Nonlocality Distillation and Trivial Communication Complexity for High-Dimensional Systems *

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A nonlocality distillation protocol for arbitrary high-dimensional systems is proposed. We study the nonlocality distillation in the 2-input d-output bi-partite case. Firstly, we give the one-parameter nonlocal boxes and their correlated distilling protocol. Then, we generalize the one-parameter nonlocality distillation protocol to the two-parameter case. Furthermore, we introduce a contracting protocol testifying that the 2-input d-output nonlocal boxes make communication complexity trivial.

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Quantum nonlocality is a significant phenomenon related to the foundation of quantum mechanics. It has attracted much interest in both theoretical and experimental works.[1−4] For example, quantum nonlocality is indispensable for device-independent quantum information processing protocols.[5−9] It is also a necessary resource for nonlocal computation.[10] Moreover, nonlocality may make stronger correlations in which communication complexity becomes trivial.[11]

In 2009, Forsteret *et al.* presented the first nonlocality distillation protocol in the framework of nonsignaling theory.[12−14] Later, Brunner *et al.* presented another protocol for deterministically distilling nonlocality, which works efficiently for a specific class of postquantum nonlocal boxes (the correlated nonlocal boxes).[15] More efficient quantum nonlocality protocols were also given later.[16] Recently, the nonlocality distillation protocol by Foster *et al.* has been confirmed by experiment in Ref. [17]. In addition to this bipartite nonlocality distillation, the distillation protocols of multipartite nonlocality and multi-setting boxes were also proposed.[18]

In the previous study,[19] we presented a nonlocality distillation protocol for two three-level (qutrit) systems. This protocol was based on a three-setting Bell inequality and worked efficiently for a specific class of 3-input 3-output nonlocal boxes. In this Letter, we present a general nonlocality distillation protocol for the 2-input d-output ($d \geq 2$) bipartite systems in the framework of the nonsignaling theory. This protocol is based on the Collins–Gisin–Linden–Massar–Popescu (CGLMP) inequality. It works efficiently for a class of 2-input d-output nonlocal boxes. We also demonstrate that the 2-input d-output nonlocal boxes reduce to the correlated nonlocal boxes by contracting protocol, which makes communication complexity trivial.

Our study focuses on the case of the 2-input d-output bipartite system. To present our main results, we first deal with the problem of how to quantify nonlocality in the 2-input d-output bipartite system. Here we use the CGLMP expression introduced in Ref. [21] in our research. For the sake of convenience, we reform the original CGLMP expression I_d by letting $A_i \to -A_i$ ($i = 1$ and 2) and $B_1 \leftrightarrow B_2$, i.e.,

$$\begin{aligned} I_d \equiv \sum_{k=0}^{[d/2]-1} \Big(1 - \frac{2k}{d-1}\Big)\{+[P(A_1 + B_2 \doteq -k) \\ + P(A_2 + B_2 \doteq k+1) + P(A_2 + B_1 \doteq -k) \\ + P(A_1 + B_1 \doteq k)] - [P(A_1 + B_2 \doteq k+1) \\ + P(A_2 + B_2 \doteq -k) + P(A_2 + B_1 \doteq k+1) \\ + P(A_1 + B_1 \doteq -k-1)]\}, \end{aligned} \tag{1}$$

where $\doteq$ denotes equality modulo d. It could be expressed as a $4 \times d^2$ matrix $\mathbb{I}_d$, by using the prescripts: (1) The subscripts of A_i and B_j $(i, j = 1, 2)$ correspond to the values of inputs x and y respectively, $x = 0 \Longrightarrow i = 1$, $x = 1 \Longrightarrow i = 2$, $y = 0 \Longrightarrow j = 1$, $y = 1 \Longrightarrow j = 2$. (2) The values of A_i and B_j correspond to the values of outputs a and b respectively, $a, b = 0, 1, 2, \ldots, d-1$. Thus we can define the nonlocality of a 2-input d-output system $\mathbb{P}_d$ as

$$NL(\mathbb{P}_d) = \sum_{i,j} \mathbb{I}_d \otimes \mathbb{P}_d. \tag{2}$$

Now, we introduce a class of one-parameter nonlocal boxes $\mathbb{P}_d^{\epsilon}$. *Definition 1*: We define the classical box $\mathbb{P}_d^{\mathrm{C}}$ as

$$\mathbb{P}_d^{\mathrm{C}} := \begin{cases} \frac{1}{d} & \text{if } a + b \doteq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Definition 2: We define the general PR-box $\mathbb{P}_d^{\mathrm{PR}}$ as

$$\mathbb{P}_d^{\mathrm{PR}} := \begin{cases} \frac{1}{d} & \text{if } a + b \doteq x \cdot y, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

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Definition 3: The general correlated nonlocal box $\mathbb{P}_d^\epsilon$ we use for distilling are defined as follows:

$$\mathbb{P}_d^\epsilon := \epsilon \mathbb{P}_d^{\rm PR} + (1-\epsilon)\mathbb{P}_d^{\rm C}, \ (\text{where } 0<\epsilon<1). \quad (5)$$

It is easy to see that $\mathbb{P}_d^{\rm C}$, $\mathbb{P}_d^{\rm PR}$, and $\mathbb{P}_d^\epsilon$ are nonsignaling. With slightly more calculations we can obtain the nonlocality of $\mathbb{P}_d^{\rm C}$, $\mathbb{P}_d^{\rm PR}$ and $\mathbb{P}_d^\epsilon$

$$NL(\mathbb{P}_d^{\rm C}) = 2, \ (\forall d \geq 2), \quad (6)$$
$$NL(\mathbb{P}_d^{\rm PR}) = 4, \ (\forall d \geq 2), \quad (7)$$
$$NL(\mathbb{P}_d^\epsilon) = 2+2\epsilon, \quad (\forall d \geq 2, \ 0<\epsilon<1). \quad (8)$$

We can find that when $\epsilon \to 0$ and $\epsilon \to 1$, $NL(\mathbb{P}_d^\epsilon)$ catches the nonlocal classical maximum and the nonlocal arithmetic maximum, respectively.

Now we introduce the nonlocality distillation protocol for the nonlocal boxes $\mathbb{P}_d^\epsilon$.

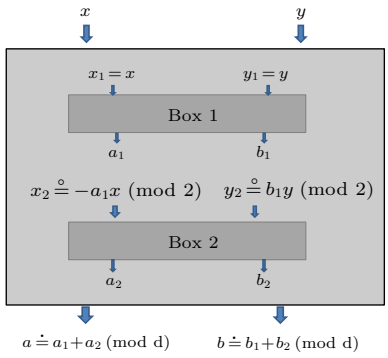


Fig. 1. The distillation protocol. Alice and Bob share two boxes. They input the values x_i and y_i into the ith box and obtain the corresponding outputs a_i and b_i, respectively. Lastly, they calculate $a \doteq a_1 + a_2$ and $b \doteq b_1 + b_2$ as the final outputs of the new box. ($\doteq$ denotes equality modulo 2.)

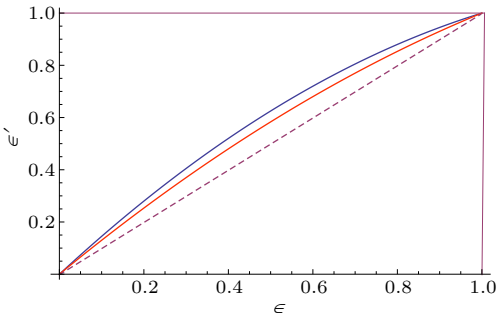


Fig. 2. The relation between ϵ and ϵ'. The blue solid is the result when d is even, and the red solid is the result when d is 3. The dashed straight line is given as a reference $\epsilon' = \epsilon$.

Distillation protocol: This protocol takes two copies of any box $\mathbb{P}_d^\epsilon$ to a new general correlated nonlocal box $\mathbb{P}_d^{\epsilon'}$. It indicates that nonlocality is distilled when $\epsilon' > \epsilon$. The protocol works in Fig. 1. After one iteration of this distillation protocol, two copies of general correlated nonlocal box $\mathbb{P}_d^\epsilon$ turn to one general correlated nonlocal box $\mathbb{P}_d^{\epsilon'}$, where

$$\epsilon' = \begin{cases} \frac{\epsilon}{2}(3-\epsilon), & \text{for even } d, \\ \frac{d-1}{d}\frac{\epsilon}{2}(1-\epsilon)+\epsilon, & \text{for odd } d. \end{cases}$$

In the even case, the expression of ϵ' is the same as that in Ref. [15]. A more intuitional expression of the relation between ϵ and ϵ' can be shown in Fig. 2.

We now give a proof of the above results. Firstly, we illustrate how this protocol works in the case $d = 3$ (see Fig. 3). The process is generally welcome for arbitrary d. We need only consider the non-zeroth term in $\mathbb{P}_3^\epsilon$, which is numbered in the figure. Our protocol should succeed if we obtain a new matrix $\mathbb{P}_3^{\epsilon'}$ under an iteration. The new matrix $\mathbb{P}_3^{\epsilon'}$ has the properties: (1) its first three rows are unchanged, (2) its fourth row could be formulated by one coefficient ϵ' just like $\mathbb{P}_3^\epsilon$, and (3) the amount flows follow the arrows shown in the figure, i.e., $\epsilon' > \epsilon$.

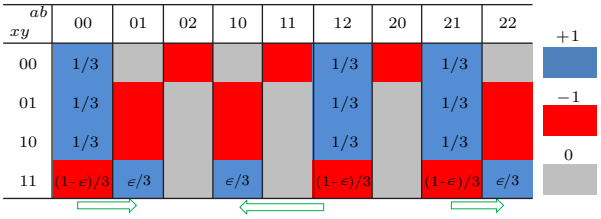


Fig. 3. An intuitional way to show how this protocol works. Different colors denote different parameters in front of the corresponding terms. The red panes denote -1, the blue panes denote $+1$, and the gray panes denote 0. The numbers in those colored blocks denote elements of $\mathbb{P}_3^\epsilon$, and the probability of blocks without a number is zero. The arrows show the orientation of amount flows.

First we calculate the element $P^{\rm new}(00|00)$ in new box $\mathbb{P}_d^{\epsilon'}$. According to the protocol,

$$P^{\rm new}(00|00) = \sum P^1(a_1b_1|x_1y_1)\cdot P^2(a_2b_2|x_2y_2),$$

where $P^i(a_ib_i|x_iy_i)$ denotes the element in box i. The summation calculates all the cases in which one restricts $a_1 + a_2 \doteq 0$ and $b_1 + b_2 \doteq 0$. Noting that only non-zero terms in the first row of $\mathbb{P}_d^\epsilon$ obey $a_1 + b_1 \doteq 0$, we obtain

$$P^{\rm new}(00|00) = \frac{1}{d}.$$

It is easy to generalize this result to other elements in the new boxes which have the form $P^{\rm new}(k-k|xy)$ $(k = 0, 1, \ldots, d-1)$, where $P^{\rm new}(k-k|11)$ is an exception. This means that the first three rows are unchanged after the distilling.

Then let us calculate $P^{\rm new}(00|11)$ firstly in a similar way. Here $P^1(a_1b_1|11)$ is not zero only in the cases of $a_1 + b_1 \doteq 0$ or $a_1 + b_1 \doteq 1$. Then we have

$$P^{\rm new}(00|11) = \sum_{a_1=0}^{d-1} P^1(a_1 - a_1|11)\cdot P^2(-a_1a_1|x_2y_2).$$

When d is even, there are $\frac{d}{2}$ terms of $P^2(-a_1a_1|00)$ and $\frac{d}{2}$ terms of $P^2(-a_1a_1|11)$ in the summation.

When d is odd, there are $\frac{d+1}{2}$ terms of $P^2(-a_1a_1|00)$ and $\frac{d-1}{2}$ terms of $P^2(-a_1a_1|11)$ in the summation. Thus we obtain

$$P^{\text{new}}(00|11) = \begin{cases} \frac{1-\frac{\epsilon(3-\epsilon)}{2}}{d}, & \text{for even } d, \\ \frac{1-[\frac{(d-1)\epsilon(1-\epsilon)}{2d}+\epsilon]}{d}, & \text{for odd } d. \end{cases}$$

We could generalize this result to elements $P^{\text{new}}(k-k|11)$ (where $k = 0, 1, \ldots, d-1$). Similar to $P^{\text{new}}(00|11)$, we have

$$P^{\text{new}}(k-k+1|11) = \begin{cases} \frac{\epsilon(3-\epsilon)}{2d}, & \text{for even } d, \\ \frac{1}{d}[\frac{d-1}{d}\frac{\epsilon(1-\epsilon)}{2}+\epsilon], & \text{for odd } d, \end{cases}$$

where $k = 0, 1, \ldots, d-1$.

Thus, we have proved that this protocol works. It turns the nonlocal box $\mathbb{P}_d^{\epsilon}$ into a new general correlated nonlocal box $\mathbb{P}_d^{\epsilon'}$ with

$$\epsilon' = \begin{cases} \frac{\epsilon}{2}\cdot(3-\epsilon), & \text{for even } d, \\ \frac{d-1}{d}\frac{\epsilon}{2}(1-\epsilon)+\epsilon, & \text{for odd } d. \end{cases}$$

Next, we give a brief description of the two-parameter nonlocal boxes and their distilling protocol. Let us introduce the two-parameter nonlocal boxes $\mathbb{P}_d^{\epsilon}$.

Definition 4: We define the noise-box $\mathbb{P}_d^N$ as

$$\mathbb{P}_d^N := \frac{1}{d^2}\cdot ONE(4, d^2),$$

where $ONE(4, d^2)$ is a $4 \times d^2$ matrix with all entries equal to 1.

Definition 5: The general correlated nonlocal box $\mathbb{P}_d^{\epsilon,\xi}$ we use for distilling is defined as follows:

$$\mathbb{P}_d^{\epsilon,\xi} := \xi\mathbb{P}_d^{\epsilon} + (1-\xi)\mathbb{P}_d^N. \tag{9}$$

Similarly, $\mathbb{P}_d^N$ and $\mathbb{P}_d^{\epsilon,\xi}$ are non-signaling. Taking $d = 3$ as an example, we have

$$\mathbb{P}_d^{\epsilon,\xi} = \begin{pmatrix} xy\,{}^{ab} & 00 & 01 & 02 & 10 \\ 00 & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} \\ 01 & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} \\ 10 & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} & \frac{1-\xi}{d^2} \\ 11 & \frac{\xi(d-\epsilon d-1)+1}{d^2} & \frac{\epsilon\xi d-\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{\epsilon\xi d-\xi+1}{d^2} \end{pmatrix}$$

$$\begin{pmatrix} 11 & 12 & 20 & 21 & 22 \\ \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} \\ \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} \\ \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} & \frac{(d-1)\xi+1}{d^2} & \frac{1-\xi}{d^2} \\ \frac{1-\xi}{d^2} & \frac{\xi(d-\epsilon d-1)+1}{d^2} & \frac{1-\xi}{d^2} & \frac{\xi(d-\epsilon d-1)+1}{d^2} & \frac{\epsilon\xi d-\xi+1}{d^2} \end{pmatrix}.$$

The area of the protocol works is illustrated in Fig. 4.

In reality, the new general correlated nonlocal box $\mathbb{P}_d^{\epsilon',\xi'}$ and the original box $\mathbb{P}_d^{\epsilon,\xi}$ must be the same boxes. This condition will be suitable if and only if the output shots $d = 2, 3$. When $d = 2$, the correlated nonlocal box becomes undistinguished. When $d = 3$, we have $\mathbb{P}_d^{\epsilon,\xi} \to \mathbb{P}_d^{\epsilon',\xi'}$ after a distillation

$$\epsilon' = \frac{\epsilon+\xi(11\epsilon-3\epsilon^2)}{9\xi}, \quad \xi' = \xi^2.$$

Quantum nonlocality of the initial and final boxes is represented in Fig. 5.

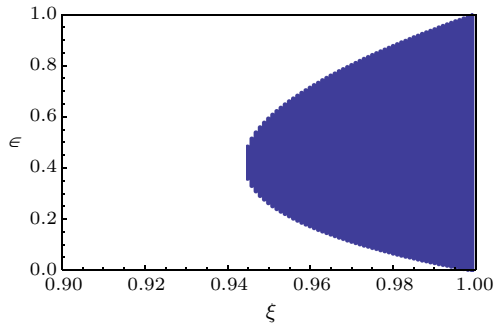


Fig. 4. The area in which our protocol works well (we choose $d = 3$). In this area, when $\mathbb{P}_d^{\epsilon,\xi} \to \mathbb{P}_d^{\epsilon',\xi'}$ by this protocol, the quantum nonlocality is distilled.

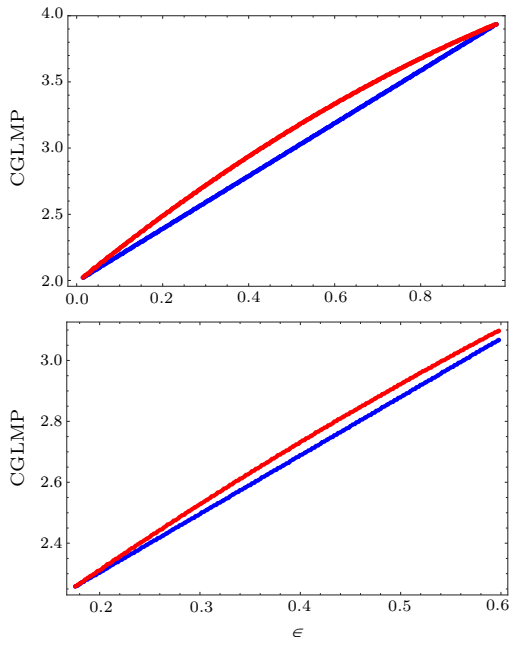


Fig. 5. Quantum nonlocality after distillation is represented by the blue line. The red line is a reference that represents quantum nonlocality of the initial box. The first figure is obtained when $\xi = 0.996$, and the second figure is obtained when $\xi = 0.96$.

The first half of the figure is obtained when $\xi = 0.996$, and the latter part is obtained when $\xi = 0.96$. We also give the distinction of quantum nonlocality in Fig. 6. We can find that as $\xi \to 1$ and $\epsilon \to 0.5$, the distillation has a satisfactory effect.

One question left is whether the correlations in Eq. (9) have a quantum realization. Our answer is negative. We shall demonstrate this by giving a contracting protocol. The contracting protocol recasts nonlocal boxes to correlated nonlocal boxes which

make communication complexity trivial. Thus the initial boxes also make communication complexity trivial and the correlations in Eq. (9) do not have a quantum realization.

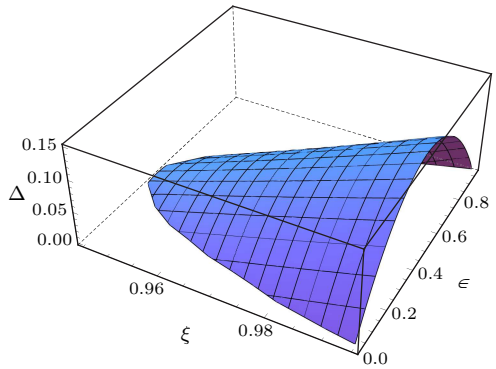


Fig. 6. The difference of quantum nonlocalities before and after distillation. For $d = 3$, the quantum nonlocality can be distilled. The vertical axis of the graph $\triangle$ denotes the increase of the nonlocality.

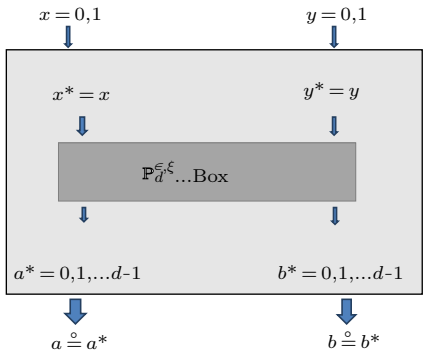


Fig. 7. The contracting protocol in the even case. The initial box is a $\mathbb{P}_d^{\epsilon,\xi}$-box, whose ports are 2-input and d-output. The correlated nonlocal box is a 2-input 2-output box, which finally outputs $a \triangleq a^*, b \triangleq b^*$.

The contracting protocol has different forms in odd d and even d cases. Firstly, we present a contracting protocol in the even case, as illustrated in Fig. 7. The initial box is a $\mathbb{P}_d^{\epsilon,\xi}$-box. We can recast it to a 2-input 2-output correlated nonlocal box, which can be expressed as

$$\mathbb{P}_{even}^{\epsilon,\xi} = \begin{pmatrix} {}_{xy}{}^{ab} & 00 & 01 & 10 & 11 \\ 00 & \frac{1+\xi}{4} & \frac{1-\xi}{4} & \frac{1-\xi}{4} & \frac{1+\xi}{4} \\ 01 & \frac{1+\xi}{4} & \frac{1-\xi}{4} & \frac{1-\xi}{4} & \frac{1+\xi}{4} \\ 10 & \frac{1+\xi}{4} & \frac{1-\xi}{4} & \frac{1-\xi}{4} & \frac{1+\xi}{4} \\ 11 & N_{11} & N_{21} & N_{31} & N_{41} \end{pmatrix}, \quad (10)$$

where $N_{11} = \frac{1-\epsilon}{2}\xi + \frac{1-\xi}{4}$, $N_{21} = \frac{\epsilon}{2}\xi + \frac{1-\xi}{4}$, $N_{31} = \frac{\epsilon}{2}\xi + \frac{1-\xi}{4}$, and $N_{41} = \frac{1-\epsilon}{2}\xi + \frac{1-\xi}{4}$. In fact, it is the two-parameter nonlocal box

$$\mathbb{P}_{\alpha,\beta} = \alpha\mathbb{P}^{\rm PR} + \beta\mathbb{P}^{\rm C} + (1-\alpha-\beta)\mathbb{P}^{\overline{\rm PR}},$$

where $\alpha = \frac{1-\xi}{2} + \xi\epsilon$, and $\beta = \xi(1-\epsilon)$.

We prove those results by direct calculations.

$$P^{\rm new}(00|00) = \sum_{a^* \triangleq 0, b^* \triangleq 0} P_d^{\epsilon,\xi}(a^*b^*|00).$$

There are $(\frac{d}{2})^2$ terms following the relationship $a^* \triangleq 0, b^* \triangleq 0$. Among them, there are $\frac{d}{2}$ terms following the relationship $a^* + b^* \triangleq 0$. Therefore, in the first three rows, the value of the $\frac{d}{2}$ terms is $\frac{\xi}{d} + \frac{1-\xi}{d^2}$. The rest of the elements in these rows are $(1-\xi)\frac{1}{d^2}$. Thus we have

$$P^{\rm new}(00|00) = \frac{1+\xi}{4},$$
$$P^{\rm new}(01|00) = \frac{1-\xi}{4}.$$

Similarly, we can obtain the rest of the elements of the first three rows. For the fourth row, the value of the terms following $a^* + b^* \triangleq 0$ is $\xi(1-\epsilon)\frac{1}{d} + \frac{1-\xi}{d^2}$, the value of the terms following $a^* + b^* \triangleq 1$ is $\xi\epsilon\frac{1}{d} + \frac{1-\xi}{d^2}$, and the value of the other terms is $(1-\xi)\frac{1}{d^2}$. Thus

$$P^{\rm new}(00|11) = P^{\rm new}(11|11) = \frac{1-\epsilon}{2}\xi + \frac{1-\xi}{4},$$
$$P^{\rm new}(01|11) = P^{\rm new}(10|11) = \frac{\epsilon}{2}\xi + \frac{1-\xi}{4}.$$

This completes the proof.

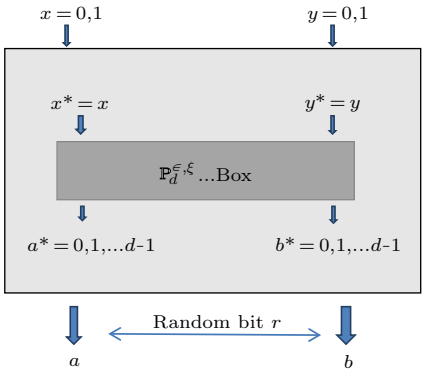


Fig. 8. The contracting protocol in the odd case. There is a sharing randomness r. Before output, Alice and Bob must measure the random number r.

The proof for the odd case is more complicated. A contracting protocol is illustrated in Fig. 8. Alice and Bob share the random number bit with $p(r = 0) = p(r = 1) = 0.5$. Before output, Alice and Bob must measure the random number r. When $r = 0$, the outcome is

$$a \triangleq a^*, \quad \begin{cases} b = 0 & (\text{when } b^* = 0), \\ b \triangleq b^* + 1 & (\text{when } b^* \neq 0). \end{cases}$$

When $r = 1$, the outcome is

$$a \triangleq a^* + 1, \quad \begin{cases} b = 1 & (\text{when } b^* = 0), \\ b \triangleq b^* & (\text{when } b^* \neq 0). \end{cases}$$

The new correlated nonlocal box can also reduce to a 2-input 2-output box, which is

$$\mathbb{P}^{\epsilon,\xi}_{\text{odd}} = \begin{pmatrix} {}_{xy}{}^{ab} & 00 & 01 & 10 & 11 \\ 00 & \frac{1+\mu}{4} & \frac{1-\mu}{4} & \frac{1-\mu}{4} & \frac{1+\mu}{4} \\ 01 & \frac{1+\mu}{4} & \frac{1-\mu}{4} & \frac{1-\mu}{4} & \frac{1+\mu}{4} \\ 10 & \frac{1+\mu}{4} & \frac{1-\mu}{4} & \frac{1-\mu}{4} & \frac{1+\mu}{4} \\ 11 & M_{11} & M_{21} & M_{31} & M_{41} \end{pmatrix}, \quad (11)$$

where $M_{11} = \frac{1-\nu}{2}\mu + \frac{1-\mu}{4}$, $M_{21} = \frac{\nu}{2}\mu + \frac{1-\mu}{4}$, $M_{31} = \frac{\nu}{2}\mu + \frac{1-\mu}{4}$, $M_{41} = \frac{1-\nu}{2}\mu + \frac{1-\mu}{4}$, $\mu = \xi + \frac{1-\xi}{d^2}$, and $\nu = \frac{\epsilon\xi d(d-1)}{\xi(d^2-1)+1}$. We can see that the box in Eq. (11) has the same form as the even case in Eq. (10). When $d \to \infty$, the difference will vanish, $\mu \to \xi, \nu \to \epsilon$.

We calculate $P^{\text{new}}(00|00)$ as an example

$$P^{\text{new}}_{\text{odd}}(00|00) = \frac{1}{2} \sum_{f(a^*)=0, f(b^*)=0, r=0} P^{\epsilon,\xi}_d(a^*b^*|00) + \frac{1}{2} \sum_{f(a^*)=0, f(b^*)=0, r=1} P^{\epsilon,\xi}_d(a^*b^*|00).$$

In the first three rows, the value of the terms following $a^* + b^* \doteq 0$ is $\frac{\xi}{d} + \frac{1-\xi}{d^2}$. The rest of the elements in these rows are $(1-\xi)\frac{1}{d^2}$. When $r = 0$, there are $(\frac{d+1}{2})^2$ terms in this summation, in which $\frac{d+1}{2}$ terms follow the relation $a^* + b^* \doteq 0$. When $r = 1$, there are $(\frac{d-1}{2})^2$ terms in this summation, in which $\frac{d-1}{2}$ terms follow the relation $a^* + b^* \doteq 0$. Therefore

$$P^{\text{new}}_{\text{odd}}(00|00) = \frac{1+\mu}{4}.$$

Similarly, we can obtain $P^{\text{new}}_{\text{odd}}(11|00)$, $P^{\text{new}}_{\text{odd}}(00|11)$, $P^{\text{new}}_{\text{odd}}(00|11)$ and $P^{\text{new}}_{\text{odd}}(01|11)$. Thus we have proved that the communication complexity collapses after distillation. Although some correlations in Eq. (15) are arbitrarily close to the set of classical correlations, all of these correlations are postquantum and make communication complexity trivial after distillation.

In summary, our work represents a primary step to explore multi-setting Bell inequalities in the task of distilling nonlocality. It may be interesting to further explore the contracting technique in proving that certain nonlocal boxes are postquantum and they are unlikely to exist in nature.

We would like to mention that Pan *et al.* also design some protocols to distill high-dimensional nonlocal boxes.[22] In their protocols, the inputs of the second box are decided by the global inputs, the outputs of the first box and the dimension d. However, in our protocols the inputs of the second box are independent of the dimension d, which seems easier to implement in experiment.

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