CHIN. PHYS. LETT. Vol. 34, No. 7 (2017) 070303

Phase Dissipation of an Open Two-Mode Bose–Einstein Condensate *

Yan-Na Li(李艳娜), Wei-Dong Li(李卫东)**

Institute of Theoretical Physics and Department of Physics, Shanxi University, Taiyuan 030006

(Received 9 February 2017)

We study the dynamics of a two-mode Bose–Hubbard model with phase dissipation, based on the master equation. An analytical solution is presented with nonzero asymmetry and phase noise. The effects of asymmetry and phase noise play a contrasting role in the dynamics. The asymmetry makes the oscillation fast, while phase noise enlarges the period. The conditions for the cases of fast decay and oscillation are presented. As a possible application, the dynamical evolution of the population for cold atomic gases with synthetic gauge interaction, which can be understood as two-mode dynamics in momentum space, is predicted.

PACS: 03.75.Kk, 05.30.Jp, 67.85.Fg, 74.50.+r DOI: 10.1088/0256-307X/34/7/070303

Trapped ultra-cold atomic gases serve an appealing model system for open quantum system,[1–4] and meanwhile for the study of the quantum dynamics of an interacting many-body system *in situ*, which brings together aspects of nonlinear dynamics,[5–10] and solid-state physics.[11,12] For ultra-cold atomic gases, the condensate is surrounded by the thermal atomic cloud. The inevitable collisions between condensates and thermal atoms play the role of main sources of decoherence, for example, elastic collision with thermal atoms causes phase dissipation,[13,14] and three-body collision causes particle loss.[15]

One possible reason for thoroughly investigating the dynamics of quantum open system could be the possibility to control those dissipations to generate interesting quantum states.[15–17] The amazing experimental precision control for ultra-cold atoms also provides an ideal system to control the dissipation process, by which some interesting quantum states could be realized.[18,19] It has been shown that the dissipative process may help to prepare some pure quantum states,[18] strong correlated states[19] and spin squeezing.[15] A positive effect of strong inelastic collisions on inducing strong correlations has been experimentally proved in Refs. [20,21].

Although very simple, the two-mode approximation helps one to explore some interesting quantum phenomena in trapped cold atomic gases, for example, macroscopic self-trapping,[7] nonlinear tunnelling[22] and Bosonic Josephson effect.[9,23] Involving phase noise and atom loss, a two-mode Bose–Hubbard model was used to investigate the dynamics of BEC in a double-well trap, and a surprising dissipation-induced coherence was explored.[24–26] Based on an effective non-Hermitian (discrete) Gross–Pitaevskii equation, the dissipation makes a substantial modification on the mean-field phase-space structure and a significant increase of the purity of the condensate.[27] Furthermore, the strength of the gain and loss plays a dominant role for the strength of the purity revivals in Ref. [28]. It is interesting to note that the two-mode approximation can also be used to understand the dynamics for cold atoms with spin–orbit (SO) coupling,[29–32] where the two hyperfine levels (two modes located around $\pm k_{\rm r}$) of atoms are coupled by a pair of Raman lasers, where $\hbar k_{\rm r}$ is the recoil momentum.

Generally, the atom loss can be large compared with phase noise, when the BEC are not very pure and has to be considered by a multiple number of modes.[14] However, in the case of two-mode approximation, as pointed out in Refs. [14,24] the atom loss and dissipation are usually extremely weak, compared with phase noise, for the dynamics of pure BEC. For example, in a 30 s experiment on Josephson oscillation,[9,33,34] phase noise for a few hundreds of milliseconds was reported, while only less than 10% of atoms are lost. Therefore, we will ignore the atom loss in the following calculation. Although usually the atomic interaction plays a crucial role in dynamics of ultra-cold atomic gases, the free atom gases, without atomic interaction is still an important model to be considered, especially for analytical solutions.[28,35,36]

In this Letter, an analytical solution for a two-mode Bose–Hubbard model with phase dissipation is presented by solving the master equation. Compared with results in Refs. [28,35,36] a bias of the two modes and phase noise are considered, while the atom exchanging with the environment is not considered. We focus on the effects of phase dissipation and asymmetry of two-mode on oscillation period and oscillation amplitude by using the analytical method. As an application, we present a dynamics prediction for cold atoms with synthetic gauge interaction.

*Supported by the National Natural Science Foundation of China under Grant No 11374197, the Program for Changjiang Scholars and Innovative Research Team in Universities of China under Grant No IRT13076, and the Hundred Talent Program of Shanxi Province.

**Corresponding author. Email: wdli@sxu.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd

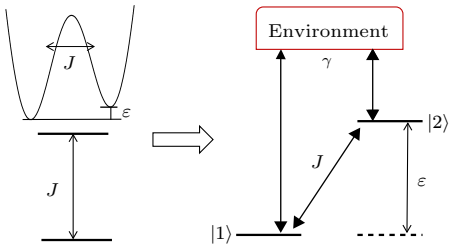

Fig. 1. A trapped two-mode Bose–Einstein condensate. We can present whether a BEC in a double well or a BEC in two different hyperfine states is used in a two-level schematic diagram.

The dynamics of a dilute ultra-cold atomic gas trapped in a double-well trap, i.e., in two-site deep optical lattice even SO coupling in momentum space, can be well described by the two-mode approximation Eq. (1). All of the mentioned setups have been easily experimentally realized.[29,30,37,38] In the literature, this model can be called the Bose–Josephson model or the two-mode Bose–Hubbard Hamiltonian in Refs. [5,39],

$$\hat{H} = \varepsilon(\hat{n}_2 - \hat{n}_1) - J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \tag{1}$$

where ε describes the asymmetry between two modes, which represents the energy difference of the double wells (or two adjacent sites in optical lattice) in real space, or the two-photon detuning in SO cold atoms, and J is related to the kinetic energy and parameterizes the tunnelling between two modes. The operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the bosonic annihilation and creation operators for mode i, and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ ($i = 1$ and 2) is the corresponding number operator. Considering the analytical calculation, we do not consider the nonlinear interaction or on-site interaction in this work. It is interesting to note that this situation can be realized by adjusting the atomic interaction by the Feshbach resonance or in the SO case, in which this interaction was safely ignored.[30,40]

The same as Ref. [24] and to clarify our calculation, we rewrite Eq. (1) as

$$\hat{H} = -2J\hat{L}_x + 2\varepsilon\hat{L}_z, \tag{2}$$

by introducing the collective operator $\hat{L}_{x,y,z}$

$$\hat{L}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2},$$
$$\hat{L}_y = i\frac{\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1}{2},$$
$$\hat{L}_z = \frac{\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1}{2},$$

which consists of a pseudo angular-momentum $SU(2)$ algebra with total quantum number $N/2$, where N is the total number of atoms. Physically, $\hat{L}_z$ is defined by the number difference between two modes and $\hat{L}_{x,y}$ are quadrature interference terms of two modes and are therefore pointing to the phase difference between the two modes.[41]

Assuming that our system is connected with a Markovian environment, the dynamics can be described with a master equation in the form of

$$\frac{\partial\hat{\rho}}{\partial t} = -i[\hat{H},\hat{\rho}] - \frac{\gamma}{2}\sum_{j=1,2}(\hat{n}_j^2\hat{\rho} + \hat{\rho}\hat{n}_j^2 - 2\hat{n}_j\hat{\rho}\hat{n}_j), \tag{3}$$

where γ denotes the phase noise due to elastic collisions with the back ground or thermal gases, or interaction with the external laser field. As mentioned above, the atom loss and dissipation are not considered in this study, which is different from Refs. [14,24].

As shown by many works,[5,6,39] the macroscopic dynamics of cold atomic gases can be well investigated using the mean-field approximation. In this case, the expectation values of angular momentum $\hat{L}_i$, $s_i(t) = 2\mathrm{Tr}(\hat{L}_i\hat{\rho}(t))$, represent their macroscopic dynamics, and the time evolution of $\dot{s}_i(t) = 2\mathrm{Tr}(\hat{L}_i\dot{\hat{\rho}}(t))$ is determined by

$$\dot{s}_x = -2\varepsilon s_y - \gamma s_x, \tag{4}$$
$$\dot{s}_y = 2Js_z + 2\varepsilon s_x - \gamma s_y, \tag{5}$$
$$\dot{s}_z = -2Js_y, \tag{6}$$

which show how asymmetry and phase noise affect the dynamical evolution. In some sense, these two parameters play different roles: asymmetry induces one kind of oscillation motion between $s_x(t)$ and $s_y(t)$, which is similar to J between $s_z(t)$ and $s_y(t)$, while phase noise makes a decay. Thus, one stationary solution is always allowed for long enough time. This is the reason why phase noise can relax the system to an equilibrium state where two modes are equally populated (see Figs. 2 and 3). The dynamics of the difference between the populations in two modes are coupled with $s_y(t)$ only, through which the effects of asymmetry and phase noise are made on $s_z(t)$. In the case of $\varepsilon \neq 0$ and $\gamma \neq 0$, Eqs. (4)–(6) allow one analytical solution

$$\begin{aligned} s_x(t) = &-6\varepsilon e^{-\frac{(2\gamma+6\lambda_1)t}{3}}\Lambda_1 d_1 - 6\varepsilon e^{-\beta t}\Lambda_2[(\gamma \\ &+3\lambda_1)(d_3e^{-i\lambda_2 t} + d_2e^{i\lambda_2 t}) \\ &-3i\lambda_2(d_2e^{i\lambda_2 t} - d_3e^{-i\lambda_2 t})], \end{aligned} \tag{7}$$
$$s_y(t) = e^{-\frac{2(\gamma+3\lambda_1)t}{3}}d_1 + e^{-\beta t}(d_2e^{i\lambda_2 t} + d_3e^{-i\lambda_2 t}), \tag{8}$$
$$\begin{aligned} s_z(t) = &3Je^{\frac{-2(\gamma+3\lambda_1)t}{3}}\Lambda_3 d_1 + 3Je^{-\beta t}\Lambda_4[(2\gamma \\ &-3\lambda_1)(d_2e^{i\lambda_2 t} + d_3e^{-i\lambda_2 t}) \\ &+3i\lambda_2(d_2e^{i\lambda_2 t} - d_3e^{-i\lambda_2 t})], \end{aligned} \tag{9}$$

where $\beta = \frac{2\gamma}{3} - \lambda_1$, Λ_j ($j = 1$, 2, 3 and 4) and

$\lambda_l (l = 1, 2)$ are defined as

$$\Lambda_1 = \frac{1}{\gamma - 6\lambda_1}, \tag{10}$$

$$\Lambda_2 = \frac{1}{(\gamma + 3\lambda_1)^2 + 9\lambda_2^2}, \tag{11}$$

$$\Lambda_3 = \frac{1}{\gamma + 3\lambda_1}, \tag{12}$$

$$\Lambda_4 = \frac{2}{(2\gamma - 3\lambda_1)^2 + 9\lambda_2^2}, \tag{13}$$

$$\lambda_1 = -\frac{\Gamma}{3 \times 2^{2/3}\Omega} + \frac{\Omega}{6 \times 2^{1/3}}, \tag{14}$$

$$\lambda_2 = -\frac{\sqrt{3}\Gamma}{3 \times 2^{2/3}\Omega} - \frac{\sqrt{3}\Omega}{6 \times 2^{1/3}}, \tag{15}$$

with $\Gamma = -\gamma^2 + 12(J^2 + \varepsilon^2)$, $\Omega = (\Theta + \sqrt{\Delta})^{1/3}$, $\Delta = \Theta^2 + 4\Gamma^3$ and $\Theta = 36J^2\gamma - 72\varepsilon^2\gamma - 2\gamma^3$, and $d_{1,2,3}$ are free parameters and are determined by initial conditions for $s_{x,y,z}(0)$. Equations (14) and (15) show that the asymmetry of Eq. (2) modify not only the decay rate of dynamics but also the period of the reviving as a complex form. It is not difficult to find that we cannot reduce Eqs. (7)–(9) to the case of symmetrical two modes (or without phase noise) by simply letting $\varepsilon = 0$ (or $\gamma = 0$). However, this does not stop us from understanding the solution by numerically solving Eqs. (4)–(6) with the fourth-order Runge–Kutta method.[42,43] Let us consider one normal case where two modes are populated at the initial time. In this case, we have our initial values

$$s_x(0) = 0, \quad s_y(0) = 0, \quad s_z(0) = c.$$

The constant $d_{1,2,3}$ can be taken as $d_1 = \Lambda_2 c/(3J\Lambda)$ and $d_{2,3} = -[\Lambda_2 c/(6J\Lambda)] \pm i[Qc/(18J\lambda_2\Lambda)]$ with $Q = \Lambda_1 - \gamma\Lambda_2 - 3\lambda_1\Lambda_2$ and $\Lambda = \Lambda_2\Lambda_3 + \Lambda_1\Lambda_4 - 3\gamma\Lambda_2\Lambda_4$.

There are two kinds of dynamic evolution, depending on λ_2. Once λ_2 is real, we will have a slow decay assistant with an oscillation between two modes, otherwise a quick decay will be found without any oscillations. From Eq. (15), the condition for a real parameter for λ_2 is $\Gamma < 0$, $\Delta \geq 0$, $\Theta \geq 0$ or $\Gamma \geq 0$. Detailed formulas are $\sqrt{12(J^2 + \varepsilon^2)} < \gamma \leq \sqrt{10J^2 - 4\varepsilon^2 - \frac{J^4 - J(J^2 - 8\varepsilon^2)^{3/2}}{2\varepsilon^2}}$ or $\gamma \leq \sqrt{12(J^2 + \varepsilon^2)}$. As an example, two typical results with $c = 0.6$ are shown in our following simulations. In these simulations, the numerical results are obtained by numerically solving Eqs. (4)–(6) using the fourth-order Runge–Kutta method (shown in Fig. 2, where $\gamma = 10^{-4}J$ is described by black up-triangle scatters, $\gamma = 1J$ is described by red pentagon scatters). The analytical solutions are obtained by Eqs. (7)–(9) (shown in Fig. 2, where $\gamma = 10^{-4}J$ is described by the black dotted line, $\gamma = 1J$ is described by the red solid line). To avoid parameter $\varepsilon = 0$ in Eqs. (7)–(9), we have made $\varepsilon = 10^{-6}J$ for the case in Fig. 2(a). A good agreement between the analytical (line) and numerical simulations (scatters) can be found in Fig. 2. In all the calculations, parameters of asymmetry ε and phase noise γ are measured in units of tunneling strength J. For the case of oscillation in Fig. 2(a), $\varepsilon \to 0$, the condition for oscillation and decay is $\gamma < 4J$. In the case of asymmetry $\varepsilon = 0.5J$ (Fig. 2(b)), $\gamma = 1J$ allows an oscillation with decay for $\lambda_2 = -2.16303$. This oscillation is plotted as a red solid line. A kind of oscillation in $s_x(t)$, induced by nonzero asymmetry, can be found in Fig. 2(b). The dynamics difference between $s_x(t)$ and $s_y(t)$ can be read from Eqs. (4) and (5). These two quantities are coupled by asymmetry and the phase noise. Furthermore, $s_y(t)$ is also coupled by $s_z(t)$ by tunneling, by which the period of $s_z(t)$ is modified, for example as $\sqrt{J^2 + \varepsilon^2}$ in the case of $\gamma = 0J$. The long time evolution makes sure that $s_z(t) \to 0$ is easy to read from Fig. 2 for the two cases.

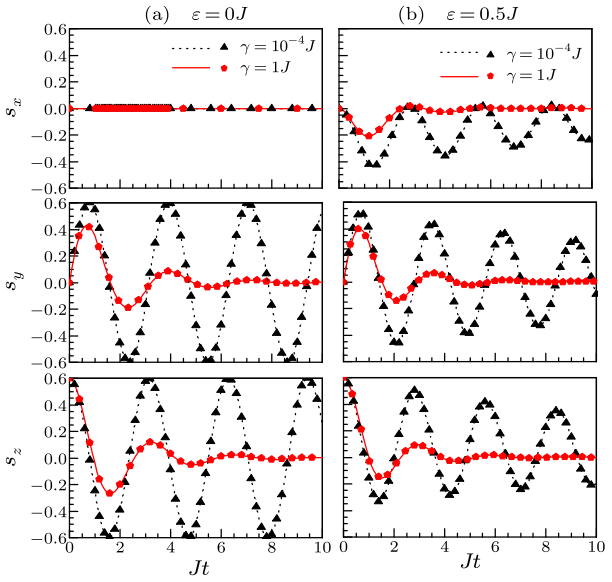

Fig. 2. (Color line) Dynamic evolution of $s_{x,y,z}$ with $J = 1$ by analytical solutions Eqs. (7)–(9) (line) and numerical simulation (scatters). (a) Symmetrical case with $\gamma = 10^{-4}J$ (black dotted line and up-triangle scatters), $\gamma = 1J$ (red solid line and pentagon scatters). (b) Asymmetrical case with the same phase noise as (a).

The characteristics of the dynamics can be well described by the period and the decay rate. Even though it is complex, and the period of $s_z(t)$ is determined by λ_2, its decay is mostly determined by β. It is easy to find that $\lambda_2 = -2\sqrt{J^2 + \varepsilon^2}$ for $\gamma = 0$, and $\lambda_2 = -\sqrt{16J^2 - \gamma^2}/2$ for $\varepsilon = 0$. These two limit cases reveal that the asymmetry makes the oscillation fast (see Fig. 2), while phase noise enlarges the period $T = 2\pi/|\lambda_2|$. On the other hand, two limits for β, which plays its role as decay rate in analytical solution, can be easily found. In the symmetrical case, $\beta = \gamma/2$, which means that the decay is solely determined by phase noise. For $\gamma = 0$, we do not find any decay since $\beta = 0$. However, this cannot be true by reading from Fig. 2 again. The reason is that β cannot

be the whole story on the effect.

As an application of our solutions, we are considering a special case, where the populations on two modes may be measured with a controllable phase noise and an asymmetrical parameter. As mentioned in abstract, the spin–orbital cold atomic gases, both fermion and boson, were experimentally reported very recently,[29–31] with the help of two Raman laser beams. In this case, the two modes prefer two different hyperfine long-lived energy states with external momentum. Therefore its asymmetrical parameter can be adjusted by an external magnetic field and the phase noise is due to the interaction with the Raman laser field. The effective Hamiltonian for this system is the same as Eq. (1).[39] The population in each mode is well defined by $n_i(t) = 2\mathrm{Tr}(\hat{n}_i\hat{\rho}(t))$, and the total number for two modes is $n = 2\mathrm{Tr}((\hat{n}_1 + \hat{n}_2)\hat{\rho})$, which actually is a constant during the time evolution. From our definition for $s_z(t)$ in Eq. (6), it is easy to arrive at

$$n_1(t) = \frac{1}{2}(1 - s_z(t)), \tag{16}$$

$$n_2(t) = \frac{1}{2}(1 + s_z(t)), \tag{17}$$

where $s_z(t)$ is expressed in Eq. (9). It is easy to predict that the long time behavior for $n_{1,2}(t)$ will tend to 0.5, due to $s_z(\infty) \to 0$.

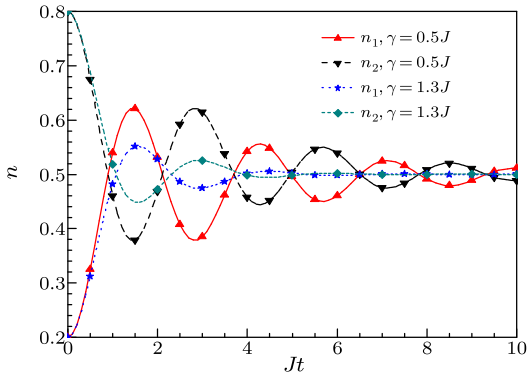

Fig. 3. Dynamical evolution of $n_{1,2}$ with $\varepsilon = 0.5J$.

Carefully adjusting the parameters, we can find a perfect oscillation, consistent with a decay in asymmetrical case ($\varepsilon = 0.5J$ in Fig. 3). A clear suppression on the oscillation amplitude can be found in Fig. 3 by increasing $\gamma = 0.5J$ to $1.3J$. Meanwhile, the period is enlarged as expected. Figure 4 shows the effect of asymmetry on the dynamical evolution. An even faster decay is read from Fig. 4(b) due to relatively large ε. Both the results in Figs. 3 and 4 show that equal populations in two modes are reached at $t \sim 8/J$, which is mostly defined by the decay rate $\sim 10/\gamma$. A possible method to generate the twin Fock state was suggested in Ref. [44].

In a real experimental condition, it is not easy to keep everything exactly the same for each experiment, for example the magnetic field. Therefore, it is reasonable to consider the fluctuations.

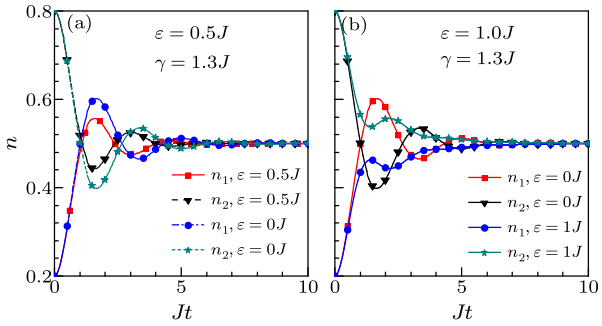

Fig. 4. Dynamical evolution of $n_{1,2}$ in the non-symmetric double well.

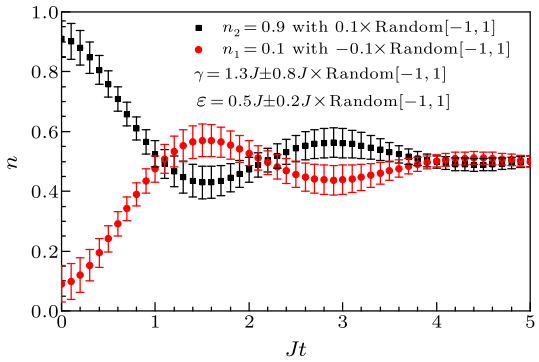

Fig. 5. The time evolution of populations in two modes with fluctuations in parameters.

We have considered the fluctuation of phase dissipation by $\gamma \sim 1.3J \pm 0.8J \times \mathrm{Random}[-1, 1]$, asymmetry by $\varepsilon \sim 0.5J \pm 0.2J \times \mathrm{Random}[-1, 1]$ and $\pm 10\%$ fluctuations for initial values of populations ($n_1 = 0.1$, $n_2 = 0.9$) in each mode, the time evolution of populations are shown in Fig. 5.

In summary, we have presented an analytical solution for the two-mode Bose–Hubbard Hamiltonian with phase noise and asymmetry. The effects of asymmetry of two modes and phase noise on the dynamical evolution are investigated, based on our analytical solution. A condition for fast decay and oscillation with decay is presented. The asymmetry makes the oscillation fast, while phase noise enlarges the period. On the other hand, the decay is determined by phase noise in the symmetrical case, while a complex effect is found for the case of $\gamma = 0$. As an application, the dynamical evolution is investigated by considering current experimental conditions.

References

[1] Breuer H P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[2] Breuer H P 2004 *Phys. Rev.* A **69** 022115
[3] Strunz W T and Yu T 2004 *Phys. Rev.* A **69** 052115

[4] Nikola Buric 2011 *Chin. Phys.* B **20** 120306
[5] Milburn G J, Corney J, Wright E M and Walls D F 1997 *Phys. Rev.* A **55** 4318
[6] Smerzi A et al 1997 *Phys. Rev. Lett.* **79** 4950
[7] Raghavan S et al 1999 *Phys. Rev.* A **59** 620
[8] Raghavan S, Smerzi A and Kenkre V M 1999 *Phys. Rev.* A **60** R1787
[9] Albiez M et al 2005 *Phys. Rev. Lett.* **95** 010402
[10] Salasnich L 1999 *Phys. Rev.* A **61** 015601
[11] Greiner M et al 2002 *Nature* **415** 39
[12] Cole W S et al 2012 *Phys. Rev. Lett.* **109** 085302
[13] Anglin J R 1997 *Phys. Rev. Lett.* **79** 6
[14] Ruostekoski J and Walls D F 1998 *Phys. Rev.* A **58** R50
[15] Li Yun, Castin Y and Sinatra A 2008 *Phys. Rev. Lett.* **100** 210401
[16] Khodorkovsky Y, Kurizki G and Vardi A 2008 *Phys. Rev. Lett.* **100** 220403
[17] Huelga S F and Plenio M B 2007 *Phys. Rev. Lett.* **98** 170601
[18] Kraus B et al 2008 *Phys. Rev.* A **78** 042307
[19] Diehl S et al 2008 *Nat. Phys.* **4** 878
[20] Syassen N et al 2008 *Science* **320** 1329
[21] Garcia-Ripoll J J et al 2009 *New J. Phys.* **11** 013053
[22] Jia X Y, Li W D and Liang J Q 2008 *Phys. Rev.* A **78** 023613
[23] Yang H W and Zuo W 2007 *Chin. Phys. Lett.* **24** 620
[24] Witthaut D, Trimborn F and Wimberger S 2009 *Phys. Rev.* A **79** 033621
[25] Wang W, Fu L B and Yi X X 2007 *Phys. Rev.* A **75** 045601
[26] Witthaut D, Trimborn F and Wimberger S 2008 *Phys. Rev. Lett.* **101** 200402
[27] Trimborn F, Witthaut D and Wimberger S 2008 *J. Phys. B: At. Mol. Opt. Phys.* **41** 171001
[28] Dennis Dast et al 2016 *Phys. Rev.* A **94** 053601
[29] Lin Y J et al 2011 *Nature* **471** 83
[30] Zhang J Y et al 2012 *Phys. Rev. Lett.* **109** 115301
[31] Wang P J et al 2012 *Phys. Rev. Lett.* **109** 095301
[32] Cheuk L W et al 2012 *Phys. Rev. Lett.* **109** 095302
[33] Gati R et al 2006 *Phys. Rev. Lett.* **96** 130404
[34] Gati R, Esteve J et al 2006 *New J. Phys.* **8** 189
[35] Rajagopal K K 2015 *Physica* A **429** 231
[36] Rajagopal K K and Muniandy S V 2015 *Physica* A **440** 118
[37] Shin Y et al 2004 *Phys. Rev. Lett.* **92** 050405
[38] Hall B V et al 2007 *Phys. Rev. Lett.* **98** 030402
[39] Anglin J R and Vardi A 2001 *Phys. Rev.* A **64** 013605
[40] Widera A et al 2006 *New J. Phys.* **8** 152
[41] Taesoo K et al 1998 *Phys. Rev.* A **57** 4004
[42] Zhang Z X, Chen L and Bao X Y 2010 *Opt. Express* **18** 8261
[43] Wang J N and Nie J S 2015 *Proc. SPIE* **9671** 96711V
[44] Holland M and Burnett K 1993 *Phys. Rev. Lett.* **71** 1355