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Unified Analysis for Calculating the Incoherent Spontaneous Emission of Cooperative Radiations *

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A unified analysis is presented to calculate the incoherent spontaneous power of cooperative radiations based on self-amplified spontaneous emission. Using quantum mechanical tools, we derive analytical expressions for the incoherent spontaneous power of undulator and Cherenkov free-electron lasers (FELs). The undulator and Cherenkov FELs are considered as two different examples for the radiation that accumulate cooperatively. In the case of the undulator FEL, we show an excellent agreement between an expression for the incoherent radiation power derived in the present work and that obtained using a completely different approach [Phys. Rev. E 65 (2002) 026501]. For the Cherenkov radiation, we demonstrate a satisfactory agreement between the incoherent power predicted in our analysis and previous experimental results.

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The generation of short, bright, intense pulses of cooperative radiation based on the self-amplified spontaneous emission (SASE) has attracted ever-growing interest. Such radiation would have many applications in academic research, commercial, medical, and biological fields. The undulator free-electron laser (UFEL) and Cherenkov FEL (CFEL) are potential candidates to generate cooperative radiation in the short-wavelength (optical to x-ray)[1–4] and long-wavelength (mm to far-infrared) ranges,[5–8] respectively. In the UFEL and CFEL, an electron beam propagates in a static magnetic undulator and above/through planar/cylindrical dielectric waveguide, respectively. In the UFEL,[1,3] the magnetic undulator results in transverse electron oscillations allowing a coupling between the transverse electron velocity and the transverse component of the radiation field (TE mode). On the other hand, in the CFEL,[7,8] the electron beam couples to the axial field component of a transverse magnetic (TM) mode of radiation. In both cases, the interaction between the electron beam and radiation field causes microbunching of the electron beam at the scale of radiation wavelength. The characteristic wavelength of the UFEL is determined by a proper slippage of radiation phase with respect to the electron trajectory (i.e., phase matching). A correct slippage ensures the transverse velocity of an electron and the electric field remains parallel. Then, an energy transfer from the electron to the field is attained. In the CFEL, the resonance wavelength is determined by a velocity synchronism of the electron beam and the guided mode (i.e., velocity matching).

The UFEL operated in the VUV and x-ray spectral ranges[4] and CFEL operated in the far-infrared[8] and optical ranges[9,10] are typically based on the SASE process. In SASE, the radiation fields are initially generated by a shot noise current due to the random position of electrons within the electron bunch. Initially, the shot noise current results in a white power spectrum. The fields excited by individual electrons add incoherently giving rise to spontaneous characteristics of the emitted radiation. The temporal and spectral characteristics of the radiation are influenced by the shot noise inherent in the electron beam. The power of incoherent emission produced by the shot noise is proportional to the electron beam current I_0. The characteristic of radiation spectra can be considerably improved using a pre-modulated electron beam at the radiation frequency.[11,12] Another technique is to provide a current profile of the electron pulse that changes on the scale of a radiation wavelength.[13] The latter case is also realized when the electron bunches is shorter than the radiation wavelength. In both cases, the instability of the cooperative radiation evolves from a coherent spontaneous emission (CSE) whose intensity is well above that expected from incoherent shot noise. The CSE is also termed as super-radiant emission.[14] The power of CSE is proportional to the square of the beam current I_0^2.

In general, the power spectra of spontaneous emission are almost represented by squared sinc-function. In the SASE operation mode, as the microbunching process evolves, the bandwidth of the power spectrum becomes narrower. In most practical cases, the lasing process is initiated by the incoherent spontaneous emission. Due to the logarithmic dependence of the

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amplification on the start-up power, the estimation of the initial level of the incoherent power is crucial to predict the power growth rate and saturation length. In Ref. [14] an analytical expression has been reported to calculate the incoherent power in the UFEL based on a classical treatment. In Ref. [14] the electromagnetic field is expressed in terms of transverse eigenmodes excited and propagated in free space. The current density as a source term of radiation is expressed in the frequency domain by means of Fourier transform. The Maxwell equations formulated in a continuum domain of frequencies are then used to describe the incoherent and coherent radiation power generation. To the best of our knowledge, no rigorous model is addressed for calculating the incoherent spontaneous power of the Cherenkov radiation that is a crucial quantity especially in the far-infrared and optical regimes.

The purpose of this work is to present analytical expressions for the incoherent spontaneous power of both UFEL and CFEL using a unified analysis. In our treatment, the spontaneous power is given in terms of the gain coefficient and the spectral width that are continuously varied in the FEL operation. Then, the obtained expressions of the spontaneous power are also generalized in the sense that they can predict the evolution of the spontaneous power along the interaction. In the case of UFEL, we show the compatibility between our results and those obtained in Ref. [14]. For the CFEL, we confirm that the theoretical expression of the spontaneous power predicts in satisfactory manner the data measured in experiments.[10]

The spatial variation of the radiation power along the longitudinal direction (z-direction) $P(z)$ can be written as[15]

$$\frac{dP(z)}{dz} = gP(z) + C_{\rm sp}, \tag{1}$$

where g is the gain coefficient that determines the growth of the stimulated emission, and $C_{\rm sp}$ is the rate of the spontaneous power. From Eq. (1), in the low-gain regime of the UFEL and CFEL, the relative power increases to the initial power during one passage, $\Delta P/P = gz$, where z is the interaction length (undulator or waveguide length). In Eq. (1), all loss mechanisms in the FEL operation such as those associated to the medium of interaction and electron beam are ignored. The derivation of Eq. (1) and expressions for g and $C_{\rm sp}$ in terms of the electric and magnetic fields of radiation are shown in Ref. [15]. In Ref. [15], it is instructive to note that the electric and magnetic fields and the beam current are expressed in the time and spatial domains. In Ref. [15], the beam current is divided into two components. The first one, related to $C_{\rm sp}$, causes the incoherent emission and is represented by a delta-function. The second component, related to g, causes the coherent emission and is formulated using the fluid model where the electron density and velocity are modulated in response to the emitted radiation. On the other hand, in Ref. [14], the fields and beam current are expressed in the frequency and spatial domains. In fact, the electric field amplitude $F(z)$ in Ref. [15] corresponds to $\tilde{C}_q(z,\omega)$ in Ref. [14] that is the field amplitude of a mode q. In Ref. [14], in the case of CSE (super-radiant emission), the beam current is assumed to be modulated (or pre-bunched) at frequency ω equal to the resonance frequency of the radiation spectrum. In the case of incoherent emission, a dc electron beam is assumed ($\omega = 0$). Considering the contribution of the relative frequencies of the beam current, the power spectral density given by Eq. (3) in Ref. [14] corresponds to the total power described by Eq. (12) in Ref. [15].

In the quantum mechanical framework, the electron transition from an initial level to a lower-level (stimulated and spontaneous emissions) is proportional to $S + 1$, where S is the number of photons. On the other hand, the electron transition from the initial level to a higher-level (the absorption process) is proportional to S. Then, we can straightforwardly write

$$C_{\rm sp} = gP(z)|_{S=1}, \tag{2}$$

where $P(z)|_{S=1}$ means that S is replaced by 1 when the power P is expressed in terms of the photons number S.

To derive an expression for radiation power P in terms of the photon number S, we should define the radiation energy in an arbitrary volume. It is beneficial to choose a volume with a length $L_{\rm r}$ equal to that of the radiation wave packet. Then, the photon number $S(z)$ in the volume of length $L_{\rm r}$ is given by

$$\hbar\omega S(z) = \int_{z-L_{\rm r}/2}^{z+L_{\rm r}/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon_0|E|^2 dxdydz = \frac{L_{\rm r}}{c}P(z). \tag{3}$$

One can write

$$L_{\rm r} = \frac{2\pi c}{\Delta\omega}, \tag{4}$$

where $\Delta\omega$ is considered as the measured frequency width. A solution for Eq. (1) can be given in the form of $P(z) = (C_{\rm sp}/g)[\exp(gz) - 1]$.[10] Then, in the spontaneous emission regime ($gz \ll 1$), the power of the spontaneous emission is $P(z) \approx C_{\rm sp}z$. Using Eqs. (3) and (4) with Eq. (2), we obtain a general expression for $P_{\rm sp}$ as

$$P_{\rm sp}(z) \approx C_{\rm sp}z, \quad \text{with} \quad C_{\rm sp} = \frac{\hbar\omega\Delta\omega}{2\pi} \times g, \tag{5}$$

which shows the dependence of the spontaneous emission rate represented by $C_{\rm sp}$ on the stimulated emission rate represented by g and the spectral bandwidth

$\Delta\omega$. It is noted that $\Delta\omega$ represents the contributions of all possible modes in the spontaneous spectrum.

It is noteworthy that the start up of incoherent spontaneous emission, from which a high-intensity cooperative radiation grows, determines the frequency and time stability of the radiation pulse. Then, the gain coefficient in the low-gain regime would be taken into account in our following treatments. It is also noted that the spontaneous process is of less importance in the high-gain regime where $\Delta\omega$ in Eq. (5) becomes considerably narrower. In what follows, we will derive analytical expressions for the incoherent spontaneous power of UFEL and CFEL using Eq. (5).

In the following we derive an expression for the spontaneous power using the low-gain coefficient of UFEL. Then, a perfect agreement with the results obtained in Ref. [14] is demonstrated. Before proceeding, let us define a key parameter, i.e., the gain length $L_{\rm g}$ that is the undulator length over which the power increases by a factor of 2.71 from the initial power: $L_{\rm g} = (1/\sqrt{3})(2m_0\gamma^3\lambda_{\rm U}/\pi\mu_0 e^2 a_{\rm U}^2\overline{N})^{1/3}$,[16] where γ is the electron energy in units of the rest energy m_0c^2, $a_{\rm U}$ is the undulator parameter, $\lambda_{\rm U}$ is the undulator period, and $\overline{N}$ is the average electron density.

In the low-gain limit (i.e., $z \leqslant 2L_{\rm g}$ or $gz \ll 1$), the gain coefficient g is given by[16]

$$g = \frac{\pi e^2 \overline{N} a_{\rm U}^2 N_{\rm U}^2 \lambda_{\rm U}}{4\varepsilon_0 m_0 c^2 \gamma^3} \times f(\xi), \tag{6}$$

where $N_{\rm U}$ is the number of undulator period. In Eq. (6), f is the well-known dispersion function of the Madey theory where

$$f = \frac{d}{d\xi}\Big(\frac{\sin^2(\xi)}{\xi^2}\Big), \tag{7}$$

$\xi = \pi N_{\rm U}(\omega - \omega')/\omega$, ω is the resonance frequency, and ω' is the measured frequency.

To separate the term of emission line shape from that of the absorption in the dispersion function f, we assume that each electron, initially labeled with an energy level b, transits to a lower-level a. Another possible transition is from the level b to a higher-level c. Then, the derivative of the function f in Eq. (7) can be rewritten as

$$f = \frac{1}{\Delta\xi}\Big[\Big(\frac{\sin^2(\xi+\Delta\xi)}{(\xi+\Delta\xi)^2}\Big)_{\rm ba} - \Big(\frac{\sin^2(\xi)}{\xi^2}\Big)_{\rm bc}\Big], \tag{8}$$

where $\Delta\xi = (\xi)_{\rm ba} - (\xi)_{\rm bc} \to 0$ and is determined by the electronic recoil energy $2(\hbar\beta)^2/\gamma m_0$, given by

$$\Delta\xi = \frac{2\pi N_{\rm U}}{\omega}\Big(\frac{\hbar\beta^2}{\gamma m_0}\Big)_{\rm rec}, \tag{9}$$

with $\beta = \omega/c$ being the radiation wavenumber. Using Eqs. (8) and (9), we can write g in Eq. (6) in the form of

$$g = \frac{\mu_0 c^2 e^2 \overline{N} a_{\rm U}^2 z}{8\hbar\omega\gamma^2}\Big[\Big(\frac{\sin^2(\xi+\Delta\xi)}{(\xi+\Delta\xi)^2}\Big)_{\rm ba} - \Big(\frac{\sin^2(\xi)}{\xi^2}\Big)_{\rm bc}\Big]. \tag{10}$$

Using Eq. (10) and considering the first term of emission mechanism shown in the bracket $\{\ldots\}$ where $\Delta\xi \ll \xi$, $C_{\rm sp}$ shown in Eq. (5) is then given by

$$C_{\rm sp} = \frac{\mu_0 c^2 e^2 \overline{N} \Delta\omega a_{\rm U}^2 z}{16\pi\gamma^2} \times \Big(\frac{\sin^2(\xi)}{\xi^2}\Big)_{\rm ba}. \tag{11}$$

Taking the peak value of $C_{\rm sp}$ shown in Eq. (11) at which $\sin^2(\xi)/\xi^2 = 1$, the peak power of the spontaneous radiation in our work,

$$P_{\rm sp}(z)|_{\max} = C_{\rm sp}|_{\max} \times z, \tag{12}$$

is exactly the same as obtained in Ref. [14] when $\Delta\omega = \omega/N_{\rm U}$.

It is noted that the relation $\Delta\omega = \omega/N_{\rm U}$ is only valid for the on-axis spectral width in the low-gain regime. However, the broadening in the spontaneous power spectra $\Delta\omega$ should be considered as a spatial-dependent parameter. In practical cases, $\Delta\omega$ can be influenced by many other parameters such as energy spread and divergence of electron beam. Also, the magnetic field imperfections result in a deviation in the ideal electron trajectory and can produce appreciable additional broadening. Furthermore, the spectral width reduces as the radiation develops along the undulator in the low-gain regime. Therefore, in principle, $\Delta\omega$ should be obtained experimentally.

In the CFEL, the power lost by the electron beam after passing time ΔT can be given by[17]

$$\Delta P = \frac{e\omega I_0 |E_z|^2}{2m_0}\eta\Delta T^3 D(\theta), \tag{13}$$

where E_z is the longitudinal electric field of the TM mode, I_0 is the dc current, and η phenomenologically added in Eq. (13) is the coupling coefficient and represents the spatial overlap for an evanescent mode and an electron beam.[18,19] The most popular form of η is $\eta = \exp[-(4\pi d)/(\lambda\gamma\overline{\beta})]$,[20] where d is the gap between the electron beam and the waveguide and $\overline{\beta} = \overline{v}_{\rm e}/c$ is the normalized electron velocity. It is noted that η is a critical parameter where the radiated power falls off exponentially as the beam to guide gap increases. In Eq. (13), $D(\theta)$ is the familiar dispersion function of the gain line-shape curve,

$$D(\theta) = \frac{d}{d\theta}\Big(\frac{\sin^2(\theta/2)}{(\theta/2)}\Big)^2, \tag{14}$$

where $\theta = (\overline{\beta}\overline{v}_{\rm e} - \omega)(L/\overline{v}_{\rm e})$, and L is the waveguide (or interaction) length. It is noted that Eq. (13) is also valid for the Smith–Purcell laser in which an electron beam passes through a periodically corrugated waveguide.[21]

Using the relation of $P = (1/2)\varepsilon_0 c|E_z|^2 A_\mathrm{b}$ in Eq. (13), where A_b is the cross-sectional area of the electron beam and following the same procedure applied for UFEL with the help of Eq. (7), the maximum incoherent power is given by

$$P_\mathrm{sp}(z)|_\mathrm{max} = \frac{e\mu_0 c\eta I_0}{2\pi A_\mathrm{b}\gamma^2}\Delta\omega L^2. \tag{15}$$

In deriving Eq. (15), we use the relation $D(\theta) = \{\mathrm{sinc}^2[(\theta/2) - \Delta\theta] - \mathrm{sinc}^2[(\theta/2) + \Delta\theta]\}/2\Delta\theta$, where $\Delta\theta = [2(\hbar\beta)^2/\gamma m_0](L/2\overline{v}_\mathrm{e})$.

Next, we compare the spontaneous power predicted by Eq. (15) with measured data in Ref. [10]. In the experiments,[10] a 40 kV electron beam propagating above a Si waveguide is used to emit radiation in the optical range (i.e., $\lambda \sim 1.5\,\mu\mathrm{m}$). Using our notations, the experimental parameters in Ref. [10] are $I_0 = 6.5\,\mu\mathrm{A}$, $A_\mathrm{b} = \pi 10^{-8}\,\mathrm{m}^2$, $\gamma \approx 1$, $\Delta\omega = 5.5 \times 10^{13}\,\mathrm{s}^{-1}$, and $\eta \approx 6 \times 10^{-4}$. Figure 1 shows a good agreement with the experimental results published in Ref. [10] when $L \leqslant 6\,\mathrm{mm}$ where the spontaneous power is $\sim$1 nW. For $L > 6\,\mathrm{mm}$, the measured data are smaller than those predicted theoretically. Two main reasons can cause this mismatch.[9,10] The first and most significant one is the misalignment of the electron beam above the guide surface: as the guide length increases, more mirror positive charges are created in the dielectric material and then the electron beam inclines toward the guide surface. The latter cause is the unavoidable loss in the dielectric material.

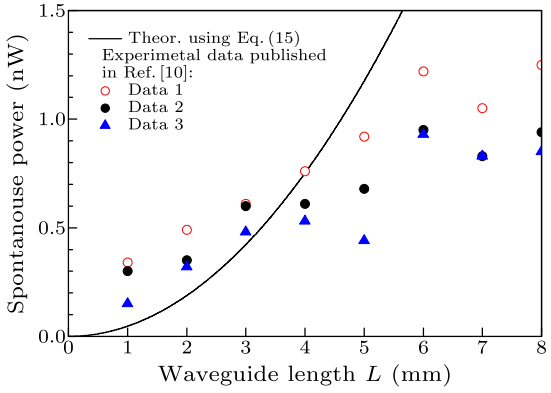

Fig. 1. Comparison between the spontaneous power calculated from Eq. (15) and experimental data measured at three different times as published in Ref. [10].

In summary, we have presented a simple unified analysis to obtain analytical expressions for the power of incoherent spontaneous emission in UFEL and CFEL. Then, our model is proved to be scrupulous and can be used to describe the incoherent emission in different cooperative FELs. In the case of UFEL, the results are in good agreement with those of previous analyses. In the case of CFEL, a good agreement with experimental results has been shown.

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