CHIN. PHYS. LETT. Vol. 33, No. 5 (2016) 059501

Dynamical Study of a Constant Viscous Dark Energy Model in Classical and Loop Quantum Cosmology *

Sara Benchikh**, Noureddine Mebarki, Dalel Aberkane

Laboratoire de Physique Mathematique et Subatomique, Mentouri University, Route Ain El Bey, Constantine 25000, Algeria

(Received 2 December 2015)

Dynamical behaviors and stability properties of a flat space Friedmann–Robertson–Walker universe filled with pressureless dark matter and viscous dark energy are studied in the context of standard classical and loop quantum cosmology. Assuming that the dark energy has a constant bulk viscosity, it is found that the bulk viscosity effects influence only the quintessence model case leading to the existence of a viscous late time attractor solution of de-Sitter type, whereas the quantum geometry effects influence the phantom model case where the big rip singularity is removed. Moreover, our results of the Hubble parameter as a function of the redshift are in good agreement with the more recent data.

PACS: 95.85.−e, 04.60.Pp, 95.36.+x DOI: 10.1088/0256-307X/33/5/059501

One of the greatest challenges in cosmology is the attempt to understand the nature and the origin of the present accelerated expansion of our universe, which has been confirmed by several recent observations, such as type Ia supernovae (SNe Ia),[1,2] cosmic microwave background (CMB) anisotropy[3−5] and large scale structure (LSS).[6,7] One alternative and possible explanation of these observations is the assumption of the existence of a new unknown undetected component with a negative pressure called dark energy (DE). It could possess an equation of state (EoS) in the form $p = \omega\rho$ with $\omega < -1/3$. The simplest DE candidate is the cosmological constant with $\omega = -1$, though favored by a number of observations[8] it still suffers from the fine-tuning and the coincidence problems.[9,10] To overcome these two problems, a variety of scalar field models have been introduced. One of them is known as the quintessence with $-1 < \omega < -1/3$,[11−13] described by an ordinary scalar field where its potential is dominant with respect to its kinetic energy. However, the recent observations allow the possibility of $\omega < -1$.[8] Thus a new class of scalar field models with negative kinetic energy, called the phantom field,[14] has been suggested to satisfy $\omega < -1$. The problem with this model in the standard Friedmann–Robertson–Walker (FRW) cosmology is that it leads to a future singularity, i.e., the so-called big rip.[15,16] Several ideas have been proposed to avoid this singularity, such as choosing certain potentials and initial conditions for which the de-Sitter universe solution turns out to be a late time attractor of the model[17] or by introducing terms of quantum effects in the action.[18] The astrophysical observations also indicate that the universe medium is not a perfect fluid[19] and its viscosity can be involved in the evolution and dynamics of the universe. In isotropic and homogeneous universes, the shear viscosity is ignored, only bulk viscosity could play a role in realistic models and its effects on the cosmological evolution have been studied from various viewpoints.[20−26] Moreover, it is commonly believed that quantum gravity effects would play an important role in the evolution of the universe. Thus it is preferred to study the properties of cosmological models in the framework of a quantum gravity theory. In this study, we work with a loop quantum cosmology (LQC) effective model,[27−29] which is a canonical quantization of homogeneous space times based upon a reduced symmetry model of loop quantum gravity.[30−32] In fact, the effects of LQC consist of modifying the standard Friedmann equations by adding a correction term of the form $-\rho^2/\rho_c$ encoding the discrete quantum geometric nature of the space time.[33] When this correction term becomes dominant, the universe begins to bounce and then expands backwards. One of the major successes in LQC is that the big bang singularity can be replaced by a large bounce[27−29] and the future singularities can be avoided. It is important to mention that all strong singularities (big rip) are absent in LQC for isotropic and homogenous universes (like FRW) where the Ricci scalar and similarly other curvature invariants are bounded for all the events where the energy density diverges.[34,35] As shown in Ref. [36], where a dynamical study of a phantom scalar field dark energy interacting with dark matter was carried out in LQC, the big rip singularity can be avoided and the role of the loop quantum gravity effect was to break the stability of the initially phantom field climbing up the potential and leave that of the rolling down the potential. Furthermore, weak

*Supported by the Algerian Ministry of Education and Research, and DGRSDT.

**Corresponding author. Email: benchikhsara@gmail.com

© 2016 Chinese Physical Society and IOP Publishing Ltd

sudden singularities may exist in LQC for potentials that are not bounded from below (above) for a canonical phantom scalar field. In this Letter, we use a dynamical study by combining LQC corrections and bulk viscous DE in one model to investigate whether there are some interesting features arising from the two effects taken together to a better understanding of the evolution and the fate of our universe, when we work with quintessence and phantom DE models separately. Finally, we compare the Hubble parameter derived from the viscous quintessence and phantom scenarios of our model as a function of the redshift with the most recent data.

Now we investigate the evolution of a flat space FRW universe model filled with pressureless DM and viscous DE in the framework of SC. We study the simple case, where DE evolves independently of DM (without interactions), the quintessence and phantom models are examined separately. In SC, the first Friedman equation is given by

$$H^2 = \frac{\rho}{3}, \tag{1}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter (a is the scale parameter), and ρ is the total energy density $\rho = \rho_{\rm DM} + \rho_{\rm DE}$. Here the dot stands for the derivative with respect to the cosmic time t. In what follows, we take the natural unit $\kappa = 8\pi G = 1$ (G is the Newton constant) and DE is assumed to obey the usual EoS: $P = \omega\rho$ with a constant value of ω in the intervals $-1 < \omega < -\frac{1}{3}$ and $\omega < -1$ corresponding to the quintessence and phantom models, respectively. In the presence of a bulk viscosity, the effective pressure of the viscous DE is defined by[37,38]

$$P_{\rm eff} = P_{\rm DE} + \Pi, \tag{2}$$

where $P_{\rm DE} = \omega\rho_{\rm DE}$ is the pressure of the DE, and $\Pi = -3\zeta H$ is the viscous pressure. The term ζ is a bulk viscous coefficient. We require that $\zeta > 0$, as a consequence of the positive entropy variation in an irreversible process.[39] In what follows, we assume that ζ is constant. Now the conservation equations for DM and DE can be expressed as

$$\dot{\rho}_{\rm DM} + 3H\rho_{\rm DM} = 0, \tag{3}$$

and

$$\dot{\rho}_{\rm DE} + 3H((1+\omega)\rho_{\rm DE} - 3\zeta H) = 0. \tag{4}$$

Differentiating Eq. (1) and employing Eqs. (3) and (4) we obtain the modified Raychaudhuri equation

$$\dot{H} = -\frac{1}{2}(\rho_{\rm DM} + (1+\omega)\rho_{\rm DE} - 3\zeta H). \tag{5}$$

Using Eqs. (1) and (5), we can define an EoS effective coefficient as

$$\omega_{\rm eff} = \frac{P_{\rm eff}}{\rho} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{6}$$

To analyze the dynamical behavior of the above system, we introduce the following dimensionless variables

$$x = \frac{\rho_{\rm DE}}{3H^2}, \quad y = \frac{1}{\frac{\zeta}{H} + 1}. \tag{7}$$

Obviously the phase space is the bounded plane region where $0 \le x \le 1$ and $0 \le y \le 1$, in terms of these variables, Eqs. (1) and (5) can be rewritten as

$$\frac{\rho_{\rm DM}}{3H^2} + x = 1, \tag{8}$$

and

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\Big(2 + \omega x - \frac{1}{y}\Big). \tag{9}$$

The deceleration parameter $q = -1 - \frac{\dot{H}}{H^2}$ and $\omega_{\rm eff}$ become

$$q = 2 + \frac{3}{2}\Big(\omega x - \frac{1}{y}\Big), \tag{10}$$

and

$$\omega_{\rm eff} = 1 + \omega x - \frac{1}{y}. \tag{11}$$

Using Eqs. (3), (4), (8) and (9) we obtain the following autonomous system of ordinary differential equations (ASODE)

$$x' = 3(x-1)\Big(\omega x - \frac{1}{y} + 1\Big), \tag{12}$$

and

$$y' = \frac{3}{2}y(y-1)\Big(2 + \omega x - \frac{1}{y}\Big). \tag{13}$$

The prime denotes a derivative with respect to the e-folding number $N = \ln a$, representing the cosmological time, in our case (the physical regions $0 \le x \le 1$ and $0 \le y \le 1$), three equilibrium critical points are found $P_1(0,1)$, $P_2(1,1)$ and $P_3\Big(1, \frac{1}{(2+\omega)}\Big)$. Their properties are determined by the sign and nature of the eigenvalues $\mu_i (i = 1, 2)$ of the Jacobi matrix J,

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_c, y_c)}.$$

where $f \equiv x'$ and $g \equiv y'$. Now if the real parts of all the $\mu_i (i = 1, 2)$ at a certain critical point are negative, it is a stable node. Otherwise, it is an unstable node. On the other hand, if at least one of the eigenvalues is zero and the others are positive or with opposite signs, the critical point is unstable. However, if the other eigenvalues are negative the linearized theory of the dynamical system may not describe correctly the nonlinear system. In this case, we must use the center manifold theorem (CMT).[40,42] The properties of the obtained critical points are summarized in Tables 1 and 2. It is important to notice: (1) At the critical point P_1, the deceleration and EoS effective parameters are $q = \frac{1}{2}$ and $\omega_{\rm eff} = 0$, respectively. Thus it

corresponds to a dark matter-dominated solution and a decelerated expansion. The related eigenvalues of the linearized stability matrix (LSME) are $\mu_1 = \frac{3}{2}$ and $\mu_2 = -3\omega$. Notice that both the eigenvalues have positive signs whether we are in the quintessence or phantom case. Therefore P_1 is unstable in concordance with the observation where this phase of a conventional dark matter-dominance is required for the formation of the observed amount of the cosmic structure. (2) The critical point P_2 corresponds either to a non-viscous dark energy-dominated solution if $\zeta = 0$, or to the viscous dark energy-dominated solution if $H \to \infty$. The deceleration and EoS effective parameters in this case are given by $q = \frac{1}{2}(1 + 3\omega)$ and $\omega_{\rm eff} = \omega$. This solution corresponds to an accelerated expansion. The LSMEs are $\mu_1 = \frac{3}{2}(1 + \omega)$ and $\mu_2 = 3\omega$. Notice that for a quintessence dark energy, this critical point has eigenvalues with opposite signs and it is then an unstable saddle point. However, for a phantom dark energy, it has negative eigenvalues. Then P_2 is a stable point corresponding to a big rip singularity attractor. (3) The critical point P_3 is not a physical solution in the case of the phantom model ($0 \leq y \leq 1$). It exists just for the quintessence dark energy (physical condition), where it is characterized by $q = -1$ and $\omega_{\rm eff} = -1$. That is due to the presence of the viscosity, the EoS effective parameters tend to -1. The eigenvalues of the Jacobian matrix at the critical point are $\mu_1 = -3$ and $\mu_2 = -\frac{3}{2}(1+\omega)$. Therefore, it is a stable solution corresponding to a viscous late time attractor of the de-Sitter type.

Table 1. The eigenvalues of the linearization matrix around given critical points P_i for a viscous SC.

Fixed-points	$x_{\rm c}$	$y_{\rm c}$	Eigenvalues
P_1	0	1	$\mu_1 = \frac{3}{2}$, $\mu_2 = -3\omega$
P_2	1	1	$\mu_1 = \frac{3}{2}(1+\omega)$, $\mu_2 = 3\omega$
P_3	1	$\frac{1}{2+\omega}$	$\mu_1 = -3$, $\mu_2 = -\frac{3}{2}(1+\omega)$

Table 2. Stability and relevant parameters of the critical points P_i for a viscous SC.

Fixed-points	q	$\omega_{\rm eff}$	Stability
P_1	$\frac{1}{2}$	0	Unstable
P_2	$\frac{1}{2}(1+3\omega)$	ω	Saddle if $\omega > -1$, stable if $\omega < -1$
P_3	-1	-1	Existed just for $\omega > -1$ (stable)

We extend the previous study by including the LQC effects. In this case, the modified flat space Friedmann equation can be written as[33]

$$H^2 = \frac{\rho}{3}\Big(1 - \frac{\rho}{\rho_{\rm c}}\Big). \tag{14}$$

where $\rho_{\rm c} \equiv \frac{\sqrt{3}}{16\pi^2\gamma^3 G^2 h}$ is the critical density in LQC, and γ is the dimensionless Barbero–Immirzi parameter (it is suggested that $\gamma = 0.2375$ by the black hole thermodynamics in LQC), we remind that the quantum correction is negligible when $\rho \ll \rho_{\rm c} \sim \rho_{\rm Pl}$ ($\rho_{\rm Pl}$ stands for the density at the Planck scale) and it dominates dynamics when $\rho \sim \rho_{\rm c}$. In particular, when $\rho = \rho_{\rm c}$ the right side of Eq. (14) vanishes yielding to $\dot{a} = 0$ and leading to a quantum bounce. Differentiating Eq. (14) and using Eqs. (3) and (4) we obtain the following modified Raychaudhuri equation

$$\dot{H} = -\frac{1}{2}(\rho_{\rm DM} + (1+\omega)\rho_{\rm DE} - 3\zeta H)\Big(1 - 2\frac{\rho}{\rho_{\rm c}}\Big). \tag{15}$$

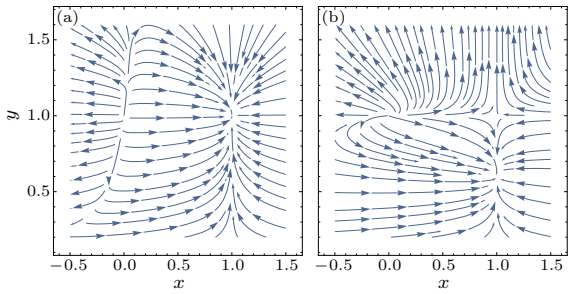

Fig. 1. Phase-plane diagrams for the SC model in the phantom case (a) with $\omega = -4/3$ and in the quintessence case (b) with $\omega = -7/20$.

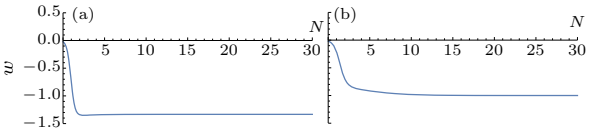

Fig. 2. The evolution of $\omega_{\rm eff}$ with the cosmological time N in the SC: (a) the phantom model with $\omega = -4/3$ and (b) the quintessence model with $\omega = -3/4$. We notice that, regardless of the initial conditions for both the cases, the universe will enter a final state dominated by a dark energy.

Using Eqs. (14) and (15) the effective EoS for the total cosmic energy in LQC can be expressed as

$$\omega_{\rm eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}\Big(\frac{\rho_{\rm c} - \rho}{\rho_{\rm c} - 2\rho}\Big). \tag{16}$$

To analyze the dynamical behavior of the universe, one has to add a new variable to the already existing variables x and y: $z = \frac{\rho}{\rho_{\rm c}}$. In this case, the effective modified Friedman Eqs. (14) and (15) can be rewritten as

$$\Big(\frac{\rho_{\rm DM}}{3H^2} + x\Big)(1 - z) = 1, \tag{17}$$

and

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\Big(\frac{1}{1-z} + \omega x - \frac{1}{y} + 1\Big)(1 - 2z). \tag{18}$$

Similarly for the deceleration and EoS parameters, one has

$$q = -1 + \frac{3}{2}\Big(\frac{1}{1-z} + \omega x - \frac{1}{y} + 1\Big)(1 - 2z), \tag{19}$$

and

$$\omega_{\rm eff} = \Big(\omega x - \frac{1}{y} + 1\Big)(1 - z). \tag{20}$$

Using Eqs. (3), (4), (17) and (18) we obtain the following ASODEs

$$x' = 3((1-2z)x-1)\Big(\omega x - \frac{1}{y}+1\Big) - \frac{3xz}{1-z}, \tag{21}$$

$$y' = \frac{3}{2}y(y-1)\Big(\frac{1}{1-z}+\omega x - \frac{1}{y}+1\Big)(1-2z), \tag{22}$$

$$z' = -3z - 3z(1-z)\Big(\omega x - \frac{1}{y}+1\Big). \tag{23}$$

Now for the physical solutions with $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, we obtain the following critical points $P_1(0,1,0)$, $P_2(1,1,0)$ and $P_3(1,1/(2+\omega),0)$. Their stability properties are determined by the eigenvalues of the Jacobi matrix M at these points (x_c, y_c, z_c),

$$M = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}_{(x_c,y_c,z_c)},$$

where $h \equiv z'$.

Table 3. The eigenvalues of the linearization matrix around a given critical points P_i for a viscous LQC.

Fixed-points	x_c	y_c	z_c	Eigenvalues
P_1	0	1	0	$\mu_1 = \frac{3}{2}$, $\mu_2 = -3\omega$, $\mu_3 = -3$
P_2	1	1	0	$\mu_1 = \frac{3}{2}(1+\omega)$, $\mu_2 = 3\omega$, $\mu_3 = -3(1+\omega)$
P_3	1	$\frac{1}{2+\omega}$	0	$\mu_1 = -3$, $\mu_2 = -\frac{3}{2}(1+\omega)$, $\mu_3 = 0$

Table 4. Stability and relevant parameters of the critical points for a viscous LQC.

Fixed-points	q	$\omega_{\rm eff}$	Stability
P_1	$\frac{1}{2}$	0	Saddle
P_2	$\frac{1}{2}(1+3\omega)$	ω	Saddle
P_3	-1	-1	Existed just for $\omega > -1$ (Stable)

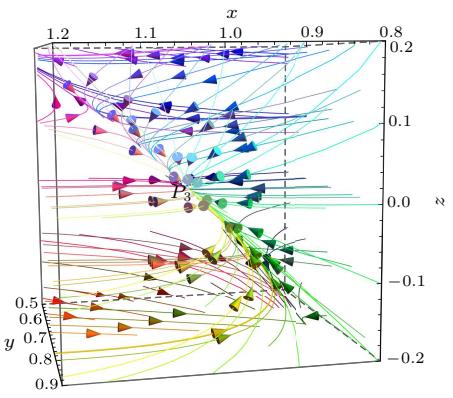

Fig. 3. The 3D phase portrait showing the stable critical point P_3 for the LQC model in the quintessence case with $\omega = -3/4$.

The properties of the resulted equilibrium points are summarized in Tables 3 and 4. We find the same critical points as in the classical case (in three-dimensional space), while their stability properties are almost different. In fact, the critical point P_1 (the dark matter-dominated solution) becomes a saddle (which was an unstable point in SC). This is essentially due to the different signs of the Jacobian matrix eigenvalues $\mu_1 = \frac{3}{2}$, $\mu_2 = -3\omega$ and $\mu_3 = -3$. Likewise the classical case, this point is still unstable for both phantom and quintessence dark energy with a decelerated expansion. For the critical point P_2 (no viscous dark energy-dominated solution) where the LSMEs are $\mu_1 = \frac{3}{2}(1+\omega)$, $\mu_2 = 3\omega$ and $\mu_3 = -3(1+\omega)$, the stability for the phantom model ($\omega < -1$) is different from the classical case. In fact, this point is converted from a stable point to a saddle point due to the fact that it possesses eigenvalues with opposite signs. Hence, the quantum correction effect breaks the stability of the point P_2. Therefore, the problem of the future singularity (big rip) does not occur in this scenario. For the quintessence case this point remains a saddle. For both the cases, we have an accelerated expansion in the vicinity of this point. The critical point P_3, which is of the de Sitter type (viscous dark energy-dominated solution), still exists just for the quintessence model, the corresponding LSMEs are $\mu_1 = -3$, $\mu_2 = -\frac{3}{2}(1+\omega)$ and $\mu_3 = 0$. Since the Jacobian matrix M has one of the eigenvalues zero and the others are negative, we cannot guarantee its stability properties directly from the eigenvalues, instead we use the center manifold theorem which allows us to simplify the dynamical system by reducing its dimensionality.[40] We find that this point is stable (see the supplemental material).

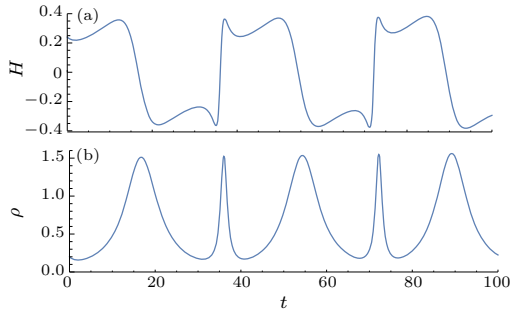

Fig. 4. (a) Time evolution of the Hubble parameter and (b) time evolution of the total energy density for the LQC model in the phantom case with $\omega = -1.2$ and $\rho_c = 1.5$. Notice that at the beginning both H and ρ increase with time when the universe undergoes acceleration until reaching a turning point at $\rho = \rho_c/2$ and $H = H_{\max}$. Beyond this point, the universe will expand but with a deceleration until a stopping phase. When ρ reaches its maximum value $\rho_{\max} \sim \rho_c$ and $H = 0$, the universe will undergo contraction until a bounce. After that the evolution of the universe mimics the previous one. Therefore the universe will oscillate forever without reaching any singularities.

In summary, in the context of both classical SC and LQC a dynamical system of a pressureless DM and viscous phantom or quintessence DE, where the viscosity coefficient is taken to be a constant, is studied. We find that in SC for the phantom dark en-

ergy ($\omega < -1$) case, there are two critical points: one is the unstable dark matter dominated solution, and the other corresponds to a big rip singularity solution whereas for the quintessence ($-1 < \omega < -1/3$) case we have obtained three critical points, two of them are unstable and the other one is stable corresponding to the de-Sitter viscous dark energy. That is, the bulk viscosity has an effect only on the quintessence model, leading to the existence of an interesting stable solution which is a viscous late time attractor of the de-Sitter type. However, by including the loop quantum geometry effects, we obtain the same critical points as in the classical Einstein cosmology (except at 3D phase space) while their stability properties are almost different. The dark matter-dominated solution for both the phantom and quintessence models is unstable in the SC case but it becomes saddle in LQC. In fact, for the phantom dominated universe solution, the quantum geometry effects break its stability and as a consequence, a future singularity (big rip) disappears. However, for the quintessence case, the stability of the nodes that corresponds to a quintessence dominated universe in LQC are the same as that in SC. Thus if we take into account the bulk viscosity and quantum geometry effects in the evolution of our universe, we have noticed that for the quintessence model our universe will enter an era dominated by a viscous dark energy of the de-Sitter type, however, for the phantom model the universe will enter an oscillatory regime, where the early and late time singularities are avoided.

References

[1] Riess A G et al 1998 *Astron. J.* **116** 1009
[2] Tonry J L et al 2003 *Astrophys. J.* **594** 1
[3] Masi S et al 2002 *Prog. Part. Nucl. Phys.* **48** 243
[4] Bennett C L et al 2003 *Astrophys. J. Suppl. Ser.* **148** 1
[5] Komatsu E et al 2011 *Astrophys. J. Suppl. Ser.* **192** 18
[6] Tegmark M et al 2004 *Phys. Rev.* D **69** 103501
[7] Seljak U et al 2005 *Phys. Rev.* D **71** 103515
[8] Wood-Vasey W M et al 2007 *Astrophys. J.* **666** 694
[9] Weinberg S 1989 *Rev. Mod. Phys.* **61** 1
[10] Carroll S M 2001 *Living Rev. Relativ.* **4** 1
[11] Fujii Y 1982 *Phys. Rev.* D **26** 2580
[12] Ratra B and Peebles J 1988 *Phys. Rev.* D **37** 321
[13] Chiba T, Sugiyama N and Nakamura T 1997 *Mon. Not. R. Astron. Soc.* **289** L5
[14] Caldwell R R 2002 *Phys. Lett.* B **545** 23
[15] Caldwell R R, Kamionkowski M and Weinberg N N 2003 *Phys. Rev. Lett.* **91** 071301
[16] Nesseris S and Perivolaropoulos L 2004 *Phys. Rev.* D **70** 123529
[17] Singh P, Sami M and Dadhich N 2003 *Phys. Rev.* D **68** 023522
[18] Nojiri S and Odintsov S D 2004 *Phys. Rev.* D **70** 103522
[19] Jaffe T R, Banday A J, Eriksen H K, Gorski K M and Hansen F K 2005 *Astrophys. J.* **629** L1–L4
[20] Zimdahl W, Schwarz D J, Balakin A B and Pavon D 2001 *Phys. Rev.* D **64** 063501
[21] Balakin A B, Pavon D, Schwarz D J and Zimdahl W 2003 *New J. Phys.* **5** 85
[22] Avelino A and Nucamendi U 2010 *J. Cosmol. Astropart. Phys.* **2010** 009
[23] Avelino A, Garcia-Salcedo R, Gonzalez T, Nucamendi U and Quiros I 2013 *J. Cosmol. Astropart. Phys.* **2013** 012
[24] Meng X and Ma Z 2012 *Eur. Phys. J.* C **72** 2053
[25] Amirhashchi H 2013 *Astrophys. Space Sci.* **345** 439
[26] Meng X H, Ren J and Hu M G 2007 *Commun. Theor. Phys.* **47** 379
[27] Ashtekar A, Bojowald M and Lewandowski J 2003 *Adv. Theor. Math. Phys.* **7** 233
[28] Bojowald M 2008 *Living Rev. Relativ.* **11** 4
[29] Ashtekar A and Singh P 2011 *Class. Quantum Grav.* **28** 213001
[30] Rovelli C 1998 *Living Rev. Relativ.* **1** 1
[31] Thiemann T 2003 *Lect. Notes. Phys.* **631** 41
[32] Ashtekar A and Lewandowski J 2004 *Class. Quantum Grav.* **21** R53
[33] Ashtekar A, Pawlowski T and Singh P 2006 *Phys. Rev.* D **74** 084003
[34] Singh P 2014 *Bull. Astron. Soc. India* **42** 121
[35] Mebarki N and Benchikh S 2014 *Proc. Sci. PoS.* **FFP14** 2
[36] Fu X, Yu H and Wu P 2008 *Phys. Rev.* D **78** 063001
[37] Eckart C 1940 *Phys. Rev.* D **58** 919
[38] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Oxford: Pergamon Press)
[39] Zimdahl W and Pavon D 2000 *Phys. Rev.* D **61** 108301
[40] Khalil H K 1996 *Progress in Nonlinear Systems* (NJ: Prentice Hall)
[41] Chan N 2012 *PhD Dissertation* (London: London University)
[42] Xiao K and Zhu J Y 2011 *Phys. Rev.* D **83** 083501