Noiseless Linear Amplification with General Local Unitary Operations *

Song Yang(杨颂)**, Ning-Juan Ruan(阮宁娟), Yun Su(苏云), Xu-Ling Lin(林栩凌), Zhi-Qiang Wu(邬志强)
Beijing Institute of Space Mechanics and Electricity, Beijing 100094

(Received 23 March 2016)

Noiseless linear amplification (NLA), first proposed by Ralpha et al., is a nondeterministic amplification process which gives gain to the Fock state $|n\rangle \to g^n|n\rangle$, with g being the amplification gain. We here give a general framework for improving the NLA scheme with arbitrary general local unitary operations. We derive the improvement in the amplification gain in 0–1 photon subspace. In particular, we study if the local unitary is composed of single mode squeezing and coherent displacement operation. Finally, numerical simulations show that local unitary operation could give a further enhancement in the amplification gain as well as the success probability, making the NLA more feasible in future optic quantum communications.

PACS: 03.67.Hk, 42.50.Ex DOI: 10.1088/0256-307X/33/7/070304

Quantum optic amplification is an important tool in building long-distance, large-scaled optic quantum communication networks.[1,2] However, due to the quantum non-cloning theorem, the deterministic and noiseless amplification is forbidden in quantum information processing. The amplification in quantum communication can be generally divided into two categories: (1) deterministic but noisy. The typical example is the one first proposed by Caves.[3] (2) Noiseless but non-deterministic. This is the so-called noiseless linear amplification (NLA), first proposed by Ralph *et al.*[4] Indeed, such a scheme is a probabilistic but heralded process, with the successful NLA heralded by the detection result in photon detectors. Since the first proposal, the NLA scheme has been widely studied, in both theory and experiments. It is already known that NLA can be used for entanglement distribution[5–7] and the Bell-inequality violation.[8] The NLA process is still helpful for increasing the maximal secure distance for quantum key distribution[9,10] and quantum communication.[11] In laboratories quite a few experiments on NLA have been successfully performed.[12–19] More recently, a coherent two-mode amplifier for single photon polarization qubit has also been demonstrated.[20] Zhang *et al.* also proposed a quantum state transformation theory for quantum state amplification.[21]

The amplification gain and success probability are no doubt the most important factors that affect the further applications in future quantum information processing. In Ref. [22], we investigate the enhancement of the NLA scheme with the local squeezing operator and it is clearly shown that the injection of local squeezing is equivalent to preparing a different entanglement state in ancillary optic mode, and the success probability and amplification are improved simultaneously. This shows the power of local squeezing in the quantum optic state preparation. However, it is an open question whether some more effective local unitary operations (such as single mode squeezing, displacement operation or both) could exist and give a more pronounced enhancement in the performance of noiseless linear amplification. To answer such a question, we give a theoretic framework for studying the NLA scheme with arbitrary single mode unitary operation. In the 0–1 subspace, which approximates the output state very well when the average photon number of input optic state is weak, we derive the enhancement of gain $g \to g\Delta$ with Δ being an enhancing factor $\Delta > 1$. Our result is then checked with a specific unitary operation composed with local squeezing $S(\xi)$ and displacement $D(\beta)$. When $\beta = 0$ and $\xi = 0$, we obtain $\Delta = 1$ and our scheme is naturally simplified to Ralph and Lund's original scheme. When $\beta = 0$ and $\xi \neq 0$, we recover our earlier squeezing-enhanced scheme in Ref. [22]. Furthermore, when $\beta \neq 0$ and $\xi \neq 0$, we obtain the gain in more general perspective. Our result shows that compared with the squeezing-enhanced scheme, local displacement is more effective, providing more powerful gain $g \sim 10(\Delta \sim 3)$ at $g = 3$ and $t = 0.1$ with only a small coherent displacement $|\beta| \sim 0.2$.

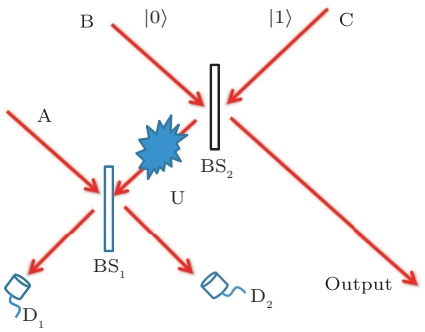

Fig. 1. (Color online) Scheme of NLA with an on-line displacement operator and two ideal photon detectors D_1 and D_2. BS_1 and BS_2 are two beam splitters with transmittance 1/2 and t ($0 \leq t \ll 1$), respectively. The Fock states $|0\rangle$ and $|1\rangle$ denote the vacuum and single photon states.[23–25] Here U is the single mode unitary operation performed on the optic mode C.

To show the enhancement of our displacement-

*Supported by the National Natural Science Foundation of China under Grant Nos 11304013, 11204197, 11204379 and 11074244, the National Basic Research Program of China under Grant No 2011CBA00200, the Doctor Science Research Foundation of Ministry of Education of China under Grant No 20113402110059, and Civil Aerospace 2013669.
**Corresponding author. Email: kellyyszx@163.com
© 2016 Chinese Physical Society and IOP Publishing Ltd

assisted NLA scheme, we again adopt the original noiseless linear amplification scheme.[4] As shown in Fig. 1, mode A is the photonic input that is to be amplified. Modes B and C are the vacuum and single photon states. BS_1 and BS_2 are two beam splitters with transmittance $1/2$ and t, respectively. Here U is the single mode unitary operation performed on optic mode C. This is new in our scheme and when $\beta = 0$ we recover the original scheme in Ref. [4]. Detectors D_1 and D_2 are two single-photon detectors. In realistic implementations, such a single-photon detector can also be replaced with photon dichotic on-off detectors.

In Fig. 1, mode B is the optic output state. Generally, the output state B is the amplified version of optic input A, with detectors D_1 and D_2 registered by 1 and 0, respectively. Now, let us take mode A as a weak coherent state, which can be well approximated with a superposition of vacuum and single photon state $|\psi_\alpha\rangle = |0\rangle + \alpha|1\rangle, (\alpha > 0)$. It should be noted that the amplification of the other kind of state can be similarly evaluated, as shown in Ref. [11].

Using a similar method to the one in Ref. [22], let us study the state evolution in Fig. 1. Let $V_{\rm BC}(t) = \exp[\arctan(\sqrt{(1-t)/t})(a_{\rm B}a_{\rm C}^\dagger - a_{\rm B}^\dagger a_{\rm C})]$ be the beam splitter with transmittance t. In fact, by acting on the state $|0\rangle_{\rm B}|1\rangle_{\rm C}$, the beam splitter generates a single photon path-entangled state $|\psi_{\rm BC}\rangle = \sqrt{t}|0\rangle_{\rm B}|1\rangle_{\rm C} - \sqrt{1-t}|1\rangle_{\rm B}|0\rangle_{\rm C}$. The single mode unitary operator U evolves the state $|\psi_{\rm BC}\rangle$ to

$$|\psi'_{\rm BC}\rangle = I \otimes U|\psi_{\rm BC}\rangle = \sqrt{t}|0\rangle_{\rm B} \otimes (U|1\rangle_{\rm C}) - \sqrt{1-t}|1\rangle_{\rm B} \otimes (U|0\rangle_{\rm C}). \quad (1)$$

In the photon number space, we can express the evolution of U when performed on vacuum state and single photon state as

$$U|0\rangle = \sum_n c_n|n\rangle \approx c_0|0\rangle + c_1|1\rangle + c_2|2\rangle, \quad (2)$$

$$U|1\rangle = \sum_n d_n|n\rangle \approx d_0|0\rangle + d_1|1\rangle + d_2|2\rangle, \quad (3)$$

where we truncated the quantum state within the few-photon subspace, which is quite enough for analyzing the amplified state.

The state in modes A and C will then be coupled with a 50:50 beam splitter and conditioned on the detection result, the input photonic state can be probabilistic amplified. In the following we take the example of detector D_1 registered by 0 photon and detector D_2 registered by 1 photon. Actually, such a photon detection process is equivalent to a process of teleportation and, simultaneously, amplification. Through mathematical calculations, the photonic state in mode B is formulated by

$$|\widetilde{\psi_{\rm B}}\rangle = {}_A\Big\langle 1_{\rm C}\Big|\langle 0\Big|U_{\rm AC}\Big(\frac{1}{2}\Big)\Big|\psi_\alpha\Big\rangle\Big|\psi'_{\rm BC}\Big\rangle \quad (4)$$

$$\approx \sqrt{\frac{t}{2}}(-d_1 + \alpha d_0)\Big(|0\rangle_{\rm B} + \Delta\sqrt{\frac{1-t}{t}}\alpha|1\rangle_{\rm B}\Big) \quad (5)$$

$$= \sqrt{\frac{t}{2}}(d_1 + \alpha d_0)(|0\rangle_{\rm B} + \Delta g\alpha|1\rangle_{\rm B}) \quad (6)$$

$$\approx \sqrt{\frac{t}{2}}(-d_1 + \alpha d_0)(|\Delta g\alpha\rangle_{\rm B}), \quad (7)$$

where $g = \sqrt{(1-t)/t}$ and

$$\Delta = \frac{c_1/\alpha - c_0}{-d_1 + \alpha d_0}. \quad (8)$$

Equation (7) tells us that the input state $|\alpha\rangle$ is now amplified to a stronger coherent state $|\Delta g\alpha\rangle_{\rm B}$. Since g is the gain factor in Ralph and Lund's scheme, by choosing appropriate β and ξ we can study analytically the improvement in the amplification gain. The probability of successful amplification can be identified with the normalization coefficient

$$P_{\rm succ} = \frac{t}{2}(\alpha d_0 - d_1)^2. \quad (9)$$

To check the correctness of our framework, we now study the amplification gain for a more specific unitary operation $U = S(\xi)D(\beta)$, with $S(\xi) = \exp[\frac{\xi}{2}(a^{\dagger 2} - a^2)]$ and $D(\beta) = \exp[\beta(a^\dagger - a)]$ being the single mode squeezed and single mode displacement operator, respectively. Here we assume that both ξ and β are real values for convenience. Specific choice of ξ and β could shed more light on the amplification gain of our scheme. (1) For $\beta = \xi = 0$, one obtains that $U = I$ is the identity matrix. Thus it easily follows

$$c_0 = 1,\ c_1 = 0,\ c_2 = 0,\ d_0 = 0,\ d_1 = 1, \quad (10)$$

and $\Delta = 1$. The amplification is exactly $g = \sqrt{(1-t)/t}$, the same as the one in Ralph and Lund's scheme.[4] (2) For $\beta = 0$ and $\xi \neq 0$, we have $c_0 = 1/\sqrt{\cosh\xi}$, $c_1 = 0$, $d_0 = 0$, $d_1 = (\cosh\xi)^{-\frac{3}{2}}$ and $\Delta = \cosh\xi > 1$ for any nonzero ξ. This is exactly the enhancement effect of local squeezing in Ref. [22]. (3) For $\beta \neq 0$ and $\xi = 0$, after direct calculation it can be obtained that $c_0 = e^{-\beta^2/2}$, $c_1 = e^{-\beta^2/2}\beta$, $c_2 = e^{-\beta^2/2}\beta^2/\sqrt{2}$ and $d_0 = -e^{-\beta^2/2}\beta$, $d_1 = e^{-\beta^2/2}(1-\beta^2)$, $d_2 = e^{-\beta^2/2}(\sqrt{2}\beta - \beta^3/\sqrt{2})$. Then, we obtain $\Delta = \frac{1-\beta/\alpha}{1+\alpha\beta-\beta^2}$. This is the enhancement introduced by a displacement operation only. (4) For $\beta \neq 0, \xi \neq 0$, we have $c_0 = e^{-\beta^2/2}(\lambda_0 - \frac{\beta^2}{\sqrt{2}}\lambda_2)$, $c_1 = \sqrt{2}e^{-\beta^2/2}\frac{\beta\lambda_2}{\sinh\xi}$, $c_2 = e^{-\beta^2/2}(1 + \frac{\beta^2}{\tanh\xi})\lambda_2$ and $d_0 = -\beta c_0 - (\sinh\xi)c_1$, $d_1 = (\cosh\xi)c_0 - \beta c_1 - \sqrt{2}(\sinh\xi)c_2$, $d_2 = \sqrt{2}(\cosh\xi)c_1 - \beta c_2$ and $\lambda_0 = \frac{1}{\sqrt{\cosh\xi}}$, $\lambda_2 = \frac{1}{\sqrt{2\cosh\xi}}\tanh\xi$ and

$$\Delta = [2\cosh\xi - 2(\beta/\alpha) - \beta^2\sinh\xi]/[2(1 - \beta^2 + \alpha\beta\cosh\xi) + \beta(\sinh\xi)(\alpha(2 - \beta^2) - 3\beta\cosh\xi)], \quad (11)$$

Equation (11) shows the enhancement of amplification gain when $U = S(\xi)D(\beta)$ is applied. In Fig. 2, we numerically evaluate the enhancement factor Δ as a function of displacement β as well as the squeezing

factor ξ. We choose the weak coherent state input $|\alpha\rangle$ such that $\alpha = 0.1$, the typical value in Ref. [4]. As is clearly shown, Δ is an increasing and decreasing function of ξ and β, respectively. For $\beta = -0.21$ and $\xi = 0.20$, one has an enhancement factor $\Delta = 3.3985$ and a gain of $g' = g\Delta = 10.19$ for the beam splitter with $t = 0.10$.

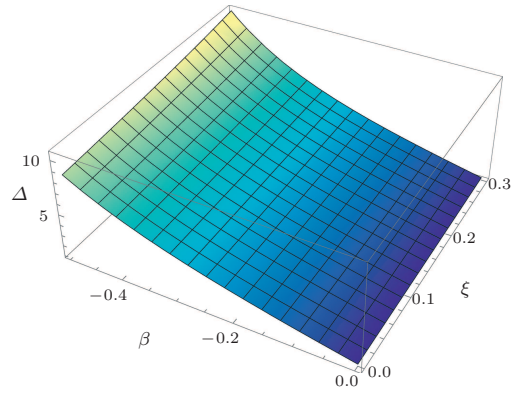

Fig. 2. (Color online) Enhancement factor as functions of displacement β and squeezing ξ. The other parameter is $\alpha = 0.1$.

The amplification process is based on the analysis within 0–1 photon subspace. It is quite important to know whether the enhancement of gain could still be observed if the infinite-dimensional photon number space is considered. We simulate the quantum state in each optic mode with 8-dimensional space which is spanned by the Fock state $\{|0\rangle, |1\rangle, \ldots, |7\rangle\}$. Such a dimension is sufficient since the input state $\alpha = 0.1$ and ξ, β are chosen to be quite small for convenience in physical implementation. In the multi-photon case, the gain g' is now determined by the ratio of single photon proportion and vacuum state proportion,

$$g' = \left| \frac{\langle 1|\widetilde{\psi_{\rm B}}\rangle}{\alpha\langle 0|\widetilde{\psi_{\rm B}}\rangle} \right|. \tag{12}$$

The enhancement factor Δ is now again defined with $\Delta = g'/g$, with $g = \sqrt{(1-t)/t}$ being again the original amplification gain when no local unitary operation is applied. In Fig. 3 we evaluate the performance of amplification gain and success probability as functions of ξ and β. The results in Fig. 3(a) once again prove the enhancement given by local unitary U. We choose $\beta = 0$, -0.1 and -0.2 and the gain g' increases more rapidly with $|\beta|$ than with ξ, which sheds more light on the optimization of amplification gain: introducing coherent displacement is more helpful (powerful) than single mode squeezing. Another important point in Fig. 3 is that the success probability $P_{\rm succ}$ is a monotonous increasing function of the amplitude of $|\beta|$. The reason can be traced back to the fact that the displacement operation could increase the average photon number in mode C and thus enhances the photon detection probability in detector D_2, i.e., the success probability of amplification.

In the above analysis, NLA is applied to the single mode coherent state. Actually, the benefit of NLA is not confined only within the single mode state. For two-mode optic states, e.g., two-mode squeezed states, our NLA scheme can be used to improve the distilled entanglement and the distillation probability, particularly for weak entanglement inputs. A scheme based on single-sided NLA distillation is shown in Fig. 4. Here A and B modes are a pure two-mode squeezed vacuum state $|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{1-\lambda^2}\lambda^n |nn\rangle$. A single-sided NLA amplifier is applied to optical mode A. The success of amplification is now heralded by detecting 0 and 1 photon in optic modes D and A, respectively. Finally, C and B are the final output entanglement. When the input entanglement is weak, we can approximate $|\psi\rangle \approx |0\rangle + \lambda|11\rangle$ and after single sided amplification, we obtain $|\psi\rangle_{\rm CB} \approx |00\rangle + g\Delta\lambda|11\rangle$, which approximates to an amplified two-mode squeezed vacuum state with squeezing parameter $g\Delta > 1$. However, with entanglement distillation with bi-sided photon subtraction, as often performed in Refs. [26–29], the distilled state obeys $|00\rangle + 2\lambda|11\rangle$, which has quite finite entanglement enhancement $\lambda \to 2\lambda$ and possesses quite low probability of success due to the extremely low reflectivity in the beam splitter used for photon subtraction.

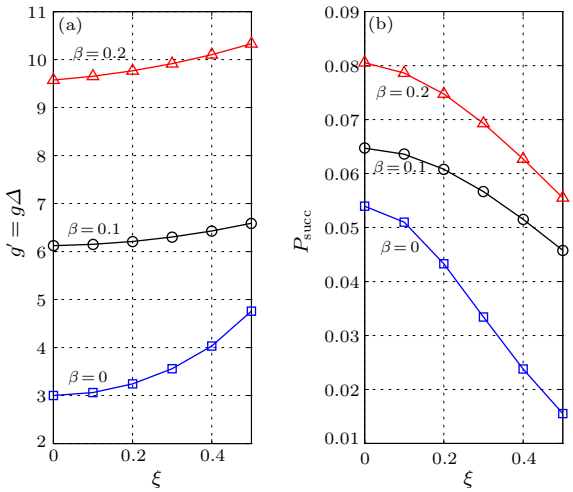

Fig. 3. (Color online) Numerical evaluation of amplification gain (a) and success probability (b) in the multi-photon space. Quantum states in each optic mode A, B and C are truncated within 8-dimensional space. The other parameters are $\alpha = 0.1$ and $t = 0.1$.

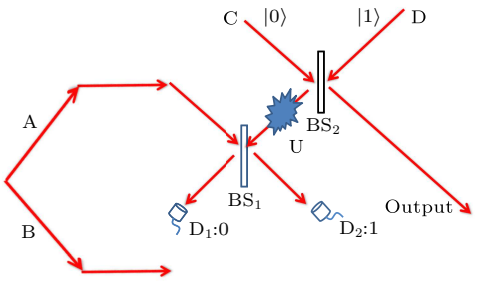

Fig. 4. (Color online) Scheme of distillation of a two-mode squeezed state with the single-sided NLA scheme.

In Fig. 5 we compare the entanglement distillation of a two-mode squeezed vacuum state with the single-

sided NLA scheme and with the bi-sided photon subtraction scheme. For convenience, we choose here $\beta = 0$ and truncate the photon number with the subspace spanned with $\{|0\rangle, |1\rangle, \ldots, |6\rangle\}$, which is large enough for numerical convenience. The entanglement is measured with logarithmic negativity, as in Refs. [26–29]. A great enhancement of entanglement as well as probability of success can be clearly observed when the input entanglement is weakly squeezed ($\lambda < 0.2$).

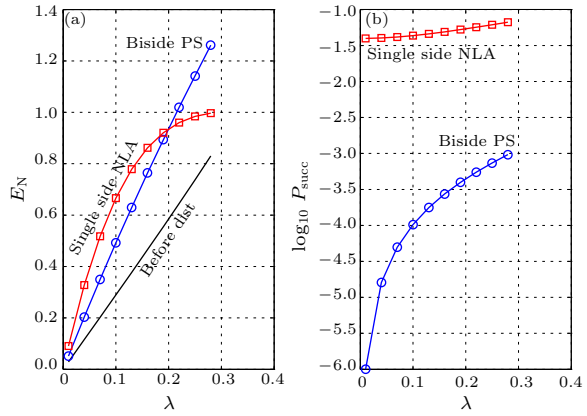

Fig. 5. (Color online) Entanglement (a) and success probability (b) in the single-sided NLA scheme and the bi-sided photon subtraction scheme. For the bi-sided PS, we choose the typical value $T = 0.90$.[30] The other parameters are $\xi = 0.3$ and $t = 0.1$.

Before concluding, a few remarks are made as follows: (1) The enhancement factor Δ in Eq. (11) shows that the use of both only local squeezing and only local displacement operation could enhance the amplification gain of the NLA scheme. However, their enhancement in probability of NLA is different. Local squeezing could improve the amplification gain while the probability of NLA is decreased. However, local squeezing could give simultaneous improvement in both gain and success probability, as clearly shown in Figs. 3(a) and 3(b). Nevertheless, although local displacement has the advantage of both increments in gain and probability of success, it has a drawback that the enhancement brought by local displacement is dependent on the input state $|\alpha\rangle$, which means that its effect may change if it is applied to amplify other states. However, the amplification with squeezing is input-independent. To this point, local squeezing is more versatile. The distinction of performance of local squeezing and local displacement in amplification once again emphasizes the importance of unitary operation in amplification. Our formalism could be beneficial in the optimization of the NLA scheme. (2) In this work, we give a formalism which helps to derive the amplification gain of an arbitrary local unitary assisted NLA scheme. We take $S(\xi)D(\beta)$ as an example and derive its amplification gain. One can also replace $S(\xi)D(\beta)$ with other local unitary operation. For example, if one uses local phase rotation instead $U = e^{i\hat{n}\phi}$, it will follow that $e^{i\hat{n}\phi}|0\rangle = |0\rangle$, $e^{i\hat{n}\phi}|1\rangle = e^{i\phi}|1\rangle$, which means that $c_0 = 1$, $c_1 = 0$, $c_2 = 0$, $d_0 = 0$, $d_1 = e^{i\phi}$ and $d_2 = 0$. There will be no enhancement since $|\Delta| = |e^{i\phi}| = 1$ which gives no improvement to the final gain.

In summary, a general framework has been provided for investigating the enhancement of the NLA scheme with arbitrary local unitary operation. We demonstrate that the noiseless linear amplification scheme, first proposed by Ralph *et al.*, can be improved with a single local squeezing operation and displacement operation. We show the enhancement brought by local unitary within 0–1 photon subspace. Finally, we relax such few-photon restriction and investigate the practical gain in multi-photon space. Numerical simulations show that the local unitary operation-enhanced NLA scheme could provide more possibility for further improving the NLA scheme. Moreover, the success probability is also superior to the squeezing-enhanced NLA scheme. Finally, we discuss the application of the NLA scheme to the distillation of two-mode squeezed vacuum state. We envisage that the local unitary-enhanced NLA scheme method could find more applications in optic quantum information processing.

References

[1] Braunstein S L and van-Loock P 2005 *Rev. Mod. Phys.* **77** 513
[2] Weedbrook C et al 2012 *Rev. Mod. Phys.* **84** 621
[3] Caves C M 1982 *Phys. Rev.* D **26** 1817
[4] Ralph T C and Lund A P 2008 *Quantum Communication Measurement and Computing Proceedings of 9th International Conference* (Calgary, Canada 19–24 August 2008) p 155
[5] Ralph T C 2011 *Phys. Rev.* A **84** 022339
[6] Zhang S L et al 2012 *Phys. Rev.* A **86** 034302
[7] Gisin N et al 2010 *Phys. Rev. Lett.* **105** 070501
[8] Brask J B et al 2012 *Phys. Rev.* A **85** 042116
[9] Fiurášek J and Cerf N J 2012 *Phys. Rev.* A **86** 060302
[10] Blandino R et al 2012 *Phys. Rev.* A **86** 012327
[11] Meng G S et al 2012 *Phys. Rev.* A **86** 042305
[12] Xiang G Y et al 2010 *Nat. Photon.* **4** 316
[13] Usuga M A et al 2010 *Nat. Phys.* **6** 767
[14] Ferreyrol F et al 2010 *Phys. Rev. Lett.* **104** 123603
[15] Ferreyrol F et al 2011 *Phys. Rev.* A **83** 063801
[16] Barbieri M et al 2011 *Laser Phys. Lett.* **8** 411
[17] Zavatta A et al 2011 *Nat. Photon.* **5** 52
[18] Osorio C I et al 2012 *Phys. Rev.* A **86** 023815
[19] Mičuda M et al 2012 *Phys. Rev. Lett.* **109** 180503
[20] Kocsis S et al 2012 *Nat. Phys.* **9** 23
[21] Hu M J and Zhang Y S 2016 arXiv:1601.05582v1 [quant-ph]
[22] Yang S et al 2013 *Phys. Rev.* A **87** 024302
[23] Hong C K and Mandel L 1986 *Phys. Rev. Lett.* **56** 58
[24] Rarity J G et al 1987 *Opt. Commun.* **62** 201
[25] Lvovsky A I et al 2001 *Phys. Rev. Lett.* **87** 050402
[26] García-Patrón R et al 2004 *Phys. Rev. Lett.* **93** 130409
[27] Opatrný T et al 2000 *Phys. Rev.* A **61** 032302
[28] Kitagawa A et al 2006 *Phys. Rev.* A **73** 042310
[29] Zhang S L and van-Loock P 2010 *Phys. Rev.* A **82** 062316
[30] Takahashi H et al 2010 *Nat. Photon.* **4** 178