm} = D_{rm}, \\ & (\hat{\boldsymbol{Z}})_{rm} = Z_{rm}, \quad (\tilde{\boldsymbol{A}})_{r1} = \tilde{A}^{(r)}, \quad (\tilde{\boldsymbol{p}})_{r1} = \tilde{p}^{(r)}, \\ & (\boldsymbol{M_r})_{rm} = \frac{\sqrt{r}}{2\sqrt{\pi}}\delta_{rm}, \quad (\boldsymbol{F_s})_{rm} \triangleq t^s\mathrm{Li}_s(-e^{\tilde{A}^{(r)}/t})\delta_{rm}, \\ & (\boldsymbol{f_s})_{rm} \triangleq t^s\mathrm{Li}_s(-e^{\tilde{A}_0^{(r)}/t})\delta_{rm}. \end{aligned} \tag{17}$$

$\mathbf{Li}_{\frac{5}{2}}$, $\mathbf{Li}_{\frac{3}{2}}$, $\boldsymbol{M_r}$, $\boldsymbol{D}$ and $\tilde{\boldsymbol{Z}}$ are square matrices (or the column matrices when they are at the most right of the related terms), $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{A}}$ are column matrices. For example, in the $SU(2)$ case

$$\boldsymbol{D} = \begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}, \qquad \boldsymbol{Z} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{\pi}} \\ \frac{4}{\sqrt{\pi}} & \frac{1}{4\sqrt{2\pi}} \end{pmatrix},$$

and for the $SU(3)$ case,

$$\boldsymbol{D} = \begin{pmatrix} 0 & 2 & \frac{2}{3} \\ 4 & 1 & \frac{16}{9} \\ 2 & \frac{8}{3} & 1 \end{pmatrix}, \ \boldsymbol{Z} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{\pi}} & \frac{1}{6\sqrt{3\pi}} \\ \frac{4}{\sqrt{\pi}} & \frac{1}{4\sqrt{2\pi}} & \frac{112}{81\sqrt{3\pi}} \\ \frac{1}{2\sqrt{\pi}} & \frac{56}{27\sqrt{2\pi}} & \frac{\sqrt{3}}{16\sqrt{\pi}} \end{pmatrix}.$$

With the help of these notations, we rewrite a unified expression of EOS for the $SU(w)$ strongly attractive Fermi gases

$$\tilde{\boldsymbol{p}} = -\boldsymbol{M_r} t^{3/2} \mathbf{Li}_{\frac{3}{2}}, \tag{18}$$

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}_{\boldsymbol{0}} - \boldsymbol{D}\tilde{\boldsymbol{p}} - \hat{\boldsymbol{Z}} t^{5/2} \mathbf{Li}_{\frac{5}{2}}. \tag{19}$$

The last term in the function $\tilde{\boldsymbol{A}}$ is negligible in pressure. Nevertheless, it is necessary in the calculation of the scaling function or phase boundaries. After a lengthy iteration, we get a close form of the EOS

$$\tilde{\boldsymbol{p}} = -\boldsymbol{M_r} \boldsymbol{F}_{\frac{3}{2}}, \tag{20}$$

$$\tilde{\boldsymbol{A}} \approx \tilde{\boldsymbol{A}}_{\boldsymbol{0}} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{f}_{\frac{3}{2}} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{f}_{\frac{1}{2}}\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{f}_{\frac{3}{2}}.$$

Furthermore, we could obtain all the thermodynamic quantities of the system in equilibrium by standard thermodynamic relations via the pressure Eq. (18), which serves as the grand thermodynamic potential of the system. In this context, the partial derivatives of the pressure by any chemical potential, external fields and temperature are essential in our approach. Thus we take the derivative of the pressure Eq. (20) with respect to the variable η: $\eta = \tilde{\mu}, h_1, h_2, \ldots$. It follows that

$$\frac{\partial \tilde{\mathbf{p}}}{\partial \eta} = -\boldsymbol{M_r}\mathbf{F}_{\frac{1}{2}} \frac{\partial \tilde{\mathbf{A}}}{\partial \eta}, \tag{21}$$

$$\frac{\partial \tilde{\mathbf{A}}}{\partial \eta} = \frac{\partial \tilde{\mathbf{A}}_0}{\partial \eta} - \mathbf{D}\frac{\partial \tilde{\mathbf{p}}}{\partial \eta} - \hat{\boldsymbol{Z}}\mathbf{F}_{\frac{3}{2}} \frac{\partial \tilde{\mathbf{A}}}{\partial \eta}. \tag{22}$$

By solving the above two linear equations, we obtain the first order derivative thermodynamic properties

$$\frac{\partial \tilde{\mathbf{p}}}{\partial \eta} = -\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}(\mathbf{I} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}} + (\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}})^2)\frac{\partial \tilde{\mathbf{A}}_0}{\partial \eta},$$

$$\frac{\partial \tilde{\boldsymbol{A}}}{\partial \eta} = (\mathbf{I} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}} + (\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}})^2)\frac{\partial \tilde{\mathbf{A}}_0}{\partial \eta}. \tag{23}$$

For the second order phase transitions, the second derivatives of the pressure, the compressibility or the susceptibility for instance, give a deep insight into the quantum criticality of the systems. Similarly, the second order thermodynamic quantities can be obtained

$$\frac{\partial^2 \tilde{\boldsymbol{p}}}{\partial \eta_1 \partial \eta_2} = \Big[-\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}(\boldsymbol{I} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}})\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{-\frac{1}{2}} - \boldsymbol{M_r}\boldsymbol{F}_{-\frac{1}{2}}\Big]\left(\frac{\partial \tilde{\boldsymbol{A}}}{\partial \eta_1}\frac{\partial \tilde{\boldsymbol{A}}}{\partial \eta_2}\right). \tag{24}$$

On the other hand, the derivative of pressure with respect to t always imposes a tedious task. After carefully solving the above equations involving the derivatives of pressure, we obtain the entropy $\tilde{\boldsymbol{s}} = \frac{\partial \tilde{\boldsymbol{p}}}{\partial t}$,

$$\frac{\partial \tilde{\boldsymbol{p}}}{\partial t} = -\boldsymbol{M_r}\frac{3}{2t}\boldsymbol{F}_{\frac{3}{2}} + \boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}\frac{\tilde{\boldsymbol{A}}}{t} - \boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}(\boldsymbol{I} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}) \times (-\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}\frac{\tilde{\boldsymbol{A}}}{t} + \boldsymbol{D}\boldsymbol{M_r}\frac{3}{2t}\boldsymbol{F}_{\frac{3}{2}}), \tag{25}$$

$$\frac{\partial \tilde{\boldsymbol{A}}}{\partial t} = \Big[\mathbf{I} + \boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}} + (\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}})^2\Big]\Big(-\boldsymbol{D}\boldsymbol{M_r}\boldsymbol{F}_{\frac{1}{2}}\frac{\tilde{\boldsymbol{A}}}{t} + \boldsymbol{D}\boldsymbol{M_r}\frac{3}{2t}\boldsymbol{F}_{\frac{3}{2}} + \boldsymbol{Z}F_{\frac{3}{2}}\frac{\tilde{\boldsymbol{A}}}{t} - \boldsymbol{Z}\frac{5}{2t}\boldsymbol{F}_{\frac{5}{2}}\Big). \tag{26}$$

This result contains not only the linear-temperature-dependent behavior of the entropy in the Luttinger liquid (for $T \ll E_F$) but also the universal quantum scalings of the entropy in the quantum critical region (for $T \gg E_F$) in the vicinity of the critical point. Similarly, the second derivative of the pressure with respect to t is given by

$$\frac{\partial^2 \tilde{\boldsymbol{p}}}{\partial t^2} = -\boldsymbol{M_r}\Big(\frac{3}{4t^2}\boldsymbol{F}_{\frac{3}{2}} + \frac{1}{t}\boldsymbol{F}_{\frac{1}{2}}\boldsymbol{B} + \boldsymbol{F}_{-\frac{1}{2}}\boldsymbol{B}^{\boldsymbol{2}} + \boldsymbol{F}_{\frac{1}{2}}\boldsymbol{D}\boldsymbol{M_r}(\frac{3}{4t^2}\boldsymbol{F}_{\frac{3}{2}} + \frac{1}{t}\boldsymbol{F}_{\frac{1}{2}}\boldsymbol{B} + \boldsymbol{F}_{-\frac{1}{2}}\boldsymbol{B}^{\boldsymbol{2}})\Big) \tag{27}$$

with $\boldsymbol{B} = \frac{\partial \tilde{\boldsymbol{A}}}{\partial t} - \frac{\tilde{\boldsymbol{A}}}{t}$. In contrast to the previous studies on multicomponent interacting fermions,[18–26] the above close forms of the thermodynamics are very useful for analyzing the behavior of the quantum liquids and critical scalings of the 1D interacting fermions with $SU(w)$ symmetry.

Quantum criticality: In the vicinities of the phase boundaries in the phase diagrams of the 1D interacting fermions with $SU(w)$ symmetries,[19] a discontinuity emerges in the polylog functions $\mathrm{Li_s(x)}$ in the EOS, namely,

$$\lim_{\mu/t\to 0^+} \mathrm{Li}_s(-e^{\mu/t}) = -\frac{(\mu/t)^s}{\Gamma(s+1)}, \quad \lim_{\mu/t\to 0^-} \mathrm{Li}_s(-e^{\mu/t}) = 0. \tag{28}$$

The sign change of the $\tilde{A}^{(r)}$ leads to a sudden change of the Polylog functions at the critical point of a phase transition, for example, see Fig. 1, where a sudden enhancement in the Wilson ratio is observed when the driving parameters is tuned across any phase boundary. The condition $A^{(r)} > 0$ implies the exist of the bound states of r-fermions in this certain quantum phase. As a consequence, in the vicinities of phase boundaries, any thermodynamic quantity can be separated into two parts: (1) *the background part* and (2) *the discontinuous part*.

The background part involves the states which do not have a sudden change and the calculation of this part is cumbersome.[38] The discontinuous part can be obtained by analyzing the first order of divergence in the EOS, i.e. the part involves a sudden change in the density of a certain branch of bound states. For example, at the phase transition from the fully paired states into the FFLO phase, the regular part mainly

relates to the thermal fluctuation presented by functions with $A^{(2)}$, whereas the singular part results in a sign change of $A^{(1)}$, which indicates the crossover of the density of unpaired fermions from zero to non-zero while the Fermi sea of the unpaired fermions starts to fill with particles.

We further calculate the phase transition driven by chemical potential or external fields. Supposing that the quantum phase transition is driven by the sign change of $\tilde{A}^{(r)}$, we observe that the thermodynamic quantities are naturally split into background and discontinuous parts as

$$\begin{aligned}\frac{\partial\tilde{\mathbf{p}}}{\partial\eta} &= -\boldsymbol{M_r F_{\frac{1}{2}}}(\mathbf{I}+\boldsymbol{DM_r F_{\frac{1}{2}}}+(\boldsymbol{DM_r F_{\frac{1}{2}}})^2)\frac{\partial\tilde{\mathbf{A}}_0}{\partial\eta}\\ &= \frac{\partial\tilde{\mathbf{p}}}{\partial\eta}\Big|_0 - \boldsymbol{M_r F_{\frac{1}{2}}}\frac{\partial\tilde{\mathbf{A}}_0}{\partial\eta},\end{aligned} \tag{29}$$

where $\frac{\partial\tilde{\mathbf{p}}}{\partial\eta}\big|_0$ denotes the backgroud part, i.e.

$$\frac{\partial\tilde{\mathbf{p}}}{\partial\eta}\Big|_0 = -\boldsymbol{M_r F_{\frac{1}{2}}}\Big(\mathbf{I}+\boldsymbol{DM_r F_{\frac{1}{2}}}+(\boldsymbol{DM_r F_{\frac{1}{2}}})^2\Big)\frac{\partial\tilde{\mathbf{A}}_0}{\partial\eta}\Big|_{F_s^{(r)}=0}.$$

Then we obtain the explicit scaling form of the first derivative thermodynamic quantities of the $SU(w)$ Fermi gases

$$\frac{\partial\tilde{p}}{\partial\eta} = \frac{\partial\tilde{p}}{\partial\eta}\Big|_0^{(r)} - \frac{\sqrt{r}}{2\sqrt{\pi}}\frac{\partial\tilde{A}_0^{(r)}}{\partial\eta}t^{1/2}\mathrm{Li}_{\frac{1}{2}}\left(-e^{r(\tilde{\mu}-\mu_{cr})/t}\right), \tag{30}$$

where the μ_{cr} is the critical chemical potential for the phase transition induced by the change of the r-atoms bond states. This method can be further applied to the second order thermodynamic quantities, explicitly

$$\begin{aligned}\frac{\partial^2\tilde{p}}{\partial\eta_1\partial\eta_2} \approx \frac{\partial^2\tilde{p}}{\partial\eta_1\partial\eta_2}\Big|_0^{(r)} &- \frac{\sqrt{r}}{2\sqrt{\pi}}\frac{\partial\tilde{A}_0^{(r)}}{\partial\eta_1}\frac{\partial\tilde{A}_0^{(r)}}{\partial\eta_2}t^{-1/2}\\ &\cdot\mathrm{Li}_{-\frac{1}{2}}(-e^{r(\tilde{\mu}-\mu_{cr})/t}).\end{aligned} \tag{31}$$

We thus read off the critical exponents from these scaling functions, i.e. the dynamic critical exponent $z=2$ and the correlation critical exponent $\nu=1/2$. In particular, the specific heat is given by

$$\begin{aligned}\frac{\tilde{c}_V}{t} \approx \frac{\partial^2\tilde{\boldsymbol{p}}}{\partial t^2} \approx \frac{\partial^2\tilde{\boldsymbol{p}}}{\partial t^2}\Big|_0 &- \boldsymbol{M_r}\Big(\frac{3}{4t^2}\boldsymbol{F_{\frac{3}{2}}} - \frac{1}{t}\boldsymbol{F_{\frac{1}{2}}}\frac{\tilde{\boldsymbol{A}}_0}{t}\\ &+ \boldsymbol{F_{-\frac{1}{2}}}\frac{(\tilde{\boldsymbol{A}}_0)^2}{t^2}\Big),\end{aligned} \tag{32}$$

which gives

$$\frac{\partial^2\tilde{p}}{\partial t^2} \approx \frac{\partial^2\tilde{p}}{\partial t^2}\Big|_0^{(r)} - \frac{\sqrt{r}}{2\sqrt{\pi}t}\mathcal{H}\left(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\right), \tag{33}$$

where the function

$$\mathcal{H}(x) = \frac{3}{4}\mathrm{Li}_{\frac{3}{2}}(-e^x) - x\mathrm{Li}_{\frac{1}{2}}(-e^x) + x^2\mathrm{Li}_{-\frac{1}{2}}(-e^x). \tag{34}$$

Solving the equation $\frac{d}{dx}\mathcal{H}(x)=0$, we obtain two solutions: $y_1=0.639844$, $y_2=0.276201$, which determine the two peaks of the specific heat at quantum criticality in the $SU(w)$ Fermi gases, i.e.

$$t_1^* = -y_1\, r(\tilde{\mu}-\mu_{cr}), \quad t_1^* = y_2\, r(\tilde{\mu}-\mu_{cr}). \tag{35}$$

The two crossover temperatures fanning out from the critical point indicate the quantum critical region beyond the quantum liquid phases, see Fig. 2. Recent studies on the quantum criticality in 1D Heisenberg spin chain[39] and 1D Bose gas[6] confirm such a novel existence of the critical cone while a quantum phase transition occurs.

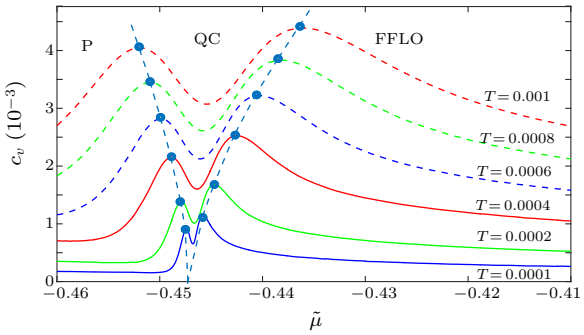

Fig. 2. The temperature rescaled specific heat c_v/T is in the unit $\frac{1}{2}|c|$ vs the dimensionless chemical potential $\tilde{\mu}=\frac{\mu}{|c|^2/2}$ for a fixed dimensionless external field $\tilde{h}=0.46$ in the $r=2$ Fermi gas (1). The phase transition occurs when the chemical potential drives the system across the phase boundary from fully paired phase into the FFLO-like state at $t=0$. The specific peaks fanning out from the critical point indicate three regions: the Luttinger liquid of bosonic bound pairs (P), quantum critical (QC) region and the phase of the FFLO-like state.

For clarity and possible experimental use, we present the explicit scaling forms of the thermodynamic properties for the 1D strongly attractive $SU(w)$ Fermi gases. The polarization of the systems is $\tilde{m}=\sum_k^w \frac{1}{2}k(w-k)\tilde{n}_k$, where $\tilde{n}_r=\frac{\partial\tilde{p}}{\partial h_r}$ presents the density of the r-atoms. For the phase transition related to the sign change of $\tilde{A}^{(r)}$, i.e. the phase transition occurs when the Fermi sea of the charge bound states of r-atoms starts to fill or to be gapped at zero temperature. Based on the above unified scaling forms (25)–(28), we can directly present the result of the scaling functions of physical quantities,

$$\begin{aligned}\tilde{n} &\approx \tilde{n}_0 - \frac{r\sqrt{r}}{2\sqrt{\pi}}t^{1/2}\mathcal{F}\Big(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\Big),\\ \tilde{m} &\approx \tilde{m}_0 - \frac{1}{2}(w-r)r\frac{\sqrt{r}}{2\sqrt{\pi}}t^{1/2}\mathcal{F}\Big(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\Big),\\ \tilde{\kappa} &\approx \tilde{\kappa}_0 - \frac{r^2\sqrt{r}}{2\sqrt{\pi}}t^{-1/2}\mathcal{G}\Big(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\Big),\\ \tilde{\chi}_r &\approx \tilde{\chi}_0 - \frac{1}{2}r(w-r)\frac{\sqrt{r}}{2\sqrt{\pi}}t^{-1/2}\mathcal{G}\Big(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\Big),\\ \frac{\tilde{c}_V}{t} &\approx \frac{\tilde{c}_{V0}}{t} - \frac{\sqrt{r}}{2\sqrt{\pi}}t^{-1/2}\mathcal{H}\left(\frac{r(\tilde{\mu}-\mu_{cr})}{t}\right),\end{aligned}$$

with

$$\mathcal{F}(x) = \mathrm{Li}_{\frac{1}{2}}(-e^x), \quad \mathcal{G}(x) = \mathrm{Li}_{-\frac{1}{2}}(-e^x),$$
$$\mathcal{H}(x) = \frac{3}{4}\mathrm{Li}_{\frac{3}{2}}(-e^x) - x\mathrm{Li}_{\frac{1}{2}}(-e^x) + x^2\mathrm{Li}_{-\frac{1}{2}}(-e^x).$$

Here every first term in the above quantities denote the background parts. Notice that for the $SU(2)$ case, we conventionally take the magnetic field $H = 2H_1$, and the dimensionless form now $\tilde{p}^{(r)} = \frac{p^{(r)}}{|c|^3/2}$, $\tilde{A}^{(r)} = \frac{A^{(r)}}{|c|^2/2}$, $\tilde{\mu} = \frac{\mu}{|c|^2/2}$, $h = \frac{H}{|c|^2/2}$, $t = \frac{T}{|c|^2/2}$. These scaling functions provide exact result of quantum critical phenomena of the 1D $SU(w)$ Fermi gases, also see the theory of quantum criticality.[40,41]

In summary, we have presented a unified approach to the thermodynamics and quantum scaling functions for the 1D strongly attractive Fermi gases with $SU(w)$ symmetry. In particular, we have obtained the two crossover temperature lines fanning out from the critical point that confirm the existence of the critical cone at quantum criticality. While the quantum liquids can be measured through the dimensionless ratios, revealing the important free fermion nature of 1D interacting fermions. Our results pave a way to experimentally study quantum criticality of the fermionic alkaline-earth atoms that display an exact $SU(w)$ spin symmetries with $w = 2I+1$.[42,43] Here I is the nuclear spin. The study of critical phenomena and quantum correlations in ultracold atoms with high symmetries has become a new frontier in atomic physics.

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References

[1] Kinoshita T, Wenger T and Weiss D S 2004 *Science* **305** 1125
[2] Paredes B, Widera A, Murg V, Mandel O, Fölling S, Cirac I, Shlyapnikov G V, Hänsch T W and Bloch I 2004 *Nature* **429** 277
[3] Kinoshita T, Wenger T and Weiss D S 2006 *Nature* **440** 900
[4] Van Amerongen A H, Van Es J J P, Wicke P, Kheruntsyan K V and van Druten N J 2008 *Phys. Rev. Lett.* **100** 090402
[5] Haller E, Gustavsson M, Mark M J, Danzl J G, Hart R, Pupillo G and Christo H 2009 *Science* **325** 1224
[6] Yang B, Chen Y Y, Zheng Y G, Sun H, Dai H N, Guan X W, Yuan Z S and Pan J W 2016 arXiv:1611.00426[cond-mat.quant-gas]
[7] Liao Y A, Rittner A S C, Paprotta T, Li W H, Partridge G B, Hulet R G, Baur S K and Mueller E J 2010 *Nature* **467** 567
[8] Wenz A N, Zürn G, Murmann S, Brouzos I, Lompe T and Jochim S 2013 *Science* **342** 457
[9] Pagano G, Mancini M, Cappellini G, Lombardi P, Schäfer F, Hu H, Liu X J, Catani J, Sias C, Inguscio M and Fallani L 2014 *Nat. Phys.* **10** 198
[10] Andrei N, Furuya K and Lowenstein J H 1983 *Rev. Mod. Phys.* **55** 331
[11] Dukelsky J, Pittel S and Sierra G 2004 *Rev. Mod. Phys.* **76** 643
[12] Essler F H L, Frahm H, Göhmann F, Klümper A and Korepin E 2005 *The One-Dimensional Hubbard Model* (Cambridge: Cambridge University Press)
[13] Takahashi M *Thermodynamics of One-Dimensional Solvable Models* 1999 (Cambridge, Cambridge University Press)
[14] Wang Y P, Yang W L, Cao J P and Shi K 2015 *Off-Diagonal Bethe Ansatz for Exactly Solvable Models* (Heidelberg: Springer-Verlag Berlin)
[15] Wang Y P 1999 *Phys. Rev.* B **60** 9236
[16] Batchelor M T, Guan X W, Oelkers N and Tsuboi Z 2007 *Adv. Phys.* **56** 465
[17] Cazalilla M A, Citro R, Giamarchi T, Orignac E and Rigol M 2011 *Rev. Mod. Phys.* **83** 1405
[18] Guan X W, Batchelor M T and Lee C H 2013 *Rev. Mod. Phys.* **85** 1633
[19] Yu Y C, Chen Y Y, Lin H Q, Roemer R A and Guan X W 2016 *Phys. Rev.* B **94** 195129
[20] Wu C, Hu J P and Zhang S C 2003 *Phys. Rev. Lett.* **91** 186402
[21] Cazalilla M A and Rey A M 2014 *Rep. Prog. Phys.* **77** 124401
[22] Schlottmann P 1993 *J. Phys.: Condens. Matter* **5** 5869
[23] Lee J Y, Guan X W and Batchelor M T 2011 *J. Phys. A* **44** 165002
[24] Jiang Y Z, He P Guan X W 2016 *J. Phys.* A **49** 174005
[25] He P, Yin X G, Guan X W, Batchelor M T and Wang Y P 2010 *Phys. Rev.* A **82** 053633
[26] Oelkers N, Batchelor M T, Bortz M and Guan X W 2006 *J. Phys.* A **39** 1073
[27] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[28] Gaudin M 1967 *Phys. Lett.* A **24** 55
[29] Sutherland B 1968 *Phys. Rev. Lett.* **20** 98
[30] Takahashi M 1970 *Prog. Theor. Phys.* **44** 348
[31] Olshanii M 1998 *Phys. Rev. Lett.* **81** 938
[32] Sommerfeld A 1928 *Z. Phys.* **47** 1
[33] Wilson K G 1975 *Rev. Mod. Phys.* **47** 773
[34] Heidelberg Y P 1998 *Int. J. Mod. Phys.* B **12** 3465
[35] Guan X W, Yin X G, Foerster A Batchelor M T, Lee C H and Lin H Q 2013 *Phys. Rev. Lett.* **111** 130401
[36] Guan X W Batchelor M T, Lee C and Zhou H Q 2008 *Phys. Rev. Lett.* **100** 200401
[37] Kuhn C C N and Foerster A 2012 *New J. Phys.* **14** 013008
[38] Guan X W and Ho T L 2011 *Phys. Rev.* A **84** 023616
[39] He F, Jiang Y Z, Yu Y C, Lin H Q and Guan X W 2017 arXiv:1702.05903[cond-mat.stat-mech]
[40] Sachdev S 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)
[41] Giamarchi T 2004 *Quantum Physics in One Dimension* (Oxford: Oxford University Press)
[42] Gorshkov A V, Hermele M, Gurarie V, Xu C, Julienne P S, Ye J, Zolller P, Demler E, Lukin M D and Rey A M 2010 *Nat. Phys.* **6** 289
[43] Cazalilla M A, Ho A F and Ueda M 2009 *New J. Phys.* **11** 103033