Travelling Wave in the Generalized Kuramoto Model with Inertia *

Di Yuan(袁地)[1**], Dong-Qiu Zhao(赵冬秋)[1], Yi Xiao(肖奕)[2], Ying-Xin Zhang(张颖新)[3]

[1]*School of Physics and Electrical Engineering, Anyang Normal University, Anyang 455000*
[2]*Department of Physics, Huazhong University of Science and Technology, Wuhan 430074*
[3]*Students' Affairs Division, Henan Vocational College of Nursing, Anyang 455000*

(Received 17 December 2015)

We study the dynamics of the generalized Kuramoto model with inertia, in which oscillators with positive coupling strength are conformists and oscillators with negative coupling strength are contrarians. By numerically simulating the model, we find that the model supports a modulated travelling wave state except for already displayed travelling wave states and π state in previous literature. The modulated travelling wave state may be characterized by the phase distributions of oscillators. Finally, the modulated travelling wave state and the travelling wave state of the model in the parameter space are presented.

PACS: 05.45.Xt, 89.75.−k DOI: 10.1088/0256-307X/33/5/050502

Synchronization phenomena in phase oscillator networks are usually addressed by considering the paradigmatic Kuramoto model since it was proposed by Kuramoto in 1975.[1] This model has been applied in many contexts ranging from the synchronous flashing of groups of fireflies,[2] the coupling of oscillatory neurons in the suprachiasmatic nucleus of the brain governing circadian rhythms,[3] the interaction of cells containing oscillatory chemically reacting constituents,[4] applauding persons in a large audience[5] to phase synchronization in electrical power distribution networks,[6−8] and many other systems.[9−18]

The original Kuramoto model consists of N phase oscillators. Each oscillator has its own natural frequency ω chosen from a given probability density $g(\omega)$ and interacts with the mean field with a global coupling strength K, which is positive, corresponding to an attractive interaction. A natural generalization of the Kuramoto model is to allow K to have either sign. The negative coupling strength accounting for a repulsive interaction has been taken into consideration in recent years. Tsimring *et al.* considered the case in which the interaction between oscillators and the mean field is a repulsive one and found that phase locking among oscillators is destroyed for an array of non-identical phase oscillators provided that the number of oscillators is sufficiently large.[19] Positive and negative communications coexist in biological systems. Some studies have evaluated the significance of these communications on the synchronization in neural networks that consist of excitatory and inhibitory neurons,[20] which interact positively and negatively with their neighboring neurons, respectively. Inhibitory interactions have been proposed to suppress undesired synchronization[21] and to destabilize synchronized neural networks.[22] Some researchers considered the local interaction among oscillators and found evidence of glassy behaviors when both positive and negative coupling strengths were allowed simultaneously.[23,24] Hong *et al.* studied the situation in which the coupling strength is regarded as an oscillator's ability to react to the mean field individually.[25,26] In their works, both positive and negative coupling strengths are present in the population. Oscillators with positive K behave like conformists by tending to fall in line with prevailing rhythm in the population, whereas those with negative K are repelled by the rhythm and act like contrarians. They found a surprising time-dependent state, a travelling wave state in which the mean field oscillates at a frequency different from the population's mean natural frequency and the phase difference between conformists and contrarians are locked at an angle away from π. Positive and negative interactions are also very common in social systems such as human society. For example, conformists positively interact with the neighbors, following the neighbors' opinion unconditionally, whereas contrarians negatively interact, always rejecting the neighbors' idea.

The modification of the Kuramoto model with an additional inertial term was firstly reported by Tanaka *et al.*[27] These researchers have been inspired in their model by a previous phase model developed by Ermentrout to mimic the synchronization mechanisms observed in the firefly Pteroptix malaccae.[28] These fireflies synchronize their flashing activity by entraining to the forcing frequency with almost zero phase

*Supported by the National Natural Science Foundation of China under Grant Nos 11447001 and 11247318, and the Key Project of Scientific and Technological Research of the Education Department of Henan Province under Grant Nos 16A140002 and 13A140025.

**Corresponding author. Email: yuandi@aynu.edu.cn

© 2016 Chinese Physical Society and IOP Publishing Ltd

lag, even for stimulating frequencies different from their own intrinsic flashing frequency. The main ingredient to allow for the adaptation of the flashing frequency to the forcing one is to include an inertial term in a standard phase model for synchronization. Furthermore, networks of coupled phase oscillators with inertia have been recently employed to investigate the self-synchronization in power grids,[29–31] as well as in disordered arrays of underdamped Josephson junctions.[32] Explosive synchronization has been reported for a complex system made up of phase oscillators with inertia, where the natural frequency of each oscillator is assumed to be proportional to the degree of its node.[33,34] Recently the dynamics of the Kuramoto model with an inertia term have also been well examined in many other studies.[35–38]

In this Letter, we examine the dynamics and synchronization properties of the generalized Kuramoto model consisting of conformists and contrarians with an inertial term for fully coupled systems. Here we focus on the model with a uniform probability density of natural frequency, which cannot be solved analytically by either the remarkable ansatz discovered by Ott *et al.*[39] or the theoretical method proposed by Watanabe *et al.*[40]

The generalized Kuramoto model with inertia consists of the following set of equations

$$m\ddot{\phi}_i + \dot{\phi}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i),\ \ i = 1, 2, \ldots, N, \tag{1}$$

where ϕ_i is the phase of the ith oscillator at time t, N is the number of phase oscillators in the system, m is the mass of an oscillator, ω_i is the natural frequency of the ith oscillator and is chosen at random from $[-\gamma, \gamma]$ with γ being the width of natural frequency distribution, K_i is the coupling strength of the ith oscillator to the mean field and is chosen from a double-δ probability density $\Gamma(K) = (1-p)\delta(K - K_-) + p\delta(K - K_+)$ with $K_- < 0$ and $K_+ > 0$ being the couplings for the contrarians and conformists, respectively, and p denotes the probability that a random oscillator is a conformist.

The collective rhythm in the model is quantified by a mean field-like quantity, that is, a complex order parameter $Re^{i\Phi}$ defined as

$$Z = Re^{i\Phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}, \tag{2}$$

where the amplitude $0 \le R \le 1$ measures the phase coherence in the population, and Φ gives the average phase. In terms of R and Φ, Eq. (1) can be rewritten as

$$m\ddot{\phi}_i + \dot{\phi}_i = \omega_i + K_i R\sin(\Phi - \phi_i),\ \ i = 1, 2, \ldots, N, \tag{3}$$

which expresses the evolution of the ith oscillator solely in terms of R and Φ. The complex order parameters in conformists and in contrarians are also important quantities to determine the dynamics in Eq. (1) and they are defined as $Z_\pm = R_\pm e^{i\Phi_\pm} = \frac{1}{N_\pm}\sum_{j\in S_\pm} e^{i\phi_j}$, where S_+ (or S_-) means the set of conformists (or contrarians), $N_\pm$ are the numbers of conformists and contrarians, respectively, and Z is related to $Z_\pm$ by $Z = \frac{N_-}{N}Z_- + \frac{N_+}{N}Z_+$.

We numerically investigate the dynamics in Eq. (1) by a fourth-order Runge–Kutta algorithm with a time step $\delta t = 0.01$ and the quantities of interest are measured after a sufficiently long transient is discarded. Throughout the work, we let $N = 10000$, $K_- = -0.5$, $K_+ = 1.0$, $m = 2.0$ and $\gamma = 0.1$ unless specified. Initially, we assign each oscillator a phase randomly drawn from $[0, 2\pi]$.

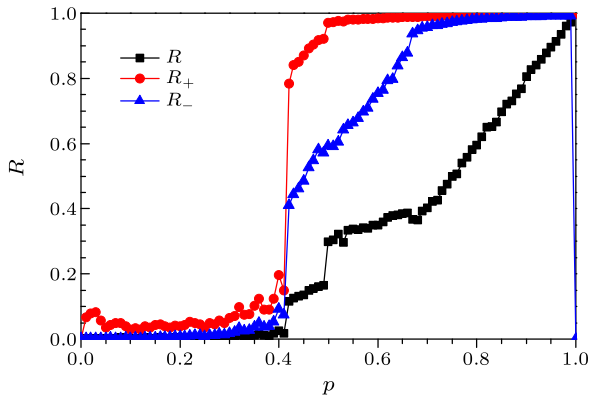

Fig. 1. (Color online) (a) The amplitudes of the order parameters R (the lower one), R_+ (the upper one) and R_- (the middle one) against the fraction of conformists in the population p. Here $K_- = -0.5$, $K_+ = 1.0$, $m = 2.0$ and $\gamma = 0.1$.

Firstly, we investigate the dynamics of Eq. (1) depending on the fraction of conformists p. For this aim, we consider the variations of the order parameters R, R_+ and R_- against p. The results are presented in Fig. 1. With the increase of p, the fraction of conformists increases and we find that the system presents several different regimes for different dynamical states. In the studies of the Kuramoto model and its different variants, the incoherence stability which signifies the onset of synchronization is an important topic. We investigate the threshold of p for the onset of synchronization. The incoherence loses its stability at $p = 0.32$, thus the incoherent state is stable in the range of $p < 0.32$, where each oscillator oscillates at its own natural frequency. In the range of $p > 0.68$, the π state is stable in which conformists and contrarians form two partially synchronized clusters and the peaks of their phase distributions are separated from each other by an angle of π. The solutions in the range of $0.5 < p < 0.68$ are labelled as travelling wave states in which the phase distributions of conformists and con-

trarians spontaneously travel along the phase axis and the peaks of the phase distributions of conformists and contrarians always maintain a separation while not at the angle of π. However, there still exists an interesting dynamical state in the range of $0.41 < p < 0.50$ in which the travelling wave is modulated; we term the state a modulated travelling wave state.

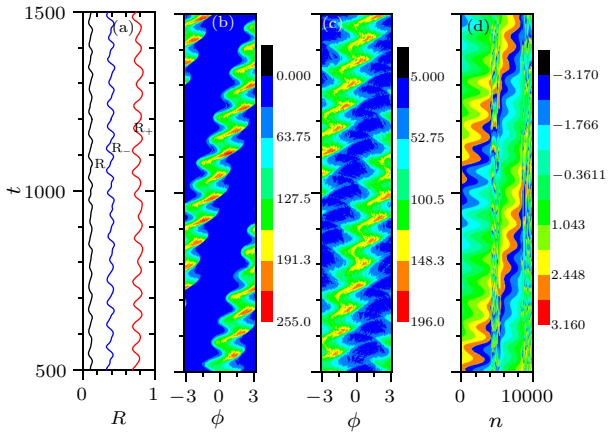

Fig. 2. (Color online) (a) The time sequences of the order parameters R (the left one in black), R_+ (the right one in red) and R_- (the middle one in blue). (b) The time evolution of the phase distribution in the subpopulation of conformists. (c) The time evolution of the phase distribution in the subpopulation of contrarians. (d) The time evolution of oscillators' phases where the oscillators are numbered according to their coupling strength and natural frequencies. Here $p = 0.42$, $K_- = -0.5$, $K_+ = 1.0$, $m = 2.0$ and $\gamma = 0.1$.

Now we explore the dynamics of the modulated travelling wave state. We present the time sequences of the order parameters after transient in Fig. 2(a) where $p = 0.42$. Different from the travelling wave state where the order parameters stay at constants, here three order parameters oscillate periodically. The temporal behavior of the order parameters can be explained by the evolutions of the phase distributions for both conformists and contrarians. As shown in Figs. 2(b) and 2(c), the phase distributions of the two subpopulations are not stationary and their time evolutions consist of two components: one is a slow one which travels along the phase axis at a constant speed, and the other is a fast one which periodically modulates the phase distributions of conformists and contrarians. Under the modulation of the fast component, the peak values of the two phase distributions vary periodically. Interestingly, Figs. 2(b) and 2(c) show that the phase distributions are always asymmetrical about the peak, which is different from the travelling wave state as shown in Refs. [25,26]. The careful comparison between Figs. 2(b) and 2(c) reveals that the separation between the peaks of the phase distributions of conformists and contrarians is not a constant and always away from π which is a remnant of the travelling wave state.

Furthermore, we investigate how conformists and contrarians organize themselves in a modulated travelling wave state. To carry it out, we number oscillators according to their coupling strength and their natural frequencies. If an oscillator i has its coupling strength K_- and an oscillator j has the coupling strength K_+, then we have $n_i > n_j$ regardless of their natural frequencies (n_i and n_j are the numbers of oscillators i and j, respectively). On the other hand, we have $n_i < n_j$ if the two oscillators have the same coupling strength while $\omega_i < \omega_j$ (ω_i and ω_j are the natural frequencies of oscillators i and j, respectively). The phase evolution of oscillators is shown in Fig. 2(d), adjacent oscillators stay closely in phase and the phase of each oscillator presents periodic change with time. Moreover, the period of the phase evolution of oscillators is the same as that of the evolutions of the phase distributions of both conformists and contrarians.

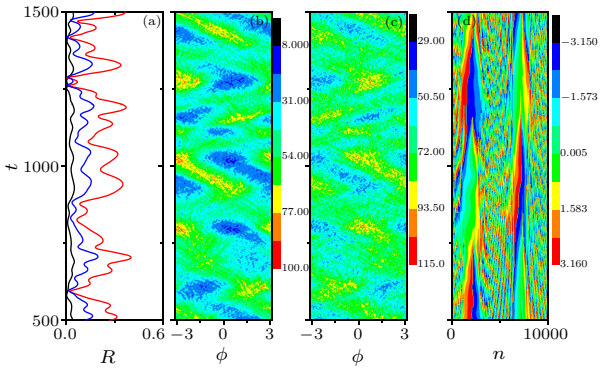

Fig. 3. (Color online) (a) The time sequences of the order parameters R (the left one in black), R_+ (the right one in red) and R_- (the middle one in blue). (b) The time evolution of the phase distribution in the subpopulation of conformists. (c) The time evolution of the phase distribution in the subpopulation of contrarians. (d) The time evolution of oscillators' phases where the oscillators are numbered according to their coupling strength and natural frequencies. Here $p = 0.40$, $K_- = -0.5$, $K_+ = 1.0$, $m = 2.0$ and $\gamma = 0.1$.

Before going further, we must point out that the modulated travelling wave state does not directly bifurcate from the incoherent state. Actually, it emerges from a travelling wave state. We present the dynamical state in Fig. 3 for $p = 0.40$ which lies between the incoherent state and the modulated travelling wave state. Though the order parameters in Fig. 3(a) display violent fluctuation due to the finite size N, the evolutions of the phase distributions in Figs. 3(b) and 3(c) show that the system is in a partially synchronized state and the synchronized oscillators do not oscillate at the mean natural frequency. The phase evolution of oscillators in Fig. 3(d) shows clearly a travelling wave state in the middle of both conformists and contrarians with a much slower speed.

To characterize the dependence of the travelling wave state and the modulated travelling wave state

on the parameters of m and p, we consider two quantities. One is the speed of a travelling wave Ω and the other is the standard deviation of instantaneous wave speed ξ. The speed of a travelling wave is defined as $\Omega = (1/N)\sum_{j=1}^{N}\langle\dot{\phi}_j\rangle_T$. When $\Omega \neq 0$ and $\xi = 0$, the system of coupled phase oscillators displays a travelling wave behavior. When $\Omega \neq 0$ and $\xi \neq 0$, the travelling wave state is modulated. The speed of travelling wave Ω in Fig. 4(a) and the standard deviation of instantaneous wave speed ξ in Fig. 4(b) are plotted as functions of m and p. As shown in Fig. 4(a), the travelling wave state occurs in a wider window with the increase of m from zero, the wave speed decreases gradually. The area for the modulated travelling wave state to occur is gradually increasable by increasing m as shown in Fig. 4(b). We notice that the modulated travelling wave state locates at the boundary of the travelling wave state by comparing Figs. 4(a) and 4(b). In the area of the modulated travelling wave state, the wave speed is extremely low. The above results are obtained at $K_- = -0.5$ and $K_+ = 1.0$. For other combinations of K_- and K_+, the dependence of the travelling wave state and the modulated travelling wave state on m and p is similar to those in Fig. 4.

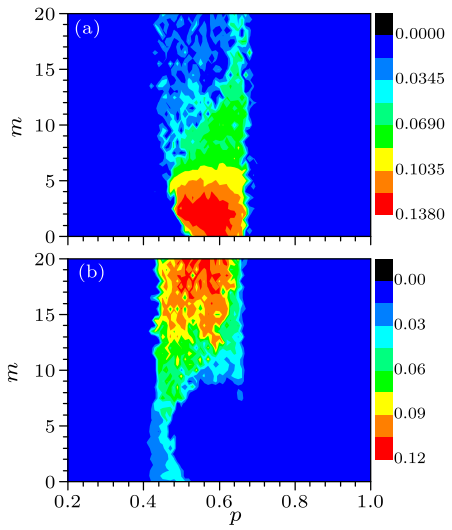

Fig. 4. (Color online) The speed of travelling wave Ω in (a) and the standard deviation of instantaneous wave speed ξ in (b) as functions of m and p. Here $K_- = -0.5$, $K_+ = 1.0$ and $\gamma = 0.1$.

In summary, we consider the generalized Kuramoto model with an inertia term, in which oscillators with positive coupling strength are conformists and oscillators with negative coupling strength are contrarians. We have explored the dynamics of the model in detail by numerical simulations and found rich dynamics which may be characterized by phase distributions of oscillators such as travelling wave states and π state. Interestingly, we find another type of nonstationary state, and we term it the modulated travelling wave state. In the modulated travelling wave state, the phase distributions of conformists and contrarians are not stationary and their time evolutions consist of two components: one is a slow one which travels along the phase axis at a constant speed, and the other is a fast one which periodically modulates the phase distributions of conformists and contrarians. Furthermore, the modulated travelling wave which may bifurcate from travelling wave is explored. Finally, we present the phase diagram of the model (1) in the parameter space, and the parameter regimes of the travelling wave state and the modulated travelling wave state can be obtained.

References

[1] Kuramoto Y 1975 *Lecture Notes in Physics* (New York: Springer)
[2] Buck J and Buck E 1976 *Sci. Am.* **234** 74
[3] Yamaguchi S et al 2003 *Science* **302** 1408
[4] Kiss I Z et al 2002 *Science* **296** 1676
[5] Néda Z et al 2000 *Phys. Rev.* E **61** 6987
[6] Heide D et al 2008 *Phys. Rev.* E **77** 056103
[7] Simonsen I et al 2008 *Phys. Rev. Lett.* **100** 218701
[8] Dorfler F et al 2013 *Proc. Natl. Acad. Sci. USA* **110** 2005
[9] Strogatz S H 2000 *Physica* D **143** 1
[10] Ott E 2002 *Chaos in Dynamical Systems* (Cambridge: Cambridge University Press)
[11] Acebrón J A et al 2005 *Rev. Mod. Phys.* **77** 137
[12] Yuan D and yang J Z 2013 *Commun. Theor. Phys.* **59** 684
[13] Lai Y C et al 2006 *Phys. Rev.* E **73** 026214
[14] Kim B et al 2013 *Phys. Rev.* E **88** 042818
[15] Zang H Y et al 2013 *Chin. Phys. Lett.* **30** 040502
[16] Wu X L and Yang J Z 2014 *Chin. Phys. Lett.* **31** 060507
[17] Feng Y E et al 2014 *Chin. Phys. Lett.* **31** 080503
[18] Ju P and Yang J Z 2015 *Chin. Phys. Lett.* **32** 030502
[19] Tsimring L S et al 2005 *Phys. Rev. Lett.* **95** 014101
[20] Borgers C et al 2003 *Neural Comput.* **15** 509
[21] Louzada V H P et al 2012 *Sci. Rep.* **2** 658
[22] Freeman G M et al 2013 *Neuron* **78** 799
[23] Daido H 1987 *Prog. Theor. Phys.* **77** 622
[24] Borgers C et al 2005 *Proc. Natl. Acad. Sci. USA* **102** 7002
[25] Hong H and Strogatz S H 2011 *Phys. Rev. Lett.* **106** 054102
[26] Hong H and Strogatz S H 2011 *Phys. Rev.* E **84** 046202
[27] Tanaka H A et al 1997 *Phys. Rev. Lett.* **78** 2104
[28] Ermentrout B 1991 *J. Math. Biol.* **29** 571
[29] Salam F et al 1984 *IEEE Trans. Circuits Syst.* **31** 673
[30] Filatrella G et al 2008 *Eur. Phys. J.* B **61** 485
[31] Rohden M et al 2012 *Phys. Rev. Lett.* **109** 064101
[32] Trees B R et al 2005 *Phys. Rev.* E **71** 016215
[33] Ji P et al 2013 *Phys. Rev. Lett.* **110** 218701
[34] Ji P et al 2014 *Phys. Rev.* E **90** 062810
[35] Gupta S et al 2014 *Phys. Rev.* E **89** 022123
[36] Olmi S et al 2014 *Phys. Rev.* E **90** 042905
[37] Jaros P et al 2015 *Phys. Rev.* E **91** 022907
[38] Peron T K D M et al 2015 *Phys. Rev.* E **91** 052805
[39] Ott E and Antonsen T M 2008 *Chaos* **18** 037113
[40] Watanabe S and Strogatz S H 1994 *Physica* D **74** 197