Consistency Conditions and Constraints on Generalized $f(R)$ Gravity with Arbitrary Geometry-Matter Coupling *

Si-Yu Wu(吴思雨)[1], Ya-Bo Wu(吴亚波)[1**], Yue-Yue Zhao(赵月月)[2], Xue Zhang(张雪)[1], Cheng-Yuan Zhang(张成园)[1], Bo-Hai Chen(陈博海)[1]

[1]*Department of Physics, Liaoning Normal University, Dalian 116029*
[2]*Department of Basic, Dalian Naval Academy, Dalian 116018*

(Received 7 December 2015)

We study the consistency conditions of the generalized $f(R)$ gravity by extending $f(R)$ gravity with non-minimal coupling to the generalized $f(R)$ with arbitrary geometry-matter coupling. Specifically, we discuss the two particular models of generalized $f(R)$ by means of consistency conditions. It is found that the second model is not physically viable so as to be ruled out. Moreover, we further constrain the first model using the Dolgov–Kawasaki stability criterion, and give the value ranges of the parameters in the first model. It is worth stressing that our results include the ones in $f(R)$ gravity with non-minimal coupling as the special case of $Q(L_{\rm m}) = L_{\rm m}$.

PACS: 98.80.−k, 98.80.Jk, 04.20.−q DOI: 10.1088/0256-307X/33/3/039801

Recent astrophysical observations[1,2] have indicated that the expansion of our universe is accelerating at present. In principle, this phenomenon can be explained by either dark energy,[3−17] in which the reason for this phenomenon is due to an exotic component with large negative pressure, or modified theories of gravity.[18−29] The $f(R)$ gravity[30] is one of the competitive candidates in modified theories of gravity. The conditions of viable cosmological models have been derived in Refs. [31,32].

One of the fundamental issues in gravity is its coupling to the matter fields. A general model of $f(R)$ gravity has been proposed in Ref. [33], which contains a non-minimal coupling between geometry and matter. The general action for non-minimally coupled $f(R)$ gravity[34] can be written as

$$S_0 = \int d^4x\sqrt{-g}[f_1(R) + f_2(R)L_{\rm m}], \tag{1}$$

where $f_1(R)$ and $f_2(R)$ are two general (sufficiently regular) functions of the curvature scalar R, and $L_{\rm m}$ denotes the matter Lagrangian. After that, a more general model was proposed in Ref. [35], i.e., the so-called generalized $f(R)$ gravity with arbitrary geometry-matter coupling, and the corresponding action is

$$S_0 = \int d^4x\sqrt{-g}[f_1(R) + f_2(R)Q(L_{\rm m})], \tag{2}$$

where $f_i(i = 1, 2)$ and $Q(L_{\rm m})$ are arbitrary functions of the Ricci scalar R and the matter Lagrangian $L_{\rm m}$, respectively. The related issues such as the energy conditions and the Dolgov–Kawasaki (DK) stability criterion for this model have been derived in Refs. [36–38]. As a maximal extension of non-minimally coupled $f(R)$ gravity, $f(R, L_{\rm m})$ gravity with arbitrary coupling[39] has been further studied in Refs. [40–42]. As we know, the modified gravity must be stable at classical and quantum level. According to Ref. [43], we can realize that the DK stability criterion of $f(R, L_{\rm m})$ gravity models is the same as the one in the generalized $f(R)$ model, i.e.,

$$f_1''(R) + 2f_2''(R)Q(L_{\rm m}) \geq 0. \tag{3}$$

Moreover, the consistency of the non-minimally coupled $f(R)$ theory has been investigated in Ref. [44]. The purpose of this work is to study the consistency conditions of the generalized $f(R)$ gravity with arbitrary geometry-matter coupling, and we further rule out or severely restrict the generalized $f(R)$ models by means of both the consistency conditions and the DK stability criterion.

In the following we study an action which is dynamically equivalent to Eq. (2) to find some constraints on the generalized $f(R)$ gravity with arbitrary geometry-matter coupling. Motivated by Ref. [45], we consider the following theory

$$S = \int d^4x\sqrt{-g}\Big[f + \frac{\partial f}{\partial\alpha}(R-\alpha) + \frac{\partial f}{\partial\beta}(Q(L_{\rm m})-\beta)\Big], \tag{4}$$

where $f = f_1 + f_2Q(L_{\rm m})$, and α and β are two scalar fields. Varying this action with respect to the two

*Supported by the National Natural Science Foundation of China under Grant Nos 11175077, 11575075 and 11547156, the Joint Specialized Research Fund for the Doctoral Program of Higher Education of Ministry of Education of China under Grant No 20122136110002, the Open Project Program of State Key Laboratory of Theoretical Physics of Institute of Theoretical Physics under Grant No Y4KF101CJ1, and the Project of Key Discipline of Theoretical Physics of Department of Education in Liaoning Province under Grant Nos 905035 and 905061.

**Corresponding author. Email: ybwu61@163.com

© 2016 Chinese Physical Society and IOP Publishing Ltd

scalar fields gives

$$\frac{\partial^2 f}{\partial \alpha^2}(R-\alpha)+\frac{\partial^2 f}{\partial \alpha \partial \beta}(Q-\beta)=0, \tag{5}$$

$$\frac{\partial^2 f}{\partial \alpha \partial \beta}(R-\alpha)+\frac{\partial^2 f}{\partial \beta^2}(Q-\beta)=0. \tag{6}$$

Assuming that the determinant of this system of equations is non vanishing, i.e.,

$$\frac{\partial^2 f}{\partial \alpha^2}\frac{\partial^2 f}{\partial \beta^2} \neq \left(\frac{\partial^2 f}{\partial \alpha \partial \beta}\right)^2, \tag{7}$$

the unique solution is

$$\alpha = R, \quad \beta = Q. \tag{8}$$

This follows that substituting back this solution into Eq. (4) immediately produces action (2), which implies that the two theories are dynamically equivalent.

Hence, two new scalar fields and a potential are defined as

$$\phi_1 = \frac{\partial f}{\partial \alpha}, \quad \phi_2 = \frac{\partial f}{\partial \beta},$$
$$V(\phi_1, \phi_2) = f - \alpha\phi_1 - \beta\phi_2, \tag{9}$$

where α and β are considered as functions of ϕ_1 and ϕ_2 in the potential V. According to these new variables, action (4) can be rewritten as

$$S = \int d^4x\sqrt{-g}[\phi_1 R + \phi_2 Q + V(\phi_1, \phi_2)], \tag{10}$$

which is similar to the action in Ref. [44] in form, while the definitions of the scalar fields ϕ_1, ϕ_2 and the potential V are here different from ones in Ref. [44].

Now we consider the variation of Eq. (10) with respect to ϕ_2 to integrate out the scalar field ϕ_2, which yields

$$Q + \frac{\partial V}{\partial \phi_2} = 0. \tag{11}$$

We assume

$$\phi_2^* = F(\phi_1, Q), \tag{12}$$

which is a solution to Eq. (11) with F functions of ϕ_1 and $L_{\rm m}$. Inserting Eq. (12) back into action (10) gives

$$S = \int d^4x\sqrt{-g}[\phi_1 R + P(Q, \phi_1)], \tag{13}$$

where

$$P(Q, \phi_1) = QF + V(\phi_1, F). \tag{14}$$

This follows that P is related to the interaction between matter and geometry as well as the interactional potential V between ϕ_1 and ϕ_2.

In Eq. (2), one can in principle consider any kind of matter Lagrangian. To continue our analysis we are thus free to choose any matter Lagrangian for convenience. In what follows, we consider $L_{\rm m}$ to be the Lagrangian of a matter scalar field, $\phi_{\rm m}$

$$L_{\rm m} \equiv X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\rm m}\partial_\nu\phi_{\rm m}. \tag{15}$$

Note that assuming $\partial_\mu\phi_{\rm m}$ to be time-like implies $X > 0$. Thus the energy density of the scalar field and the speed of sound are given by[46,47]

$$\epsilon = 2XP_{,X} - P, \quad c_{\rm s}^2 = \frac{P_{,X}}{2XP_{,XX} + P_{,X}}, \tag{16}$$

where $_{,X}$ denotes the differentiation with respect to X, P is functions of X and ϕ, and ϵ is the energy density of the scalar field. It should be positive, while $c_{\rm s}$ is the speed of sound, which should not be faster than the speed of light. Therefore, we should impose the further condition

$$\epsilon > 0, \quad 1 \geq c_{\rm s}^2 \geq 0, \tag{17}$$

which must be satisfied by every physically viable model. As we will see, once we choose a particular generalized $f(R)$ model in action (2), the conditions (17) will be used to constrain the free parameters of the theory.

Specifically, in the following we consider the two particular models of generalized $f(R)$ in Eq. (2). *Model 1*:

$$f_1(R) = \frac{R}{16\pi G}, \quad f_2(R) = R^n, \quad Q(L_{\rm m}) = L_{\rm m}^\gamma, \tag{18}$$

where n and γ are two free real parameters. According to Eq. (18), the corresponding fields ϕ_1, ϕ_2 and the potential $V(\phi_1, \phi_2)$ to model 1 can be given by

$$\phi_1 = M_{\rm pl}^2 + n\alpha^{n-1}\beta, \quad \phi_2 = \alpha^n, \quad V = (M_{\rm pl}^2 - \phi_1)\phi_2^{\frac{1}{n}}, \tag{19}$$

where $M_{\rm pl}^2 = \frac{1}{16\pi G}$ is the Planck mass. At this point we must find a solution to Eq. (11), in this case it reads

$$L_{\rm m}^\gamma + (M_{\rm pl}^2 - \phi_1)\phi_2^{\frac{1-n}{n}} = 0. \tag{20}$$

The solution to Eq. (20) can be obtained as follows:

$$\phi_2^* = \left(\frac{L_{\rm m}^\gamma}{\phi_1 - M_{\rm pl}^2}\right)^{\frac{n}{1-n}}, \quad (n \neq 1). \tag{21}$$

Furthermore, according to Eq. (14) we can obtain

$$\begin{aligned} P &= L_{\rm m}^\gamma\left(\frac{L_{\rm m}^\gamma}{\phi_1 - M_{\rm pl}^2}\right)^{\frac{n}{1-n}} \\ &\quad - n(\phi_1 - M_{\rm pl}^2)\left[\left(\frac{L_{\rm m}^\gamma}{\phi_1 - M_{\rm pl}^2}\right)^{\frac{n}{1-n}}\right]^{\frac{1}{n}} \\ &= (\phi_1 - M_{\rm pl}^2)\frac{n}{n-1}L_{\rm m}^{\frac{\gamma}{1-n}} - n(\phi_1 - M_{\rm pl}^2)^{\frac{n}{n-1}}L_{\rm m}^{\frac{\gamma}{1-n}} \\ &= (1-n)(\phi_1 - M_{\rm pl}^2)^{\frac{n}{n-1}}X^{\frac{\gamma}{1-n}}. \end{aligned} \tag{22}$$

Hence, we have

$$P_{,X} = \gamma(\phi_1 - M_{\rm pl}^2)^{\frac{n}{n-1}} X^{\frac{\gamma+n-1}{1-n}}, \tag{23}$$

$$P_{,XX} = \frac{\gamma+n-1}{1-n}\gamma(\phi_1 - M_{\rm pl}^2)^{\frac{n}{n-1}} X^{\frac{\gamma+2n-2}{1-n}}. \tag{24}$$

Therefore, by means of Eq. (16) we can obtain the results

$$\begin{aligned}\epsilon &= (2\gamma - 1 + n)(\phi_1 - M_{\rm pl}^2)^{\frac{n}{n-1}} X^{\frac{\gamma}{1-n}} \\ &= \frac{2\gamma - 1 + n}{1-n} P,\end{aligned} \tag{25}$$

$$\begin{aligned}c_{\rm s}^2 &= \frac{X^{\frac{\gamma+n-1}{1-n}}}{2X\left(\frac{\gamma+n-1}{1-n}\right)X^{\frac{\gamma+2n-2}{1-n}} + X^{\frac{\gamma+n-1}{1-n}}} \\ &= \frac{1-n}{2\gamma+n-1}, \ (n \neq 1).\end{aligned} \tag{26}$$

Specifically, if we let $\gamma = 1$ (i.e., $Q(L_{\rm m}) = L_{\rm m}$), we can obtain

$$\epsilon = \frac{1+n}{1-n} P, \quad c_{\rm s}^2 = \frac{1-n}{1+n}. \tag{27}$$

Evidently, Eq. (27) is just the result given in Ref. [44].

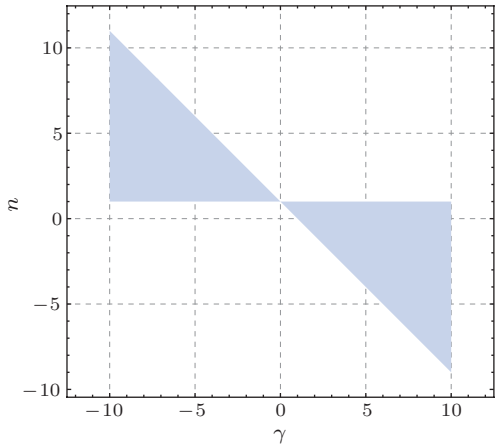

Fig. 1. The constraints of the consistency conditions on the parameters n and γ in the model with $f_1 + f_2 Q(L_{\rm m}) = \frac{R}{16\pi G} + R^n L_{\rm m}^{\gamma}$.

To avoid superluminal signals, physically we require $1 \geq c_{\rm s}^2 \geq 0$. The constraints on the parameters n and γ is illustrated in Fig. 1, from which we can see that the value ranges of the parameters of n and γ are divided into the two fields. One is in the field of $\gamma < 0$ and $1 < n \leq 1 - \gamma$, and the other is $\gamma > 0$ and $1 - \gamma \leq n < 1$. In other words, the constraints on the parameters γ and n for model 1 are

$$\begin{aligned}&\gamma < 0, \quad 1 < n \leq 1 - \gamma, \\ &\gamma > 0, \quad 1 - \gamma \leq n < 1.\end{aligned} \tag{28}$$

In addition, according to the DK stability criterion (3), we can obtain that the corresponding result of model 1 is as follows:

$$n(n-1)R^{n-2}L_{\rm m}^{\gamma} \geq 0, \tag{29}$$

where $L_{\rm m} = X > 0$ and n is an integer, while $n \neq 1$. If we consider $R > 0$,[48] we have $n(n-1) \geq 0$ in Eq. (29). In this case, we obtain

$$n \leq 0 \ \text{ or } \ n > 1. \tag{30}$$

It follows that the corresponding DK stability criterion to Eq. (30) has no restriction on the parameter γ, while it gives new constraints on the parameter n when $\gamma > 0$. Thus by combining the consistency condition (28) with the DK stability criterion (30) we can finally obtain the constraints on the parameters γ and n as follows:

$$\begin{aligned}&\gamma < 0, \quad 1 < n \leq 1 - \gamma, \\ &\gamma \geq 1, \quad 1 - \gamma \leq n \leq 0.\end{aligned} \tag{31}$$

Finally, we illustrate the constraints on the parameters n and γ in Fig. 2.

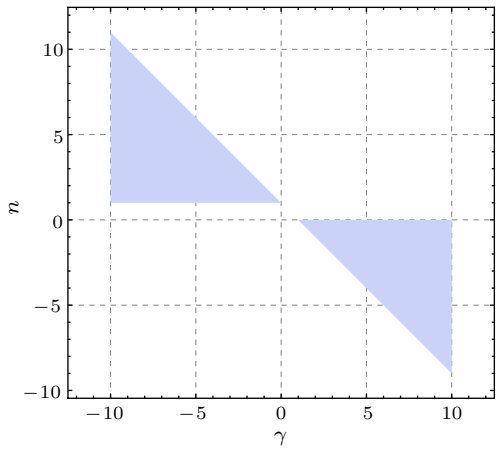

Fig. 2. The constraints of the consistency conditions and DK stability criterion on the parameters n and γ in the model with $f_1 + f_2 Q(L_{\rm m}) = \frac{R}{16\pi G} + R^n L_{\rm m}^{\gamma}$.

Model 2:

$$f_1(R) = \frac{R}{16\pi G}, \ f_2(R) = e^{\lambda R}, \ Q(L_{\rm m}) = L_{\rm m}^{\gamma}, \tag{32}$$

where λ and γ are the two free parameters. The two scalar fields and the potential can be given by

$$\phi_1 = M_{\rm pl}^2 + \lambda e^{\lambda\alpha}\beta, \ \phi_2 = e^{\lambda\alpha}, \ V = -\frac{\phi_1 - M_{\rm pl}^2}{\lambda}\ln\phi_2, \tag{33}$$

and Eq. (11) reads

$$L_{\rm m}^{\gamma} + \frac{M_{\rm pl}^2 - \phi_1}{\lambda\phi_2} = 0, \tag{34}$$

with solution

$$\phi_2^* = \frac{\phi_1 - M_{\rm pl}^2}{\lambda L_{\rm m}^{\gamma}}. \tag{35}$$

At this point we derive P as

$$\begin{aligned} P &= L_{\rm m}^{\gamma}\cdot\frac{\phi_1-M_{\rm pl}^2}{\lambda L_{\rm m}^{\gamma}}-\frac{\phi_1-M_{\rm pl}^2}{\lambda}\ln\phi_2 \\ &= \frac{\phi_1-M_{\rm pl}^2}{\lambda}(1-\ln\phi_2) \\ &= \frac{\phi_1-M_{\rm pl}^2}{\lambda}\Big(1+\ln\frac{\lambda X^{\gamma}}{\phi_1-M_{\rm pl}^2}\Big). \end{aligned} \tag{36}$$

Finally it is enough to evaluate the condition on the sound speed to realize that this model is immediately ruled out. In fact from Eq. (16) we find

$$c_{\rm s}^2=\frac{\frac{1}{X}}{2X\Big(-\frac{1}{X^2}\Big)+\frac{1}{X}}=-1, \tag{37}$$

which is obviously non-physical.

In summary, we have studied the consistency conditions of the generalized $f(R)$ gravity by extending $f(R)$ with non-minimal coupling to the generalized $f(R)$ gravity with arbitrary geometry-matter coupling. Specifically, by means of consistency conditions we have discussed the constraints on the parameters of the two particular models of the generalized $f(R)$. The research results show that model 2 is not physically viable so as to be ruled out. Moreover, we have further constrained model 1 using the DK stability criterion, and have given the value ranges of the parameters in model 1, i.e., $\gamma<0$, $1<n\leq 1-\gamma$ and $\gamma\geq 1$, $1-\gamma\leq n\leq 0$. It is worth stressing that our results include the ones in Ref. [44] as the special case of $Q(L_{\rm m})=L_{\rm m}$.

References

[1] Riess A G, Filippenko A V, Challis P et al 1998 *Astron. J.* **116** 1009
[2] Perlmutter S, Aldering G, Goldhaber G et al 1999 *Astron. J.* **517** 565
[3] Caldwell R R, Dave R and Steinhardt P J 1998 *Phys. Rev. Lett.* **80** 1582
[4] Sen A 2002 *J. High Energy Phys.* **0207** 065
[5] Li M 2004 *Phys. Lett.* B **603** 1
[6] Huang Q G and Li M 2004 *J. Cosmol. Astropart. Phys.* **2004** 013
[7] Gong Y G 2004 *Phys. Rev.* D **70** 064029
[8] Zhu Z 2004 *Astron. Astrophys.* **423** 421
[9] Cai R G and Wang A 2005 *J. Cosmol. Astropart. Phys.* **0503** 002
[10] Cai R G 2007 *Phys. Lett.* B **657** 228
[11] Wang B, Gong Y and Su R K 2005 *Phys. Lett.* B **605** 9
[12] Zhao W and Zhang Y 2006 *Phys. Rev.* D **73** 123509
[13] Hu B and Ling Y 2006 *Phys. Rev.* D **73** 123510
[14] Zhang X 2007 *Phys. Lett.* B **648** 1
[15] Wei H and Cai R G 2008 *Phys. Lett.* B **660** 113
[16] Zhu W T, Wu P X and Yu H W 2013 *Chin. Phys. Lett.* **30** 059501
[17] Wu Y B et al 2014 *Chin. Phys. Lett.* **31** 029801
[18] Brans C and Dicke R H 1961 *Phys. Rev.* **124** 925
[19] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
[20] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370
[21] Dvali G et al 2000 *Phys. Lett.* B **485** 208
[22] Carroll S M 2001 *Living Rev. Relativ.* **4** 1
[23] Maartens R 2004 *Living Rev. Relativ.* **7** 7
[24] Carroll S M et al 2005 *Phys. Rev.* D **71** 063513
[25] Zhang X and Ma Y 2011 *Phys. Rev. Lett.* **106** 171301
[26] Meng X H and Wang Y B 2011 *Eur. Phys. J.* C **71** 1755
[27] Huang Q G, Zhang K C and Zhou S Y 2013 *J. Cosmol. Astropart. Phys.* **1308** 050
[28] Cai R G, Yang Q and Zhang Y L 2014 *J. High Energy Phys.* **1412** 147
[29] Wu Y B et al 2013 *Chin. Phys. Lett.* **30** 069801
[30] Sotiriou T P and Faraoni V 2010 *Rev. Mod. Phys.* **82** 451
[31] Nojiri S and Odintsov S D 2006 *Phys. Rev.* D **74** 086005
[32] Santos J et al 2007 *Phys. Rev.* D **76** 083513
[33] Bertolami O et al 2007 *Phys. Rev.* D **75** 104016
[34] Koivisto T 2006 *Class. Quantum Grav.* **23** 4289
[35] Harko T 2008 *Phys. Lett.* B **669** 376
[36] Bergliaffa S E P 2006 *Phys. Lett.* B **642** 311
[37] Wang J et al 2010 *Phys. Lett.* B **689** 133
[38] Wang J et al 2010 *Eur. Phys. J.* C **69** 541
[39] Harko T and Lobo F S N 2010 *Eur. Phys. J.* C **70** 373
[40] Harko T and Lobo F S N 2012 *Phys. Rev.* D **86** 124034
[41] Harko T, Lobo F S N and Minazzoli O 2013 *Phys. Rev.* D **87** 047501
[42] Wang J and Liao K 2012 *Class. Quantum Grav.* **29** 215016
[43] Wu Y B et al 2014 *Mod. Phys. Lett.* A **29** 1450089
[44] Tamanini N and Koivisto T S 2013 arXiv:1308.3401v1 [gr-qc]
[45] Bertolami O and Paramos J 2008 *Class. Quantum Grav.* **25** 245017
[46] Garriga J and Mukhanov V F 1999 *Phys. Lett.* B **458** 219
[47] Armendariz-Picon C, Mukhanov V and Steinhardt P J 2001 *Phys. Rev.* D **63** 103510
[48] Albareti F D et al 2013 *J. Cosmol. Astropart. Phys.* **2013** 009