From Nothing to Something Ⅱ: Nonlinear Systems via Consistent Correlated Bang *

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The Chinese ancient sage Laozi said that everything comes from 'nothing'. In the work [Chin. Phys. Lett. 30 (2013) 080202], infinitely many discrete integrable systems have been obtained from nothing via simple principles (Dao). In this study, a new idea, the consistent correlated bang, is introduced to obtain nonlinear dynamic systems including some integrable ones such as the continuous nonlinear Schrödinger equation, the (potential) Korteweg de Vries equation, the (potential) Kadomtsev–Petviashvili equation and the sine-Gordon equation. These nonlinear systems are derived from nothing via suitable 'Dao', the shifted parity, the charge conjugate, the delayed time reversal, the shifted exchange, the shifted-parity-rotation and so on.

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Around the 6th century BC, in chapter 42 of a Chinese classical text 'Daodejing', Laozi said that 'Dao sheng yi, yi sheng er, er sheng san, san sheng wanwu, ...'.[1] The correct translation of Laozi's ideology should have the form,[2] 'nothing' produces the first via 'Dao', then the first produces the second, the second produces the third, and the third produces everything, ... In other words, the essence of Laozi's philosophy is that everything comes from nothing through a suitable Dao. Clearly, the big bang theory on our universe and the Dirac sea in quantum physics are two of the best manifestations of Laozi's philosophy. In Ref. [2] Laozi's idea has been really used to produce infinitely many discrete integrable models from nothing.

A naturally important question has been left in the first work: how to find continuous models including some integrable models directly from nothing via suitable Dao? To partially answer this question, we will introduce a new idea called 'consistent correlated bang' (CCB) which splits one trivial equation (nothing) into two or more equations while the split equations are correlated and consistent. It will be interesting that the nontrivial equation systems such as the well known nonlinear Schrödinger equation (NLS), the (potential) Korteweg de Vries (KdV) equation, the (potential) Kadomtsev–Petviashvili (KP) equation and the sine-Gordon (sG) equation, may be naturally included in the consistent and correlated banged systems.

General CCB. As shown in Ref. [2], mathematically, the nothing can be written as $0 = 0$, or for convenience in the following, we write it as

$$L\psi = 0, \quad \psi = 0, \tag{1}$$

where L may be simply taken as a linear operator. Different choices of L in Eq. (1) will relate to different selections of Dao.

To obtain some nontrivial systems from trivial relation (1), we take three steps: (i) bang, (ii) correlation and (iii) consistence.

(i) Bang. Firstly, we write $\psi = \sum_{i=1}^{n} \psi_i$ and the right 0 of Eq. (1) as $0 = \sum_{i=1}^{n} G_i$. Thus Eq. (1) becomes

$$L\sum_{i=1}^{n}\psi_i = \sum_{i=1}^{n} G_i, \quad \sum_{i=1}^{n}\psi_i = 0, \quad \sum_{i=1}^{n} G_i = 0. \tag{2}$$

It is clear that the trivial Eq. (1) can be banged to an equation set

$$\Delta \equiv \{\Delta_i (\equiv L\psi_i - G_i = 0),\ i = 1, 2, \ldots, n\}, \tag{3}$$

with

$$\sum_{i=1}^{n}\psi_i = 0, \tag{4}$$

and

$$\sum_{i=1}^{n} G_i = 0. \tag{5}$$

(ii) Correlation. Because ψ_i and G_i are banged from $\psi(= 0)$ and 0, respectively, it is reasonable that there are some relationships among them. In other words, they are correlated to each other and may be connected by

$$\psi_i = \hat{f}_i\psi_1, \ G_i = \hat{g}_i G_1, \tag{6}$$

for suitable operators $\{f_i, g_i\},\ i = 1, 2, \ldots, n$.

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(iii) Consistency. Because of the correlated condition (6), we can find some necessary consistent conditions.

The first set of consistent conditions possesses the form

$$g_i L = L f_i,\ i = 1, 2, \ldots, n, \tag{7}$$

which can be directly obtained by applying $\hat{g}_i$ on

$$\Delta_1 \equiv L\psi_1 - G_1 = 0. \tag{8}$$

The second type of consistent conditions reads

$$\hat{f}_i \Delta = \Delta,\ i = 1, 2, \ldots, n, \tag{9}$$

and

$$\hat{g}_i \Delta = \Delta,\ i = 1, 2, \ldots, n. \tag{10}$$

The conditions (9) and (10) can equivalently express that the sets

$$\mathcal{G} = \{\hat{g}_i,\ i = 1, 2, \ldots, n\}, \tag{11}$$

and

$$\mathcal{F} = \{\hat{f}_i,\ i = 1, 2, \ldots, n\ \} \tag{12}$$

are all groups with n elements.

Substituting Eq. (6) into Eq. (5) leads to the third type of consistent condition

$$\sum_{i=1}^{n} g_i G_1 = 0. \tag{13}$$

Under the above consistent conditions, only one equation of Δ, that is, Eq. (8) is independent while others can be reproduced by applying $\hat{f}_i$ or $\hat{g}_i$ to it.

Nonlinear systems from CCB. To show some concrete examples, we restrict the special cases for $n = 2$. In this case, the only second order group is the cyclic group that means

$$\mathcal{F} = \{\hat{f}, I(= \hat{f}^2)\},\ \mathcal{G} = \{\hat{g}, I(= \hat{g}^2)\}, \tag{14}$$

where I is the identity operator.

The banged system in the $n = 2$ situation becomes

$$\Delta_1 \equiv L\psi_1 - G_1 = 0, \tag{15}$$
$$\Delta_2 \equiv L\psi_2 - G_2 = 0, \tag{16}$$

while the correlated conditions are

$$\psi_2 = \hat{f}\psi_1, \tag{17}$$
$$G_2 = \hat{g}G_1. \tag{18}$$

The consistent conditions are simplified to

$$\hat{g}L = L\hat{f}, \tag{19}$$
$$\hat{f}^2 = \hat{g}^2 = 1, \tag{20}$$

and

$$G + gG = 0,\ (G_1 \to G). \tag{21}$$

For simplicity, we take

$$\hat{g} = \delta\hat{f},\ \delta^2 = 1. \tag{22}$$

Substituting Eq. (22) into the consistent condition (19), we have

$$\hat{f}L\hat{f}^{-1} = \delta L, \tag{23}$$

which means that $\hat{f}$ is a symmetry transformation of L for $\delta = 1$ and an antisymmetry transformation for $\delta = -1$.

Now, we can select some possible L to derive nontrivial models.

NLS and derivative NLS equations. To derive the NLS equation, we select L possessing the form

$$L = i\partial_t + \partial_x^2 + \partial_y^2 + \partial_z^2 = i\partial_t + \nabla^2, \tag{24}$$

which means that Eq. (1) is a trivial linear Schrödinger equation without potential

$$i\psi_t + \nabla^2\psi = 0. \tag{25}$$

After writing $\psi = \psi_1 + \psi_2$ and $0 = G - G$ ($G_1 = G$ and $G_2 = -G$), the linear Schrödinger equation (25) can be banged to

$$i\psi_{1t} + \nabla^2\psi_1 = G, \tag{26}$$

and

$$i\psi_{2t} + \nabla^2\psi_2 = -G. \tag{27}$$

For the given operator Eq. (24), the possible symmetry transformations with $\hat{f}^2 = 1$ can be selected as

$$\hat{f} = \hat{T}_i\hat{P}_j,\ i = 1, 2,\ j = 1, 2, 3, 4, \tag{28}$$

with

$$\hat{T}_1 t = t,\ \hat{T}_2 = \hat{C}\hat{T}_{\rm d},\ \hat{C}\phi = \phi^*,\ \hat{T}_d t = -t + t_0, \tag{29}$$

$$\hat{P}_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{30}$$

$$\hat{P}_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \tag{31}$$

$$\hat{P}_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} M3_{11} & M3_{12} & M3_{13} \\ M3_{21} & M3_{22} & M3_{23} \\ M3_{31} & M3_{32} & M3_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} z_0 \cos\theta\cos\gamma \\ z_0 \cos\theta \sin\gamma \\ -z_0 \end{pmatrix}, \tag{32}$$

$$\hat{P}_4 \begin{pmatrix} lx \\ y \\ z \end{pmatrix} = \begin{pmatrix} M4_{11} & M4_{12} & M4_{13} \\ M4_{21} & M4_{22} & M4_{23} \\ M4_{31} & M4_{32} & M4_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x_1 \\ -y_0 \\ -z_0 \end{pmatrix}, \tag{33}$$

where $M3_{11} = 1-(1+\sin\theta)\cos^2\gamma$, $M3_{12} = -\sin\gamma\cos\gamma(1+\sin\theta)$, $M3_{13} = \cos\theta\cos\gamma$, $M3_{21} = -\sin\gamma\cos\gamma(1+\sin\theta)$, $M3_{22} = (1+\sin\theta)\cos^2\gamma - \sin\theta$, $M3_{23} = \cos\theta\sin\gamma$, $M3_{31} = \cos\theta\cos\gamma$, $M3_{32} = \cos\theta\sin\gamma$, $M3_{33} = \sin\theta$, $M4_{11} = (1-\sin\theta)\cos^2\gamma - 1$, $M4_{12} = \sin\gamma\cos\gamma(1-\sin\theta)$, $M4_{13} = \cos\theta\cos\gamma$, $M4_{21} = \sin\gamma\cos\gamma(1-\sin\theta)$, $M4_{22} = (\sin\theta-1)\cos^2\gamma - \sin\theta$, $M4_{23} = \cos\theta\sin\gamma$, $M4_{31} = \cos\theta\cos\gamma$, $M4_{32} = \cos\theta\sin\gamma$, $M4_{33} = \sin\theta$, $x_1 = y_0\sin\gamma + z_0(1+\sin\theta)$, θ, γ, y_0, t_0 and z_0 are arbitrary constants, $\hat{T}_1$ and $\hat{P}_1$ are identity transformations for the time and space, respectively, $\hat{C}$ is a charge conjugate operator, $\hat{T}_{\rm d}$ is a delayed time reversal operator, $\hat{P}_2$ is a shifted parity operator for all space components, $\hat{P}_3$ is a combination of the rotation and the shifted parity for one space component, and $\hat{P}_4$ is a combination of the rotation and the shifted parity for two space components.

Applying the $\hat{f}$ operator Eq. (28) under the condition $\hat{f}\psi_2 = \psi_1$ to Eq. (27) and comparing the result with Eq. (26), we have the consistent condition for G,

$$\hat{f}G = -G, \tag{34}$$

which is the same as Eq. (21).

For any given ψ_1-independent $G = G(x,y,z,t)$ with Eq. (34), the banged system is trivially integrable due to its linearity.

It is interesting that the consistent condition (34) permits various nonlinear realizations. For instance, if we employ

$$\hat{f} = \hat{C}\hat{T}_{\rm d}\hat{P}_2, \tag{35}$$

we can take ($x_1 = x$, $x_2 = y$ and $x_3 = z$)

$$\begin{aligned} G =& \alpha(\psi_1^2\psi_2^* - \psi_2^2\psi_1^*) \\ &+ i\sum_{j=1}^{3}\beta_j\{\psi_1\psi_{1x_j}\psi_2^* - \psi_2\psi_{2x_j}\psi_1^*\}. \end{aligned} \tag{36}$$

Thus the banged system Eq. (26) becomes

$$\begin{aligned} i\psi_{1t} + \nabla^2\psi_1 =& \alpha(\psi_1^2\psi_2^* - \psi_2^2\psi_1^*) \\ &+ i\sum_{j=1}^{3}\beta_j\{\psi_1\psi_{1x_j}\psi_2^* - \psi_2\psi_{2x_j}\psi_1^*\}. \end{aligned} \tag{37}$$

Finally, substituting $\psi_2 = -\psi_1$ into Eq. (37), we have

$$i\psi_{1t} + \nabla^2\psi_1 + \alpha|\psi_1|^2\psi_1 + i\sum_{j=1}^{3}\beta_j|\psi_1|^2\psi_{1x_j} = 0, \tag{38}$$

which is a generalization of the NLS equation ($\beta_j = 0$) and the derivative NLS (DNLS) equation. The NLS and the DNLS systems are integrable only in the (1+1)-dimensional case.

KdV, potential KdV and modified KdV equations. If the operator L is fixed as

$$L = \partial_t + \partial_x^3, \tag{39}$$

then one can find possible antisymmetry ($\delta = -1$ in Eq. (23)) operators of L in the forms

$$\hat{f} = \hat{C}^a\hat{T}_{\rm d}\hat{P}_{\rm s}^x, \ a = 0 \text{ or } 1, \tag{40}$$

where $\hat{C}$, $\hat{T}_{\rm d}$ and $\hat{P}_{\rm s}^x$ are charge conjugate, delayed time reversal and x-space shifted parity operators, respectively.

For the antisymmetry operator(s) Eq. (40), the consistent condition (21) becomes

$$\hat{f}G = G, \tag{41}$$

and the banged system has the form

$$\psi_{1t} + \psi_{1xxx} = G. \tag{42}$$

Similar to the NLS case, there are various possible nonlinear realizations of Eq. (41). Here is a possible simple differential polynomial realization

$$G = \alpha(\psi_1^2\psi_{2x} - \psi_2^2\psi_{1x}) - \beta\psi_{1x}\psi_{2x}. \tag{43}$$

Substituting Eq. (43) into Eq. (42), we have

$$\psi_{1t} + \psi_{1xxx} + \alpha(\psi_2^2\psi_{1x} - \psi_1^2\psi_{2x}) + \beta\psi_{1x}\psi_{2x} = 0. \tag{44}$$

Using the trivial solution $\psi_2 = -\psi_1 = -u$ yields

$$u_t + u_{xxx} + 2\alpha u^2 u_x - \beta u_x^2 = 0, \tag{45}$$

which is a generalization of the potential KdV ($\alpha = 0$) and modified KdV ($\beta = 0$). The KdV equation cannot be directly obtained from the realization of Eq. (41) though it can be considered as simple transformations from the potential KdV and the modified KdV systems.

Nonlinear Klein–Gordon systems. To find nonlinear (1+1)-dimensional Klein–Gordon equation systems, we can use the following L operator

$$L = \partial_t^2 - \partial_x^2. \tag{46}$$

Higher-dimensional nonlinear Klein–Gordon equation systems can also be obtained using the transformation $\partial_x^2 \to \nabla^2$.

For the operator (46), its symmetry operator $\hat{f}$ with $\hat{f}^2 = 1$ possesses the form

$$\begin{aligned} &\hat{f}_{\rm s} = \hat{C}^{a_1}\hat{T}_{\rm d}^{a_2}\hat{P}_{\rm s}^{a_3}\hat{R}^{a_4}(\hat{E}_1\hat{E}_2)^{a_5}, \\ &a_i = 0 \text{ or } 1, \ i = 1,2,\ldots,5, \end{aligned} \tag{47}$$

where $\hat{C}$ is the charge conjugate, $\hat{T}_{\rm d}$ is the delayed time reversal ($\hat{T}_{\rm d}t = -t+t_0$), $\hat{P}_{\rm s}$ is the shifted x-parity ($\hat{P}_{\rm s}x = -x + x_0$), $\hat{R}$ is the shifted-parity-rotation operator defined by ($\delta^2 = \delta_1^2 = 1$),

$$\begin{aligned} \hat{R}\begin{pmatrix} x \\ t \end{pmatrix} =& \begin{pmatrix} -\delta\cosh\theta & \delta_1\sinh\theta \\ -\delta_1\sinh\theta & \delta\cosh\theta \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} \\ &+ \begin{pmatrix} x_1(1+\delta\cosh\theta) \\ \delta_1\sinh\theta x_1 \end{pmatrix}, \end{aligned} \tag{48}$$

$\hat{E}_1$ is the shifted exchange with the definition

$$\hat{E}_1 \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} x_2 \\ -x_2 \end{pmatrix}, \tag{49}$$

and $\hat{E}_2$ is the shifted exchange parity with

$$\hat{E}_2 \begin{pmatrix} cx \\ t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} x_3 \\ x_3 \end{pmatrix}, \tag{50}$$

while t_0, x_i, $i = 0, 1, 2, 3$ and θ are arbitrary constants.

For the operator (46), there are not only the symmetry operators (47), but also the antisymmetry operators with the form ($\hat{f} \to \hat{f}_{\rm a}$),

$$\begin{aligned} &\hat{f}_{\rm a} = \hat{C}^{a_1} \hat{T}_{\rm d}^{a_2} \hat{P}_{\rm s}^{a_3} \hat{R}^{a_4} \hat{E}_j, \\ &a_i = 0 \text{ or } 1, \ i = 1, 2, 3, 4, \ j = 1 \text{ or } 2. \end{aligned} \tag{51}$$

For the operator (46), the banged system possesses the form

$$\psi_{1tt} - \psi_{1xx} = G. \tag{52}$$

Under the consistent conditions

$$\hat{f}_s G = -G, \tag{53}$$

or

$$\hat{f}_a G = G. \tag{54}$$

For the consistent conditions (53) and (54) we select the nonlinear realizations

$$G = \alpha \sin(\psi_1 - \psi_2) + \beta_1(\psi_1 - \psi_2) + \beta_2(\psi_1 - \psi_2)^3, \tag{55}$$

and

$$G = K(\psi_1 - \psi_2), \quad K(-u) = K(u), \tag{56}$$

where K is an even function of a single argument $\psi_1 - \psi_2$. Under the selections (55) and (56), the related banged systems possess the forms

$$\begin{aligned} \psi_{1tt} - \psi_{1xx} = &\alpha \sin(\psi_1 - \psi_2) + \beta_1(\psi_1 - \psi_2) \\ &+ \beta_2(\psi_1 - \psi_2)^3, \end{aligned} \tag{57}$$

and

$$\psi_{1tt} - \psi_{1xx} = K(\psi_1 - \psi_2). \tag{58}$$

With the help of the trivial solution condition $\psi_2 = -\psi_1 = -u$, we have

$$u_{tt} - u_{xx} = \alpha \sin(2u) + 2\beta_1 u + 8\beta_2 u^3, \tag{59}$$

and

$$u_{tt} - u_{xx} = K(2u). \tag{60}$$

It is clear that Eq. (59) is a generalization of the well-known sine-Gordon ($\beta_1 = \beta_2 = 0$) and ϕ^4 ($\alpha = 0$) models. If $K(2u)$ is taken as $K(2u) \sim \cos(ku)$, Eq. (60) is also equivalent to a sine-Gordon equation after a simple field shift ($ku \to ku + \pi/2$).

Potential KP equation. To derive the potential KP equation, we can take a linear operator with the weak two-dimensional approximation

$$L = \partial_t \partial_x + \partial_x^4 + \partial_y^2. \tag{61}$$

It is clear that the operator L given in Eq. (61) possesses the symmetry operator

$$\hat{f} = \hat{C}^{a_1} \hat{P}_{\rm ys}^{a_2} (\hat{T}_{\rm d} \hat{P}_{\rm xs})^{a_3}, \ a_i = 0 \text{ or } 1, \ i = 1, 2, 3. \tag{62}$$

In this case the consistent banged system possesses the form

$$\psi_{1xt} + \psi_{1xxxx} + \psi_{1yy} = G, \ \hat{G} = -G. \tag{63}$$

The following nonlinear realization

$$G = -\alpha(\psi_{1x}\psi_{2x})_x \tag{64}$$

leads to the potential KP equation ($\psi_2 = -\psi_1 = -u$)

$$(u_t + u_{xxx} + \alpha u_x^2)_x + u_{yy} = 0. \tag{65}$$

In summary, a large number of nonlinear systems, including some possible integrable ones such as the well-known NLS, (potential) KdV, (potential) KP and sine-Gordon systems may be derived from nothing (trivial $0 = 0$ equation) by introducing the new idea 'consistent correlated bang' and other suitable Dao which are implied by some symmetric or antisymmetric operators. The main operators we introduced are the time-dependent Schrödinger operator (24) without potential, the linear dispersive wave operator (39) without dissipative, the linear wave operator (the operator of boson field without mass) (46), the dispersive wave operator with the weak two-dimensional approximation (61) and so on. These operators possess some types of symmetries. The main symmetries are charge conjugate, (delayed-) time reversal, (shifted-) parity, (shifted-parity-) rotations, (shifted-) exchanges and so on.

In this study, some single component nonlinear systems are derived. In fact, multiple component nonlinear systems may also be obtained by similar means with the help of the finite dimensional symmetry group, which are related to some suitable multicomponent operators.

Furthermore, the idea used here can also be applied for nontrivial systems to obtain more interesting nonlinear models which are defined as the Alice–Bob systems.[4] The Alice–Bob systems may be used to describe multi-place physics.[5]

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