CHIN. PHYS. LETT. Vol. 33, No. 5 (2016) 050302

Entanglement Dynamics of Two Qubits Coupled Independently to Cavities in the Ultrastrong Coupling Regime: Analytical Results *

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(Received 17 January 2016)

Dynamics of quantum entanglement of two qubits in two identical quantum Rabi models is studied analytically in the framework of corrections to the rotating-wave approximations. A closed-form expression for the entanglement dynamics initiated from the well-known Bell states is derived, which is very close to the numerical exact results up to the ultrastrong coupling regime. It is found that the vanishing entanglement can be purely induced by the counter-rotating terms, and can be enhanced with the atom–cavity coupling.

PACS: 03.65.Ud, 03.65.Yz, 42.50.Ct DOI: 10.1088/0256-307X/33/5/050302

Quantum entanglement is fundamental in quantum physics for both understanding the nonlocality of quantum mechanics[1] and its role in quantum computations and communications.[2] However, a realistic quantum system interacts inevitably with the surrounding environment, which always leads to decoherence of the quantum states. As a result, an initially entangled two-qubit system becomes totally disentangled after evolving for a finite time. This phenomenon is called entanglement sudden death (ESD)[3] and has been demonstrated experimentally.[4]

Among all quantum mechanical descriptions of the quantum system and its environment, the Rabi model is the simplest, and can be regarded as a paradigm for study of the entanglement evolution. In the rotating-wave approximation (RWA), entanglement dynamics for two qubits in Rabi models has been well studied 10 y before.[5–8] Analytical results for the entanglement have been easily derived if RWA is employed. Since then, the exact entanglement dynamics has been studied by Chen *et al.* based on the numerical solution to the full model without the RWA.[9] These results have stimulated extensive research in the dynamics of entanglement of qubits in the Rabi model and related models.[10–21]

In the past decade, the theoretical study to the full Rabi model has achieved great progress, stimulated by the development of experimental techniques.[22] In some circuit quantum electrodynamics (QED) systems,[23–25] the coupling between the two-level artificial atom and the single bosonic mode has entered ultrastrong coupling regime ($g/\omega \approx 0.1$, where ω is the frequency of the cavity). The evidence for the breakdown of the RWA in the strong-coupling regime has been provided.[23] The counter-rotating terms (CRTs) should be considered.[26] However, if the CRTs are involved, the photonic number is not conserved, thus the photonic Fock space then has infinite dimensions, and any solution to the Rabi model is highly nontrivial. In recent years, some analytical[27,28] and numerical[29] exact solutions have been successfully proposed. However, the eigensolutions can only be expressed in terms of infinite polynomials and therefore infinite photonic Fock states are involved. In this way, the detailed mechanism based on the exact solution is difficult to unfold.

More recently, He *et al.* proposed corrections to the RWA, called the CRWA.[30] The effect of the CRTs emerges clearly even in the first-order correction. All eigenvalues and eigenfunctions are derived analytically. The vacuum Rabi splitting, collapse and revival in the population dynamics have been studied successfully up to the remarkable coupling strength of $g/\omega \approx 0.2$, suggesting that CRWA could be convincingly applied to recent circuit QED systems operating in the ultrastrong coupling regime. Note that the present maximum value for the coupling strength in a superconducting qubit coupled to a resonant circuit[23] has reached $g/\omega \approx 0.12$. The advantage for the CRWA lies transparently in important mechanisms of interest provided by the compact eigenstates with only three photonic number states. Now with CRWA eigenvalues and eigenvectors at hand, we can revisit many physical problems that have been studied by using the RWA. In this work, we focus on the entanglement dynamics for two identical qubits interacting with two spatially separated cavities in the framework of CRWA. Analytical results for the entanglement dynamics in terms of CRWA are derived, and the comparisons with the RWA and numerically exact ones are performed.

The Hamiltonian of two equivalent independent quantum Rabi models reads

$$H = \frac{\Delta}{2}\sum_{j=\mathrm{A,B}}\sigma_z^j + \omega\sum_{j=\mathrm{A,B}}a_j^\dagger a_j + g\sum_{j=\mathrm{A,B}}(a_j^\dagger + a_j)\sigma_x^j, \tag{1}$$

where Δ is the energy splitting of the two-level atom, $\sigma_i^j (i = x, y, z)$ is the Pauli operator in the jth cavity, $a_j^\dagger$ creates one photon from the jth cavity with frequency ω, a_j is the corresponding annihilator, and g describes the atom–cavity coupling strength. To be more concise, we only study the resonance case

*Supported by the National Natural Science Foundation of China under Grant Nos 11174254 and 11474256.
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($\Delta = \omega$) in this study. It is straightforward to extend to off-resonance cases. The energy scale is set $\omega = 1$ here.

In the CRWA,[30] only three bare photonic states are kept in the eigenstates. For later use, we borrow the information from Eqs. (8) and (9) in Ref. [30] and present explicitly the first three eigenstates and the corresponding energies. The first two excited states are given by $k = 1$ and 2 for $n = 0$, thus the energies are given by

$$E_{\pm} = \frac{1}{2} \pm g - \frac{1}{2}g^2 \mp \frac{1}{8}g^3, \tag{2}$$

where $E_{\pm}$ corresponds to the energy of the first excited ($-$) and the second excited states ($+$). The corresponding two eigenstates take the same form as

$$|\pm\rangle = \begin{pmatrix} c_0^{\pm}|0\rangle + c_2^{\pm}|2\rangle \\ c_1^{\pm}|1\rangle \end{pmatrix}, \tag{3}$$

where the coefficients c's are expanded in g up to g^2.

The ground state and energy are given by Eqs. (10) and (11) in Ref. [30],

$$|GS\rangle = \begin{pmatrix} d_1|1\rangle \\ d_0|0\rangle \end{pmatrix}, \tag{4}$$

$$E_0 = -\frac{1}{2} - \frac{1}{2}g^2 + \frac{1}{8}g^4, \tag{5}$$

where d's are also up to g^2.

Interestingly, the first two terms of the eigenenergies in CRWA are none other than the RWA ones. The additional terms to the RWA wavefunction appearing in the excited atomic levels are just produced by the CRTs, $a^{\dagger}\sigma_+$ and $a\sigma_-$. It is expected that these simplest excitations will dominate many physical processes beyond the RWA in the wide coupling regime, and can be easily employed to analyze the relevant physical mechanism.

Two initial Bell states with anticorrelated and correlated spins, which are denoted by Bell states 1 and 2, respectively, can be expressed as

$$|\Psi_{\rm Bell}^{(1)}\rangle = \sin\alpha|\downarrow\uparrow\rangle + \cos\alpha|\uparrow\downarrow\rangle, \tag{6}$$

$$|\Psi_{\rm Bell}^{(2)}\rangle = \sin\alpha|\downarrow\downarrow\rangle + \cos\alpha|\uparrow\uparrow\rangle, \tag{7}$$

where spin-up denotes the atomic upper level and spin-down denotes the atomic lower level. The initial states of the cavity are usually set to be the vacuum state. In this way, in each Rabi model, the initial states only involve two states $|\uparrow, 0\rangle$ and $|\downarrow, 0\rangle$.

In the framework of the CRWA, by Eq. (3), we know that the initial state $|\uparrow, 0\rangle$ only includes the first and second excited states as

$$|\uparrow, 0\rangle = c_0^{+}|+\rangle + c_0^{-}|-\rangle. \tag{8}$$

The time-dependent state is easily derived as

$$\begin{aligned} |\Psi(t)\rangle_A &= \exp(-iHt)|\uparrow, 0\rangle \\ &= \exp(-iE_+t)c_0^{+}|+\rangle + \exp(-iE_-t)c_0^{-}|-\rangle, \\ &= \exp(-iE_+t)c_0^{(+)} \begin{pmatrix} c_0^{+}|0\rangle + c_2^{+}|2\rangle \\ c_1^{+}|1\rangle \end{pmatrix} \\ &\quad + \exp(-iE_-t)c_0^{(-)} \begin{pmatrix} c_0^{-}|0\rangle + c_2^{-}|2\rangle \\ c_1^{-}|1\rangle \end{pmatrix}. \end{aligned} \tag{9}$$

Similarly, by Eq. (4), the initial state $|\downarrow, 0\rangle$ only includes the ground state

$$|\downarrow, 0\rangle = d_0|GS\rangle,$$

then we can derive the time evolution of the states as

$$|\Psi(t)\rangle_B = d_0 \exp(-iE_0t) \begin{pmatrix} d_1|1\rangle \\ d_0|0\rangle \end{pmatrix}. \tag{10}$$

The effect of CRTs must originate from the bare photonic states appearing in the upper atomic levels in Eqs. (9) and (10).

For two-qubit states, entanglement can be quantified by the concurrence.[31] We can write the reduced density matrix at time t as

$$\rho^{\rm AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix},$$

and calculate the dynamics of the concurrence as

$$C^{\rm AB}(t) = 2\max[0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}].$$

We can then calculate the entanglement dynamics in the following. The concurrence initially in the anti-correlated Bell state described by Eq. (6) can be written as

$$C^{\rm AB}(t) = 2\max[0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}], \tag{11}$$

due to the fact that $|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}$ is always less than 0. Note that density matrix elements under RWA are given by Ref. [5],

$$\begin{aligned} \rho_{11}^{\rm RWA} &= 0, \\ \rho_{44}^{\rm RWA} &= \sin^2(gt), \\ \rho_{23}^{\rm RWA} &= \frac{1}{4}\cos^2(gt)\sin(2\alpha). \end{aligned}$$

Since $\rho_{11}^{\rm RWA} = 0$, the concurrence is only determined by $|\rho_{23}|$ and then is always greater than or equal to 0 at some discrete times. Thus there is no ESD. If the effect of CRTs is taken into account, the above picture would be modified.

What will emerge if the CRTs are set in? We will employ CRWA to calculate the entanglement dynamics. The corresponding density matrix elements under CRWA up to g^2 terms are easily obtained as

$$\begin{aligned} \rho_{11} &= \frac{1}{4}g^2\cos^2(\gamma t), \\ \rho_{44} &= \Big(1 - \frac{5g^2}{4}\Big)\sin^2(\gamma t), \\ \rho_{23} &= \frac{1}{4}\Big(\frac{g^2}{2} + (1 - g^2)\cos^2(\gamma t)\Big)\sin(2\alpha), \end{aligned}$$

where $\gamma = (E_+ - E_-)/2 = g\left(1 - \frac{g^2}{8}\right)$. We can immediately see that the frequency is reduced from g to γ due to the CRT effect.

Inserting the above equations into Eq. (11), we can obtain the analytical entanglement dynamics (up to g^2) for initial anti-correlated Bell states

$$C^{\rm AB}(t) = 2\max\left\{0, \frac{1}{4}[g^2 + 2(1-g^2)\cos^2(\gamma t)]\sin(2\alpha) - \frac{1}{4}|\sin(2\gamma t)|g\right\}. \quad (12)$$

Omitting g^2, we have

$$C^{\rm AB}(t) = |\sin(2\alpha)|\cos^2(\gamma t), \quad (13)$$

which is exactly the same as the RWA results.[5] Note in Eq. (12) that all the terms higher than linear g-terms originate from the effects of CRTs.

The condition for the ESD can be easily obtained as

$$\frac{1-g^2}{g}\sin(2\alpha) < \frac{|\sin(\gamma t)|}{\cos(\gamma t)},$$

thus in some periods of time, ESD will occur. For the initial anti-correlated Bell state, ESD will never happen in the RWA but it will appear if the CRT effect is considered.

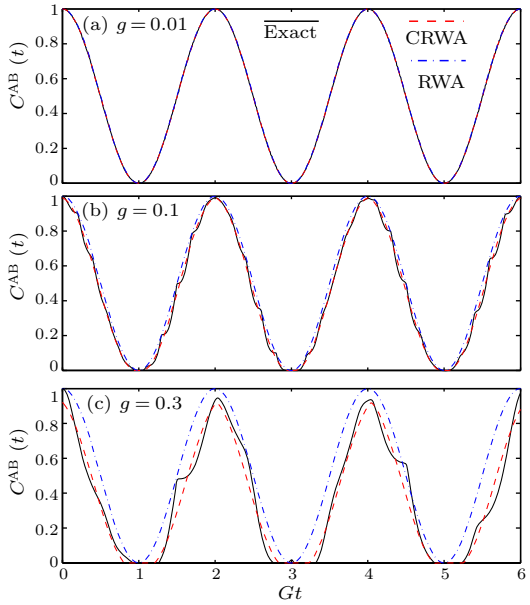

Fig. 1. (Color online) The evolution of the concurrence $C^{\rm AB}(t)$ initially in the anti-correlated Bell state with $\alpha = \pi/4$ calculated by exact diagonalization (solid), CRWA (dashed) and RWA (dashed-dotted) for (a) $g = 0.01$, (b) $g = 0.1$ and (c) $g = 0.3$.

The evolution of concurrence $C^{\rm AB}(t)$ initially from the anti-correlated Bell state in the CRWA is shown in Fig. 1 from weak to ultrastrong coupling regimes $g = 0.01$, 0.1 and 0.3. For comparison, we also list the RWA results and the numerical exact ones by exact diagonalizations. It is expected that both CRWA and RWA should give quite accurate results in the weak coupling, as indeed demonstrated in Fig. 1(a) for $g = 0.01$. It is obvious that the results of CRWA and RWA deviate from the exact ones as the coupling strength increases. The remarkable differences between the RWA and CRWA results are exhibited in Fig. 1(c) for $g = 0.3$. Interestingly, the present CRWA results agree quite well with the numerical exact ones up to the ultrastrong coupling regime, indicating that the first correction to RWA plays a dominant role in a wide coupling regime. As is also shown in Figs. 1(b) and 1(c), ESD obviously occurs in the ultrastrong coupling regime ($g >$ or ≈ 0.1) for an increasing period of time with g in the numerical calculations. CRWA can reveal this feature that RWA fails. The absence of ESD in the RWA and the enhancement of ESD with coupling in the CRWA demonstrate that the CRTs can induce the ESD, which should be considered in any quantum information processing.

It is notable in Fig. 1(c) for $g = 0.3$ that there are sudden-change-like phenomena when Gt are about 1.5 and 4.5 in the solid line from numerical exact diagonalizations, while lines obtained from the RWA and CRWA are smooth at these points. The reason is as follows: the exact eigenstates include infinite photonic Fock states, while the eigenstates in the RWA (CRWA) only contain two (three) Fock states. As coupling increases to $g = 0.3$, we find that the fourth and fifth excited states still contribute to the initial state $|\uparrow, 0\rangle$ in the exact solution, while are definitely ignored in both RWA and CRWA due to analytical approximations. Here ρ_{11} is completely determined by the upper atomic state and shows a rapid oscillation behavior in the exact solution, therefore it results in sudden-change-like phenomena at some time points. Such a rapid oscillation does not appear in both the RWA and CRWA due to the absence of the high excited states.

The concurrence initially in the correlated Bell state can be written as

$$C^{\rm AB}(t) = 2\max[0, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}], \quad (14)$$

due to the fact that $|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}$ is always less than 0. The corresponding density matrix elements under RWA are

$$\rho_{14}^{\rm RWA} = \sin\alpha\cos\alpha\cos^2(gt)e^{-i2t},$$
$$\rho_{22}^{\rm RWA} = \cos^2(gt)\sin^2(gt)\cos^2(\alpha),$$
$$\rho_{33}^{\rm RWA} = \rho_{22}^{\rm RWA}.$$

Accordingly, the corresponding density matrix elements under CRWA up to g^2 are

$$\rho_{14} = \frac{1}{2}\Big(-\frac{g^2}{4} + \Big(1 - \frac{g^2}{4}\Big)\cos^2(\gamma t) - \frac{ig}{2}\sin(2\gamma t) - \frac{g^2}{4}\sin^2(\gamma t)\Big)\sin(2\alpha)e^{-2it}, \quad (15)$$

$$\rho_{22} = \Big(\frac{3g^2}{4} + \Big(1 - \frac{3g^2}{2}\Big)\cos^2(\gamma t)\Big) \cdot \sin^2(\gamma t)\cos^2(\alpha) + \frac{g^2}{4}\sin^2(\alpha), \quad (16)$$

$$\rho_{33} = \rho_{22}, \quad (17)$$

where $\gamma = g(1 - \frac{g^2}{8})$. Then we can easily obtain the entanglement dynamics up to g^2,

$$C^{\mathrm{AB}}(t) = 2\max[0, |\rho_{14}| - |\rho_{22}|]. \quad (18)$$

Omitting g^2, we have

$$C^{\mathrm{AB}}(t) = \cos^2\gamma t[|\sin(2\alpha)| - \sin^2(\gamma t)\cos^2\alpha], \quad (19)$$

which is also exactly the same as the RWA results.[5]

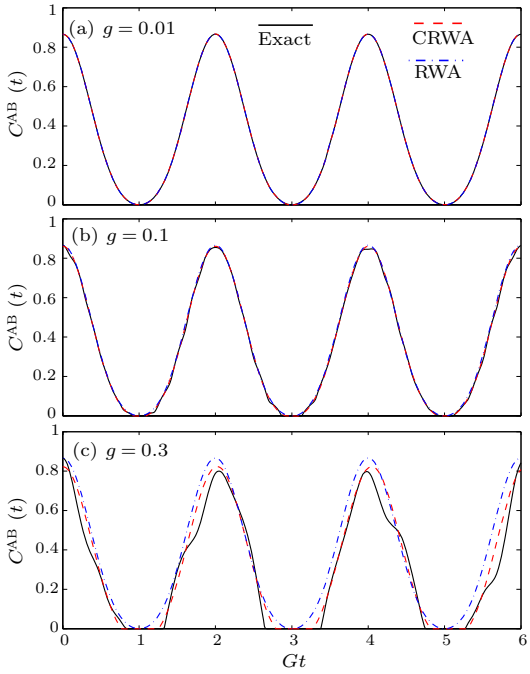

Fig. 2. (Color online) The evolution of the concurrence $C^{\mathrm{AB}}(t)$ initially in the correlated Bell state with $\alpha = \pi/3$ calculated by exact diagonalization (solid), CRWA (dashed) and RWA (dashed-dotted) for (a) $g = 0.01$, (b) $g = 0.1$ and (c) $g = 0.3$.

In the RWA, set $\alpha = \pi/3$, there is absolutely no ESD according to Eq. (12). What will happen if the CRTs are considered? We present the time evolution of concurrence $C^{\mathrm{AB}}(t)$ initially in the correlated Bell state from numerical exact calculations, CRWA, and RWA in Fig. 2. Similarly, for very weak coupling $g = 0.01$, all three methods give similar results. As the coupling increases, the RWA results become worse. As the coupling becomes strong enough, that is, $g = 0.3$, the CRWA results are still close to the exact ones, especially in the ESD regime, and the RWA results deviate from the exact ones considerably. The ESD occurs obviously at this coupling strength, while RWA cannot describe it well.

In summary, we have extended the previous CRWA technique to study the entanglement dynamics for two qubits coupling individually to single-mode cavities. The analytical results for the entanglement dynamics are explicitly given, which are very close to the numerical exact ones up to the ultrastrong coupling regime. The CRWA can account for the ESD occurring in the ultrastrong coupling regime initiated from the well-known Bell states, while in the RWA scheme, the ESD is not observed, indicating that the ESD can be purely induced by the CRTs. The period of ESD is prolonged with the system–cavity coupling, which will be important for the quantum information processing.

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