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Electron–Cyclotron Laser Using Free-Electron Two-Quantum Stark Radiation in a Strong Uniform Axial Magnetic Field and an Alternating Axial Electric Field in a Voltage-Supplied Pill-Box Cavity

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We consider the radiation from the beam electrons traveling in a strong uniform axial magnetic field and an axial alternating electric field of wavelength λ_w generated by a voltage-supplied pill-box cavity. The beam electrons emit genuine laser radiation that propagates only in the axial direction through free-electron two-quantum Stark radiation. We find that laser radiation takes place only at the expense of the axial kinetic energy when $\lambda_w \ll c/(\omega_c/\gamma)$, where ω_c/γ is the relativistic electron–cyclotron frequency. We formulate the laser power based on quantum-wiggler electrodynamics, and envision a laser of length 10 m with estimated power 0.1 GW/(kA) in the 10^{-4} cm wavelength range.

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A laser, which is originally an acronym of 'light amplification by stimulated emission of radiation', now refers to a device which generates a strong monochromatic axially propagating radiation without a waveguide support. A bound electron emits radiation into a wide solid angle through spontaneous radiation, and amplifies the incident radiation through stimulated emission so that the radiation power increases exponentially with its travel distance in the bound-electron medium. As a result, amplification becomes predominantly greatest in the axial direction. The spectral width of bound-electron laser light depends on the width of the energy level, and can be approximated to be monochromatic. The spectral width of free-electron laser depends on the periodicity of the wiggler. Unlike bound-electron spontaneous radiation, free-electron radiation is emitted only in a particular direction, e.g., only in the axial direction, and the spectral width can be narrow, depending on the modal spectrum of the wiggler. Even though the density of free-electron emitters is smaller than that of bound-electron emitters, free-electron radiation takes place in a large number of cascade processes.[1] Accordingly, even though free-electron radiation may not be stimulated emission, the radiation power of free-electron radiation can be comparable or even much greater than that of bound-electron radiation. Hence, a free-electron radiation device can be a laser.

All strong free-electron radiations which cannot be accounted for with classical electrodynamics or conventional quantum electrodynamics are of free-electron two-quantum Stark (FETQS) radiation.[1–7] The free-electron laser (FEL) is a device utilizing FETQS radiation. The FEL has a tremendous advantage compared with the bound-electron laser due to the fact that the former does not require any pumping mechanism.[1,2] The device commonly called the FEL uses a plane-polarized or helical magnetic wiggler, which is not cylindrically symmetric, as the predominant field to make the electrons radiate. However, even from the common sensical concept, we realize that the beam electrons in such a device can emit photons only in the oblique direction.

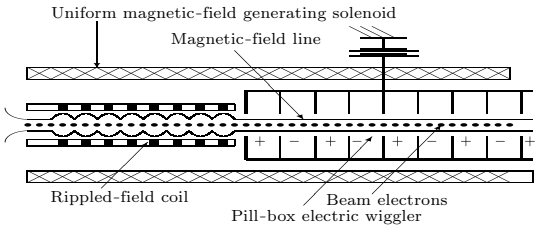

Fig. 1. Schematic diagram of an electron–cyclotron laser by using the voltage-supplied pill-box electric wiggler.

If the radiation produced by a device travels only in the axial direction, the field configuration in the region where the laser radiation takes place must be cylindrically symmetric such as the electric-wiggler region in Fig. 1. The cascade of anode–cathode plates takes the shape of a circular pill-box cavity.[8] The electron–cyclotron maser (ECM)[9–14] uses a uniform axial magnetic field as the zeroth-order field which exerts the electrons to radiate only in the axial direction. The reason why we do not call the ECM the electron–cyclotron laser (ECL) is that the radiation in such a device is mostly in the microwave region and covers a broad band of wavelength, that is, radiation is not monochromatic.

The potential of the artificially made (or externally-produced) electric field in the pill-box cav-

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ity can be split into two terms,

$$\phi(\boldsymbol{r},t)^{\rm ex}=\phi_{\rm o}\cos(k_{\rm w}z)+\phi_{\rm g}(\boldsymbol{r}), \tag{1}$$

where $\phi_{\rm o}\gg\phi_{\rm g}$, and the electric field represented by $-\boldsymbol{\nabla}\phi_{\rm g}(\boldsymbol{r})$ is the unwanted, while technically unavoidable electric field predominantly acting in the transverse direction, which causes generation of useless radiation ('garbage radiation'). In the ECL, radiation is emitted mainly in the axial direction on the grounds that both the uniform magnetic field and the field of the electric wiggler lie in the axial direction. Due to the fact that $\phi_{\rm o}\gg\phi_{\rm g}$, the radiation in the axial direction is practically intact in the presence of the mostly transverse electric field represented by $\phi_{\rm g}(\boldsymbol{r})$.

Unlike the bound electron, the momentum and energy of a free electron are inter-related through $E=(m^2c^4+c^2\boldsymbol{p}^2)^{1/2}$. Hence, one of Heisenberg's uncertainty principles, $\Delta E\Delta t\sim h$, does not apply to the free electron. The uncertainty in the electron axial position at which the electron is stimulated by the incident radiation of wavelength λ is of the order of λ. Hence, from the other uncertainty principle, i.e., $\Delta p_z\Delta z\sim h$, the uncertainty in the electron momentum during the stimulation is given by $\Delta p_z\sim h/\lambda$. Then we find that the uncertainty in the electron energy during the stimulation is given by

$$\Delta E=\Delta(m^2c^4+c^2p_z^2+c^2p_\perp^2)^{1/2}\sim\frac{c^2p_z\Delta p_z}{E}\sim h\nu, \tag{2}$$

where the fact that $p_z/E\to1/c$ as $\gamma\to\infty$ has been used. Accordingly, during the stimulation, the electron does not know whether it is at the upper level or the lower level of the transition. Hence, the electron will emit the photon of frequency ν or will absorb the incident photon on the random probability. This means that we cannot see net stimulated radiation of frequency ν, and the electron displays doing only spontaneous radiation, which has nothing to do with any external influence.

The spin state (up, or down) is conserved in the transitions by the electromagnetic wave, the magnetic wiggler, spontaneous radiation, and the electric wiggler (the so-called spin $\uparrow\to\downarrow$ and $\downarrow\to\uparrow$ transitions do not occur). The electron can be either spin-up or spin-down state with equal probability. Hence, the state of an electron which gyrates around an axial uniform magnetic field without any emission can be expressed as

$$\begin{aligned}\Phi(\boldsymbol{r},t|l,p_z)=&\varphi(\boldsymbol{r}_\perp|l,p_z)\exp\left(\frac{ip_zz}{\hbar}\right)\\&\times\exp\left(\frac{-iE(l,p_z)t}{\hbar}\right),\end{aligned} \tag{3}$$

where $\varphi(\boldsymbol{r}_\perp|l,p_z)$ can be represented by

$$\varphi(\boldsymbol{r}_\perp|l,p_z)=G(l,p_z,)\begin{bmatrix}[mc^2+E(l,p_z)]\psi_{l-1}(\boldsymbol{r}_\perp)\\0\\p_zc\psi_{l-1}(\boldsymbol{r}_\perp)\\\epsilon l^{1/2}\psi_l(\boldsymbol{r}_\perp)\end{bmatrix}. \tag{4}$$

Here the meanings of the notations can be found in Refs. [1,2,6].

In the following the interaction Hamiltonian is expressed in a way of operating only on the electron state which is represented by $\Phi(\boldsymbol{r},t|l,p_z)$ such that we need not specify the radiation state. Let the potential of the electric wiggler be written as

$$\phi(\boldsymbol{r},t)=\phi_{\rm o}\cos(k_{\rm w}z). \tag{5}$$

Then the Hamiltonian for the interaction of the test electron with the electric wiggler whose potential is given by Eq. (5) can be written as

$$\mathcal{H}_{\rm w}(\boldsymbol{r})=\mathcal{H}_{\rm w}^{\rm o}\exp(-ik_wz)+S(-k_{\rm w}), \tag{6}$$

where $\mathcal{H}_{\rm w}^{\rm o}=e\phi_{\rm o}/2$, and $S(-x)$ is of the same form as the proceeding term except that x is replaced by $-x$.

We choose spherical coordinates such that the wave vector and polarization of radiation can be expressed as

$$\begin{aligned}\hat{\boldsymbol{k}}(\theta,\psi)=&\boldsymbol{k}(\theta,\psi)/k=\sin\theta\cos\psi\hat{\boldsymbol{x}}\\&+\sin\theta\sin\psi\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}},\end{aligned} \tag{7}$$

$$\hat{\varepsilon_1}(\theta,\psi)=-\sin\psi\hat{\boldsymbol{x}}+\cos\psi\hat{\boldsymbol{y}}, \tag{8}$$

$$\hat{\varepsilon_2}(\theta,\psi)=-\cos\theta\cos\psi\hat{\boldsymbol{x}}-\cos\theta\sin\psi\hat{\boldsymbol{y}}+\sin\theta\hat{\boldsymbol{z}}. \tag{9}$$

The Hamiltonian for spontaneous emission of a photon of wave vector $\boldsymbol{k}(\theta,\psi)$ polarized in the $\hat{\varepsilon_\xi}$ (ξ=1 or 2) direction can be written as[15]

$$\mathcal{H}_\xi(\boldsymbol{r},t)=\mathcal{H}_\xi^{\rm o}\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)], \tag{10}$$

where

$$\begin{aligned}&\boldsymbol{k}=k\hat{\boldsymbol{k}}(\theta,\psi),\quad\omega=ck,\\&\mathcal{H}_\xi^{\rm o}=e(\hat{\varepsilon_\xi}\cdot\boldsymbol{\alpha})\left(\frac{2\pi\hbar c}{k\mathcal{V}}\right)^{1/2},\quad\boldsymbol{\alpha}=\begin{bmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{bmatrix},\end{aligned}$$

and $\boldsymbol{\sigma}$ is the Pauli spin-matrix vector.

The probability amplitude of the transition from state 1 with the Landau quantum number l_1 and the z-momentum p_{1z} to state 2 with the Landau quantum number l_2 and the z-momentum p_{2z} by spontaneous FETQS emission of a photon of wave-vector $\boldsymbol{k}$, polarized in parallel to the ξ-direction, at time T, can be

written as

$$A(1\to 2|\boldsymbol{k},\xi:T)=\frac{\mathcal{H}^{\rm o}_{\rm w}}{(i\hbar)^2}\sum_i\int_0^T dt\int_0^t dt'(\exp(i\,\Upsilon_1)$$
$$\times\langle\Phi(\boldsymbol{r},t|l_2,p_{2z})|\exp(-ik_w z)|\Phi(\boldsymbol{r},t|l_i,p_{iz})\rangle$$
$$\times\langle\Phi(\boldsymbol{r},t'|l_i,p_{iz})|\mathcal{H}^{\rm o}_\xi\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t')]$$
$$\times|\Phi(\boldsymbol{r},t'|l_1,p_{1z})\rangle+\exp(i\,\Upsilon_2)$$
$$\times\langle\Phi(\boldsymbol{r},t|l_2,p_{2z})|\mathcal{H}^{\rm o}_\xi\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$$
$$\times|\Phi(\boldsymbol{r},t|l_i,p_{iz})\rangle$$
$$\times\langle\Phi(\boldsymbol{r},t'|l_i,p_{iz})|\exp(-ik_w z)|\Phi(\boldsymbol{r},t'|l_1,p_{1z})\rangle)$$
$$+S(-\boldsymbol{k}_{\rm w})=\delta_{k_y,0}f(k_x,k_z,l_1,p_{1z}),\qquad(11)$$

where Υ_1 and Υ_2 are two mutually independent random phases.[5–7] Since any direction perpendicular to the waveguide axis can be the y-direction, $k_y=0$ means $k_\perp=0$, and hence $\boldsymbol{k}=k\hat{\boldsymbol{z}}$.

Consistent with $\boldsymbol{k}=k\hat{\boldsymbol{z}}$, the volume $\mathcal{V}$, where the photon emitted from an electron can be found, must be equal to $L_z r_{\rm g}^2$, with L_z being the z size of the space in which the emitted photon can be found, and $r_{\rm g}$ the gyration radius. Taking these facts into consideration, Eq. (11) can be rewritten as

$$A(1\to 2|k\hat{\boldsymbol{z}},\xi:T)=-\frac{e^2\phi_{\rm o}\beta_\perp}{4\hbar^2 r_{\rm g}}\Big(\frac{2\hbar}{ckL_z}\Big)^{1/2}$$
$$\times\delta_{p_{2z},p_{1z}-\hbar k-\hbar k_{\rm w}}$$
$$\times\Big[\delta_{l_2,l_1+1}\int_0^T\exp\{ic[(1-\beta_z)k-\beta_z k_{\rm w}+\omega_{\rm c}/\gamma]t\}dt$$
$$\times\Big(\frac{\exp(i\,\Upsilon_1)}{(1-\beta_z)k+\omega_{\rm c}/c\gamma}+\frac{\exp(i\,\Upsilon_2)}{(-\beta_z k_{\rm w})}\Big)$$
$$+\delta_{l_2,l_1-1}\int_0^T\exp\{ic[(1-\beta_z)k-\beta_z k_{\rm w}-\omega_{\rm c}/c\gamma]t\}dt$$
$$\times\Big(\frac{\exp(i\,\Upsilon_1)}{(1-\beta_z)k-\omega_{\rm c}/c\gamma}+\frac{\exp(i\,\Upsilon_2)}{(-\beta_z k_{\rm w})}\Big)\Big]+S(-k_{\rm w}).\qquad(12)$$

We can find from Eq. (12) that the transition rate from state 1 with the Landau quantum number l_1 and the z-momentum p_{1z} to state 2 with the Landau quantum number l_2 and the z-momentum p_{2z} by spontaneous FETQS emission of a photon of wave-vector $k\hat{\boldsymbol{z}}$ can be written as

$$\tau(1\to 2|k)=\frac{\sum_{\xi=1}^2|A(1\to 2|k\hat{\boldsymbol{z}},\xi:T)|^2}{T}$$
$$=\frac{\pi\alpha^2\phi_{\rm o}^2\beta_\perp^2}{\hbar r_{\rm g}^2 k\beta_z^2 k_{\rm w}^2(1-\beta_z)L_z}\delta_{p_{2z},p_{1z}-\hbar k-\hbar k_{\rm w}}$$
$$\times\Big[\delta_{l_2,l_1+1}\delta\Big(k-\frac{\beta_z k_{\rm w}-\omega_{\rm c}/c\gamma}{(1-\beta_z)}\Big)$$
$$+\delta_{l_2,l_1-1}\delta\Big(k-\frac{\beta_z k_{\rm w}+\omega_{\rm c}/c\gamma}{(1-\beta_z)}\Big)\Big]+S(-k_{\rm w}),\qquad(13)$$

where $\alpha=e^2/\hbar c=1/137$ is the fine-structure constant. Equation (13) shows that when $\omega_{\rm c}/c\gamma\gg k_{\rm w}$, radiation takes place with a change in the transverse kinetic energy. However, when $k_{\rm w}\gg\omega_{\rm c}/c\gamma$, the term proportional to δ_{l_2,l_1-1} is of nearly equal weight as the term proportional to δ_{l_2,l_1+1} and hence, radiation does not produce any change in the transverse kinetic energy on probabilistic average.

In most practical ECM situations, $\beta_\perp\ll 1/\gamma$ holds so that $1-\beta_z\approx 1/2\gamma^2$ is satisfied. In such a case, we find from Eq. (13) that the radiation wavelength is given by

$$\lambda=\frac{\lambda_{\rm w}}{2\gamma^2}.\qquad(14)$$

For a given wiggler wavenumber $k_{\rm w}$ of the axial wiggler, the z-momentum state is in one-to-one correspondence with the wave-vector state of the emitted photon by $p_{2z}=p_{1z}-\hbar k-\hbar k_{\rm w}$. The number of the wave-vector state between k and $k+dk$ is $L_z dk/2\pi$. The transverse velocity $\boldsymbol{v}_\perp$ consists of the gyration velocity around the magnetic field line $\boldsymbol{v}_{\rm g}$ and the rotation velocity $\boldsymbol{v}_{\rm w}$ around the waveguide axis, i.e., $\boldsymbol{v}_\perp=\boldsymbol{v}_{\rm g}+\boldsymbol{v}_r$.[2] Any tendency to increase the magnitude of the gyration radius is inhibited by the binary electron–electron collision so that the gyration radius cannot be much longer than the mean inter-electron distance $l_{\rm e-e}\approx n^{-1/3}$. Note that the momentum transfer through the binary electron–electron collision causes the increase of the drift velocity around the waveguide axis, which is hydrodynamic in nature, that is, of the average quantity over the electrons about the gyration center whereas the gyration velocity is of single-electron nature. Hence, we can assume that the gyration radius is equal to the mean inter-electron distance, i.e., $r_{\rm g}=n^{-1/3}$. In this work, we are primarily concerned with the short-wavelength laser. Hence, we assume that $N_{\rm c}=\lambda n^{1/3}\lesssim 1$. Then laser radiation takes place as a single-photon process (Eq. (4) of Ref. [16]). Using these facts, assumption and $r_{\rm g}=\gamma v_{\rm g}/\omega_{\rm c}$, the radiation power from an electron with momentum $\boldsymbol{p}_1$ can be written as

$$P(\boldsymbol{p}_1)=\sum_{l_2,p_{2z}}\tau(1\to 2|k)\hbar ck$$
$$=\sum_{l_2}\int dk\,\frac{\tau(1\to 2|k)\hbar ckL_z}{2\pi}=\frac{2\alpha^2\phi_{\rm o}^2\omega_{\rm c}^2\beta_\perp^2}{ck_{\rm w}^2\beta_{\rm g}^2}$$
$$=\frac{2\alpha^2 c\phi_{\rm o}^2 n^{2/3}\gamma^2\beta_\perp^2}{k_{\rm w}^2},\qquad(15)$$

which shows that the radiation power from an electron is proportional to its transverse kinetic energy. This fact has been firmly verified and most ECMs used various methods to increase the transverse energy, that is, a periodic ripple in B_z as in Fig. 1,[9,10,12] a rising B_z at the electron emitting surface,[11] and passage of the electron beam through a thick scattering foil.[13]

Once the electron has sufficient transverse energy, it need not be replenished as long as $k_w \gg \omega_c/c\gamma$ holds.

When the uncertainty in the electron energy, ΔE, is not much larger than $h\nu$, which holds for short-wavelength radiation of $N_c \lesssim 1$, the maximum achievable radiation power from an electron is given (Eq. (4) of Ref. [16] or Eq. (1.3) of Ref. [1]) as follows:

$$P^{\max} \sim h\nu^2 = hc^2/\lambda^2 = \frac{4hc^2\gamma^4}{\lambda_w^2}, \tag{16}$$

where Eq. (14) has been used. Equation (16) shows $P^{\max} \propto \gamma^4$. Thus this fact manifests the reason why we use highly relativistic electron beams for building the high-power laser.

Equations (15) and (16) indicate that the profile of the radiation power P from an electron versus the wiggler potential ϕ_o will show the plateau region starting at ϕ_o^{plateau} given by

$$\phi_o^{\text{plateau}}(\gamma) = \frac{2\sqrt{2}\pi(hc)^{1/2}\gamma}{\alpha n^{1/3}\beta_\perp \lambda_w^2}. \tag{17}$$

From Eq. (17), we find that if $\lambda_w = 1\,\text{cm}$, $\gamma = 50$ ($E = 24.5\,\text{MeV}$), $\omega_c = 8.8 \times 10^{10}\,\text{rad/s}$ ($B = 5\,\text{kG}$), $\beta_\perp = 0.0001$, then $\phi_o^{\text{plateau}} = 12\,\text{V}$.

From the above we find that if $N_c \lesssim 1$, the laser power $\mathcal{P} = N_t P$, where N_t is the total number of electrons, is given by

$$\mathcal{P}(\phi_o, \gamma) = \begin{cases} \frac{2\alpha^2 (J/e)^{5/3}\gamma^2\beta_\perp^2 L_c \phi_o^2}{c^{2/3} S_b^{2/3} k_w^2}, & \phi_o < \phi_o^{\text{plateau}}(\gamma), \\ \frac{4hc(J/e)L_c\gamma^4}{\lambda_w^2}, & \text{otherwise}, \end{cases} \tag{18}$$

where L_c is the total length of pill-box cavity, i.e., the wiggler length, S_b is the electron-beam cross section, and J is the beam current. If $L_c = 10\,\text{m}$, $\mathcal{P}^{\max}$ per 1 kA is 124 MW. The energy extraction efficiency, which is defined as the ratio of the energy of the injected electron to the energy radiated from the electron, is written as $\eta = h\nu^2(L_c/c)/\gamma mc^2$. For the foregoing wiggler and beam parameters, η is 0.4.

In conclusion, we have found that highly relativistic electrons can radiate only through spontaneous emission. Since the axial uniform magnetic field and the pill-box longitudinal electrostatic field shown in Fig. 1 are cylindrically symmetric, we have conjectured that the electron beam shown in Fig. 1 emits a strong radiation in the axial direction through FETQS radiation, and a calculation by using QWD has shown that the electron beam would indeed do so. The radiation power is proportional to the electron's transverse kinetic energy. Hence, the beam electrons must increase their transverse kinetic energy before they enters into the lasing region. We have found that if $k_w \gg \omega_c/c\gamma$, the radiation energy does not come at the expense of the electron's transverse kinetic energy. We have considered the facts that the electron transverse motion consists of the gyration around the magnetic field line and the rotation around the waveguide axis, and the gyration radius cannot much exceed the mean inter-electron distance. It is found that with the present standard technological art of the beam and active pill-box construction, we can build an electron–cyclotron laser of manageable size ($\sim$10 m), which can yield the radiation power of 0.1 GW/(kA) in the wavelength range of 10^{-4} cm.

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