Traveling Wave Solutions of the Incompressible Ideal Hall Magnetohydrodynamics *

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The solutions of incompressible ideal Hall magnetohydrodynamics are obtained by using the traveling wave method. It is shown that the velocity and magnetic field parallel to the wave vector can be arbitrary constants. The velocity and magnetic field perpendicular to the wave vector are both helical waves. Moreover, the amplitude of the velocity perpendicular to the wave vector is related to the wave number and the circular frequency. In addition, further studies indicate that, no matter whether the uniform ambient magnetic field exists or not, the forms of the traveling wave solutions do not change.

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The Alfvén wave of arbitrary amplitude[1,2] is well known to be the exact solution of the incompressible ideal MHD equations provided that the velocity and magnetic field fluctuations are either parallel or anti-parallel. It has been extensively used to investigate dynamo activity,[3–5] turbulence in various space[6–11] and astrophysical situations. However, the MHD model cannot distinguish the relative motions of different species and cannot capture the small-scale dynamics of plasma. To capture the MHD behavior and some two-fluid effects, a more appropriate theory is to keep the Hall current term in Ohm's law to obtain the generalized Ohm's law (in Gaussian unit): $\boldsymbol{E}+(\boldsymbol{U}\times\boldsymbol{B})/c=(\boldsymbol{J}\times\boldsymbol{B})/enc$, where $\boldsymbol{E}$ is the electric field, $\boldsymbol{U}$ is the velocity, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{J}$ is the electric current density, e is the magnitude of the electron charge, n is the particle density, and c is the speed of light in a vacuum. The Hall term is important when the characteristic length scales are of the order of ion inertial length $\lambda_{\rm i}$ ($\lambda_{\rm i}=c/\omega_{\rm pi}$, where $\omega_{\rm pi}=(4\pi n_{\rm i}e^2/m_{\rm i})^{1/2}$ is the ion plasma frequency with n being the plasma density, and $m_{\rm i}$ being the ion mass), and the characteristic frequencies are comparable to the ion gyrofrequency. In the framework of the Hall MHD, the electron inertia is eliminated from the electron momentum equation and the magnetic field is frozen to the electron fluid but not to the ion fluid. So far, the ideal and dissipative Hall MHD model has been investigated by many researchers.[12–16] In 2005, Mahajan *et al.*[17] studied the exact solutions of the incompressible ideal Hall MHD equations. They showed that the exact solutions are helical waves and can make the nonlinear terms disappear from the equations. They also compared the linear and the nonlinear modes of the incompressible ideal Hall MHD and commented on their probable role in describing turbulent fluctuations in the solar wind.[18,19] In 2008, Servdio *et al.*[20] investigated the statistical properties of the incompressible ideal Hall MHD model. They assumed that the fluctuating velocity and magnetic field have the periodical solutions and can be expanded by Fourier series. Then, they obtained the spectra of three quadratic invariants (energy, generalized helicity and magnetic helicity) using the classical Gibbs ensemble method and discussed the cascade properties of the Hall MHD turbulence. Chiral symmetry breaking of Hall MHD turbulence was investigated by Meyrand *et al.*[21] They introduced the normalized magnetic polarization to separate the Hall MHD into the electron MHD regime (small scales) and the ion MHD regime (large scales). The ion MHD regime displays left-handed fluctuations with $k^{-11/3}$ spectrum and the electron MHD regime displays right-handed fluctuations and has a classical $k^{-7/3}$ spectrum. However, the normalized magnetic polarization they defined is based on the linear solutions of the incompressible ideal Hall MHD equations. We know that the traveling wave method is a basic and effective method for solving nonlinear equations. In 1993, Li *et al.*[22] obtained the traveling wave solutions of the two-dimensional KdV–Burgers equation. They proved that the solutions can be expressed as a sum of shock wave solutions of a 2D Burgers equation and solitary wave solutions of a 2D KdV equation. In 1997, Parkes *et al.* obtained the explicit traveling solitary wave solutions of a compound KdV–Burgers equation.[23] The traveling wave solutions to the KdV equation has been discussed widely. However, the traveling wave method is rarely used to investigate the MHD equations. In this work, we try to find out the traveling wave solutions to the incompressible ideal Hall MHD equations.

The equations of the incompressible, ideal Hall MHD in the usual dimensionless variables are given by

$$\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{U}\cdot\nabla\boldsymbol{U}=-\nabla p+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{U}\times\boldsymbol{B})-\varepsilon\nabla\times[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}], \tag{2}$$

$$\nabla\cdot\boldsymbol{U}=0, \tag{3}$$

$$\nabla\cdot\boldsymbol{B}=0, \tag{4}$$

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where $\boldsymbol{U}$ and $\boldsymbol{B}$ represent the normalized velocity and magnetic field, respectively. The normalized parameters are the Alfvén speed $V_{\rm A} = B_0/\sqrt{4\pi\rho}$ and B_0 (where B_0 is the uniform ambient magnetic field, and ρ is the uniform mass density). The Coulomb gauge is used for the magnetic vector potential, and p is the pressure. The parameter ε is the ratio between the ion skin depth $\lambda_{\rm i}$ and the unit characteristic length scale L_0. The time unit is the Alfvén time at L_0. Assuming that there is no mean flow and the magnetic field can be split into the ambient and fluctuating parts,

$$\boldsymbol{B} = \boldsymbol{b} + \hat{\boldsymbol{e}}_B, \quad \boldsymbol{U} = \boldsymbol{u}, \tag{5}$$

where $\boldsymbol{u}$ and $\boldsymbol{b}$ are the fluctuating velocity and magnetic field, respectively, $\hat{\boldsymbol{e}}_B$ is the uniform ambient magnetic field. If we assume that $\hat{\boldsymbol{e}}_B$ lies in the (y, z) plane of the three-dimensional Cartesian coordinates and forms an angle θ with respect to $\hat{\boldsymbol{e}}_z$, then $\hat{\boldsymbol{e}}_B$ can be written as $\cos\theta\hat{\boldsymbol{e}}_z + \sin\theta\hat{\boldsymbol{e}}_y$, where $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{e}}_z$ are three orthogonal unit vectors of the three-dimensional Cartesian coordinates.

To find the traveling wave solutions of the incompressible ideal Hall MHD equations, the fluctuating velocity and magnetic field are assumed to have the following forms

$$\boldsymbol{u} = \boldsymbol{u}(\xi), \ \boldsymbol{b} = \boldsymbol{b}(\xi), \quad \xi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t, \tag{6}$$

where ξ is a new independent variable, $\boldsymbol{k}$ is the wave vector, and ω is the circular frequency. Then we obtain the following relationships

$$\frac{\partial}{\partial t} = -\omega\frac{d}{d\xi}, \quad \nabla = \boldsymbol{k}\frac{d}{d\xi}. \tag{7}$$

For simplicity, the cold plasma approximation is considered, that is, the ion and electron temperature are both equal to zero. Therefore, we can see that the gradient of pressure disappears immediately from Eq. (1). After that, we substitute Eq. (7) into Eqs. (1)–(4) and obtain the ordinary differential equations about the new independent variable ξ,

$$(\boldsymbol{k}\cdot\boldsymbol{u} - \omega)\boldsymbol{u}' = \boldsymbol{b}'(\boldsymbol{k}\cdot\boldsymbol{B}) - \boldsymbol{k}(\boldsymbol{b}'\cdot\boldsymbol{B}), \tag{8}$$

$$-\omega\boldsymbol{b}' = \boldsymbol{k}\times[(\boldsymbol{u}' - \varepsilon\boldsymbol{k}\times\boldsymbol{b}'')\times\boldsymbol{B} + (\boldsymbol{u} - \varepsilon\boldsymbol{k}\times\boldsymbol{b}')\times\boldsymbol{b}'], \tag{9}$$

$$\boldsymbol{k}\cdot\boldsymbol{u}' = 0, \tag{10}$$

$$\boldsymbol{k}\cdot\boldsymbol{b}' = 0. \tag{11}$$

If the wave vector $\boldsymbol{k}$ is in the direction of $\hat{\boldsymbol{e}}_z$, u_z and b_z will be both constants due to Eqs. (10) and (11). Here u_z and b_z are the z-axis components of the fluctuating velocity and the magnetic field, respectively. Furthermore, Eq. (9) can be simplified by Eqs. (10) and (11),

$$(\boldsymbol{k}\cdot\boldsymbol{u} - \omega)\boldsymbol{b}' = (\boldsymbol{u}' - \varepsilon\boldsymbol{k}\times\boldsymbol{b}'')(\boldsymbol{k}\cdot\boldsymbol{B}). \tag{12}$$

Then, Eqs. (8) and (12) can be expanded as

$$b_x b'_x + b_y b'_y + b'_y\sin\theta = 0, \tag{13}$$

$$(ku_z - \omega)u'_x = b'_x k(b_z + \cos\theta), \tag{14}$$

$$(ku_z - \omega)u'_y = b'_y k(b_z + \cos\theta), \tag{15}$$

$$(ku_z - \omega)b'_x = k(b_z + \cos\theta)(u'_x + \varepsilon k b''_y), \tag{16}$$

$$(ku_z - \omega)b'_y = k(b_z + \cos\theta)(u'_y - \varepsilon k b''_x), \tag{17}$$

where u_x and b_x are the x-axis components of fluctuating velocity and magnetic field, respectively, u_y and b_y are the y-axis components of fluctuating velocity and magnetic field, respectively. From Eqs. (14)–(17), we can see that if $ku_z - \omega = 0$, we can obtain that b_x, b_y, u_x and u_y are constants. That is, when the z-axis components of velocity of magnetic fluid and the wave velocity are equal, the wave relative to the fluid does not spread. Thus we can only consider that $ku_z - \omega \neq 0$. From Eqs. (14) and (15), the expressions of u'_x and u'_y can be obtained,

$$u'_x = \frac{k(b_z + \cos\theta)}{ku_z - \omega}b'_x, \tag{18}$$

$$u'_y = \frac{k(b_z + \cos\theta)}{ku_z - \omega}b'_y. \tag{19}$$

Substituting Eqs. (18) and (19) into Eqs. (16) and (17), respectively, we obtain

$$[(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]b'_x = (b_z + \cos\theta)(ku_z - \omega)\varepsilon k^2 b''_y, \tag{20}$$

$$[(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]b'_y = -(b_z + \cos\theta)(ku_z - \omega)\varepsilon k^2 b''_x. \tag{21}$$

Integrating Eqs. (20) and (21), we have

$$[(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]b_x = (b_z + \cos\theta)(ku_z - \omega)\varepsilon k^2 b'_y + c_1, \tag{22}$$

$$[(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]b_y = -(b_z + \cos\theta)(ku_z - \omega)\varepsilon k^2 b'_x + c_2, \tag{23}$$

where c_1 and c_2 are integral constants. Then, substituting Eq. (22) into Eq. (21) we obtain the second order linear constant coefficient differential equation about b_x,

$$[(b_z + \cos\theta)(ku_z - \omega)\varepsilon k^2]^2 b''_x + [(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]^2 b_x = [(ku_z - \omega)^2 - k^2(b_z + \cos\theta)^2]c_1. \tag{24}$$

Thus the x-axis component of the fluctuating magnetic field can be written as

$$b_x = b_0\cos(\Omega\xi - \phi) + A_x. \tag{25}$$

Similarly, the y-axis component of the fluctuating magnetic field can be obtained from Eqs. (20) and (23),

$$b_y = b_0\sin(\Omega\xi - \phi) + A_y, \tag{26}$$

where b_0 is the arbitrary amplitude of the fluctuating magnetic field, ϕ is the initial phase, A_x and A_y are both the integral constants, and Ω has the following form

$$\Omega = \frac{(ku_z-\omega)^2 - k^2(b_z+\cos\theta)^2}{(b_z+\cos\theta)(ku_z-\omega)\varepsilon k^2}. \tag{27}$$

If Eqs. (25) and (26) are substituted into Eq. (13), the integral constants A_x and A_y are obtained,

$$A_x = 0, \quad A_y = -\sin\theta. \tag{28}$$

Therefore, the x- and y-axis components of the fluctuating magnetic field $\boldsymbol{b}$ can be rewritten as

$$b_x = b_0\cos(\Omega\xi-\phi), \tag{29}$$
$$b_y = b_0\sin(\Omega\xi-\phi) - \sin\theta. \tag{30}$$

Then substituting Eqs. (29) and (30) into Eqs. (14) and (15), respectively, we can obtain the x- and y-axis components of the fluctuating velocity $\boldsymbol{u}$,

$$u_x = \frac{k(b_z+\cos\theta)}{ku_z-\omega}[b_0\cos(\Omega\xi-\phi)] + u_{0x}, \tag{31}$$
$$u_y = \frac{k(b_z+\cos\theta)}{ku_z-\omega}[b_0\sin(\Omega\xi-\phi) - \sin\theta] + u_{0y}, \tag{32}$$

where u_{0x} and u_{0y} are the integral constants. Thus the traveling wave solutions to the incompressible ideal Hall MHD equations has the following forms

$$\begin{aligned}\boldsymbol{B} =& b_0\cos(\Omega\xi-\phi)\hat{\boldsymbol{e}}_x + b_0\sin(\Omega\xi-\phi)\hat{\boldsymbol{e}}_y \\ &+ (b_z+\cos\theta)\hat{\boldsymbol{e}}_z, \end{aligned}\tag{33}$$

$$\begin{aligned}\boldsymbol{U} =& \left\{\frac{k(b_z+\cos\theta)}{ku_z-\omega}[b_0\cos(\Omega\xi-\phi)] + u_{0x}\right\}\hat{\boldsymbol{e}}_x \\ &+ \left\{\frac{k(b_z+\cos\theta)}{ku_z-\omega}[b_0\sin(\Omega\xi-\phi)]\right. \\ &\left. - \frac{k(b_z+\cos\theta)}{ku_z-\omega}\sin\theta + u_{0y}\right\}\hat{\boldsymbol{e}}_y \\ &+ u_z\hat{\boldsymbol{e}}_z. \end{aligned}\tag{34}$$

Furthermore, if the uniform ambient magnetic field does not exist, the traveling solutions of the incompressible ideal Hall MHD equations can also be obtained,

$$\boldsymbol{B} = b_0\cos(\Omega\xi-\phi)\hat{\boldsymbol{e}}_x + b_0\sin(\Omega\xi-\phi)\hat{\boldsymbol{e}}_y + \tilde{b}_z\hat{\boldsymbol{e}}_z, \tag{35}$$

$$\begin{aligned}\boldsymbol{U} =& \left\{\frac{k\tilde{b}_z}{ku_z-\omega}[b_0\cos(\Omega\xi-\phi)] + u_{0x}\right\}\hat{\boldsymbol{e}}_x \\ &+ \left\{\frac{k\tilde{b}_z}{ku_z-\omega}[b_0\sin(\Omega\xi-\phi)] + \tilde{u}_{0y}\right\}\hat{\boldsymbol{e}}_y \\ &+ u_z\hat{\boldsymbol{e}}_z. \end{aligned}\tag{36}$$

Due to the fact that the z-axis component of the fluctuating magnetic field is an arbitrary constant, we can let $\tilde{b}_z$ equal to $b_z+\cos\theta$ in Eq. (33). Due to the fact that $k(b_z+\cos\theta)\sin\theta/(ku_z-\omega)$ is constant, $\tilde{u}_{0y}$ can equal to $u_{0y}-[k(b_z+\cos\theta)\sin\theta/(ku_z-\omega)]$ in Eq. (34). Comparing Eqs. (33) and (34) with Eqs. (35) and (36), we can conclude that the uniform ambient magnetic field does not affect the properties of the traveling wave solutions due to the diamagnetism of the fluctuating magnetic field along the y-axis.

In summary, using the traveling wave method, we have obtained the solutions to the incompressible ideal Hall MHD equations. The fluctuating velocity and magnetic field components along the direction of the wave vector are arbitrary constants. The perpendicular components of the velocity and magnetic field are both helical waves. Moreover, the amplitude of the velocity perpendicular to the wave vector is related to the wave number and the circular frequency. If the uniform ambient field exists, the fluctuating magnetic field can cancel out the y-axis component of the uniform ambient field. Therefore, we can conclude that, no matter whether the uniform ambient field exists or not, the traveling wave solutions to the Hall MHD equations do not change their forms. It should be noted that the importance of knowing an exact nonlinear solution can never be overestimated. The traveling wave solutions of the incompressible ideal Hall MHD can provide a basis for a variety of investigations related to plasma instability and turbulence, in particular, for the enquiries that probe into the small-scale domain of turbulence spectra.

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