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A Relation of the Noncommutative Parameters in Generalized Noncommutative Phase Space *

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We introduce the deformed boson operators which satisfy a deformed boson algebra in some special types of generalized noncommutative phase space. Based on the deformed boson algebra, we construct coherent state representations. We calculate the variances of the coordinate operators on the coherent states and investigate the corresponding Heisenberg uncertainty relations. It is found that there are some restriction relations of the noncommutative parameters in these special types of noncommutative phase space.

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In recent years, there has been great interest in research of noncommutative space.[1−10] The ideas of noncommutative spacetime and field theories defined on such a structure started in 1947.[11] It came up again next in the 1980s, when Connes formulated the mathematically rigorous framework of noncommutative geometry.[12] A noncommutative spacetime also appeared in string theory, namely in the quantization of open string.[1] The noncommutativity of spacetime also plays an important role in quantum gravity.[13,14] The concept of noncommutative spacetime is also applied in condensed matter physics, such as the integer quantum Hall effect.[15] Since the noncommutativity between spatial and time coordinates may lead to some problems with unitary and causality,[16] usually only spatial noncommutativity is considered. Although in string theory only the coordinate space exhibits a noncommutative structure, some researchers have studied models in which a noncommutative geometry is defined on the whole phase space.[17−22] Noncommutativity between momenta arises naturally as a consequence of noncommutativity between positions, as momenta are defined to be the partial derivatives of the action with respect to the position coordinates.

One usually uses some transformations in phase space to change a quantum mechanical problem in the noncommutative phase space (NCPS) into a corresponding problem in the normal phase space. This is easy to carry out in lower-dimensional NCPS, while it is very complicated in the case of higher-dimension. In quantum mechanics, it is well known that sometimes it is much more convenient to solve the physical problem by using some special quantum representations. Since there is the noncommutativity among different position (or momentum) component operators, the usual position or momentum representations in quantum mechanics on noncommutative spaces do not exist. To study quantum systems in NCPS, one needs to construct some appropriate representations, for example, the coherent state representations. These new representations will help us to investigate the properties of NCPS. In fact, some researchers have already studied the coherent states in NCPS.[23−25] In Ref. [24], they used squeezed state representations to derive a restriction relation of the noncommutative parameters of 4D NCPS. In the present work, by virtue of the coherent state we derive some new relations of the noncommutative parameters in some special types of higher-dimensional generalized NCPS.

There are many possible types of higher-dimensional generalized NCPS. In general, the coordinate operators $\hat{x}_i$, $\hat{p}_i$ in $2n$-dimensional NCPS may satisfy the following extended Heisenberg algebra

$$[\hat{x}_i,\hat{p}_j]=i\delta_{ij}\hbar,\ [\hat{x}_i,\hat{x}_j]=i\mu_{ij},\ [\hat{p}_i,\hat{p}_j]=i\nu_{ij},\quad (1)$$

where μ_{ij} and ν_{ij} are real parameters, $\mu_{ij}=-\mu_{ji}$, and $\nu_{ij}=-\nu_{ji}$.

For simplicity and symmetry reasons, we only consider the following special type of NCPS in the following,

$$[\hat{x}_i,\hat{p}_j]=i\delta_{ij}\hbar,\ [\hat{x}_i,\hat{x}_j]=i\varepsilon_{ij}\mu,\ [\hat{p}_i,\hat{p}_j]=i\varepsilon_{ij}\nu,\quad (2)$$

where μ and ν are some positive constants, and (ε_{ij}) is some $n\times n$ skew-symmetric matrix.

Define the deformed boson operators

$$\hat{a}_i=\frac{1}{\sqrt{2\hbar}}\Big(\sqrt[4]{\frac{\nu}{\mu}}\hat{x}_i+i\sqrt[4]{\frac{\mu}{\nu}}\hat{p}_i\Big),\quad i=1,\ldots,n,\quad (3)$$

and their Hermitian conjugate operators $\hat{a}_i^\dagger$, which satisfy commutation relations

$$[\hat{a}_i,\hat{a}_i^\dagger]=1,\ [\hat{a}_i,\hat{a}_j^\dagger]=i\varepsilon_{ij}\theta,\quad i\neq j,\ i,j=1,\ldots,n,\quad (4)$$

where $\theta=\sqrt{\mu\nu}/\hbar$, and other commutators vanish. Here we would like to emphasize that the expressions in Eq. (3) only work for the case of μ and ν both

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nonzero, and this is just the situation we are considering. When $\mu = \nu = 0$, it will return to the case of normal commutative space, and Eq. (4) reduces to the ordinary boson algebraic relations. Let M be the matrix with entries $M_{ij} = \delta_{ij} + i\varepsilon_{ij}\theta$, then the relations (4) can be rewritten as

$$[\hat{a}_i, \hat{a}_j^\dagger] = M_{ij}, \quad i, j = 1, \ldots, n. \tag{5}$$

Since ε is a skew-symmetric matrix, one can use some orthogonal transformation to transform $\{\hat{a}_1, \ldots, \hat{a}_n\}$ into $\{\hat{A}_1, \ldots, \hat{A}_n\}$ (this is equivalent to applying the similar orthogonal transformations to transform the coordinates $\{x_1, \ldots, x_n\}$, $\{p_1, \ldots, p_n\}$ into $\{X_1, \ldots, X_n\}$, $\{P_1, \ldots, P_n\}$, respectively),[26] and the new operators $\hat{A}_i$, $\hat{A}_i^\dagger$ satisfy the following commutation relations

$$\begin{aligned} &[\hat{A}_i, \hat{A}_i^\dagger] = 1, \ [\hat{A}_1, \hat{A}_2^\dagger] = i\lambda_1\theta, \ [\hat{A}_3, \hat{A}_4^\dagger] = i\lambda_2\theta, \\ &\ldots, \ [\hat{A}_{2k-1}, \hat{A}_{2k}^\dagger] = i\lambda_k\theta, \ \left(k \le \frac{n}{2}\right), \end{aligned} \tag{6}$$

and other commutators vanish. Here $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k > 0$, and $\pm i\lambda_i$ are the nonzero eigenvalues of the matrix ε.

The corresponding new coordinate operators $\hat{X}_i$, $\hat{P}_i$ and $\hat{A}_i$ satisfy the relations similar to Eq. (3), and $\hat{X}_i$, $\hat{P}_i$ satisfy the following extended Heisenberg algebra

$$\begin{aligned} &[\hat{X}_i, \hat{P}_j] = i\delta_{ij}\hbar, \ [\hat{X}_1, \hat{X}_2] = i\lambda_1\mu, \\ &[\hat{X}_3, \hat{X}_4] = i\lambda_2\mu, \ldots, \ [\hat{X}_{2k-1}, \hat{X}_{2k}] = i\lambda_k\mu, \\ &[\hat{P}_1, \hat{P}_2] = i\lambda_1\nu, \ [\hat{P}_3, \hat{P}_4] = i\lambda_2\nu, \\ &\ldots, \ [\hat{P}_{2k-1}, \hat{P}_{2k}] = i\lambda_k\nu, \end{aligned} \tag{7}$$

and other commutators vanish.

Note that the transformations we have used here must be orthogonal, thus not all of λ_i will be zero. Of course one can use some transformation that is not orthogonal to make λ_i be zero, although that is not the case we will consider.

Define a unitary displacement operator in $2n$-dimensional NCPS as

$$\hat{D}_n(\alpha_1, \ldots, \alpha_n) = \exp(\sum_{i=1}^{n} \alpha_i \hat{a}_i^\dagger - \text{H.c.}), \tag{8}$$

where H.c. means Hermitian conjugate. Using the relations (4), after a straightforward calculation one can obtain

$$\hat{D}_n^\dagger \hat{a}_i \hat{D}_n = \hat{a}_i + \alpha_i + i\theta \sum_{j=1}^{n} \varepsilon_{ij}\alpha_j. \tag{9}$$

The relations (4) show that the annihilation operators $\hat{a}_i$ commute with each other, thus they have simultaneous eigenstates, e.g., the coherent states. Starting from the vacuum state $|0\rangle$, which is normalized and destroyed by the annihilation operators $\hat{a}_i$, and using the displacement operator Eq. (8), one can construct a coherent state in NCPS

$$|\alpha_1, \ldots, \alpha_n\rangle \equiv \hat{D}_n|0\rangle. \tag{10}$$

This coherent state is an eigenstate of $\hat{a}_i$,

$$\hat{a}_i|\alpha_1, \ldots, \alpha_n\rangle = (\alpha_i + i\theta \sum_{j=1}^{n} \varepsilon_{ij}\alpha_j)|\alpha_1, \ldots, \alpha_n\rangle, \tag{11}$$

and we also have

$$\begin{aligned} &\hat{a}_i^\dagger|\alpha_1, \ldots, \alpha_n\rangle \\ &= \Big(\frac{\partial}{\partial\alpha_i} + \frac{1}{2}(\alpha_i^* - i\theta \sum_{j=1}^{n} \varepsilon_{ij}\alpha_j^*)\Big)|\alpha_1, \ldots, \alpha_n\rangle. \end{aligned} \tag{12}$$

The inner product of two such coherent states is

$$\begin{aligned} &\langle\alpha_1', \ldots, \alpha_n'|\alpha_1, \ldots, \alpha_n\rangle \\ &= \exp\Big\{-\frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\alpha_i^*\alpha_j + \alpha_i'^*\alpha_j' - 2\alpha_i'^*\alpha_j)\Big\}. \end{aligned} \tag{13}$$

This inner product means that these coherent states are normalized, but not orthogonal to each other, and moreover, they are over-complete. The corresponding resolution of the identity is

$$\frac{1}{\pi^n}\int \prod_{i=1}^{n} d^2\alpha_i \, |\alpha_1, \ldots, \alpha_n\rangle \langle\alpha_1, \ldots, \alpha_n| = \frac{1}{\det M}, \tag{14}$$

where the integral is over the complex planes, $d^2\alpha_i = d\text{Re}(\alpha_i)\, d\text{Im}(\alpha_i)$, and $\det M = \prod_{i=1}^{k}(1 - \lambda_i^2\theta^2)$.

Now, let us investigate the variances of the operators $\hat{X}_i$ and $\hat{P}_i$ on the coherent state $|\alpha_1, \ldots, \alpha_n\rangle$. The expectation value of the operator $\hat{X}_1$ is

$$\langle\hat{X}_1\rangle = \sqrt{\frac{\hbar}{2}}\sqrt[4]{\frac{\mu}{\nu}}\langle\alpha_1, \ldots, \alpha_n|(\hat{A}_1 + \hat{A}_1^\dagger)|\alpha_1, \ldots, \alpha_n\rangle. \tag{15}$$

After a straightforward calculation, one can obtain

$$(\triangle\hat{X}_1)^2 \equiv \langle\hat{X}_1^2\rangle - \langle\hat{X}_1\rangle^2 = \frac{\hbar}{2}\sqrt{\frac{\mu}{\nu}}. \tag{16}$$

Similarly, we have

$$(\triangle\hat{X}_i)^2 = \frac{\hbar}{2}\sqrt{\frac{\mu}{\nu}}, \ (\triangle\hat{P}_i)^2 = \frac{\hbar}{2}\sqrt{\frac{\nu}{\mu}}. \tag{17}$$

Thus there is

$$(\triangle\hat{X}_1)^2(\triangle\hat{X}_2)^2 = \frac{\hbar^2}{4}\frac{\mu}{\nu}. \tag{18}$$

From the commutation relations (7), we have Heisenberg's uncertainty relation

$$\triangle\hat{X}_1\triangle\hat{X}_2 \ge \frac{\lambda_1\mu}{2}, \tag{19}$$

thus

$$(\triangle\hat{X}_1)^2(\triangle\hat{X}_2)^2 = \frac{\hbar^2}{4}\frac{\mu}{\nu} \ge \frac{\lambda_1^2\mu^2}{4}. \tag{20}$$

We have

$$\mu\nu \leq \frac{\hbar^2}{\lambda_1^2}, \text{ or } \theta^2 \leq \frac{1}{\lambda_1^2}. \quad (21)$$

This is a relation of the noncommutative parameters and the Planck constant.

For example, in the 4D NCPS, we have

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \ [\hat{x}, \hat{y}] = i\mu, \ [\hat{p}_x, \hat{p}_y] = i\nu, \quad (22)$$

and other commutators vanish. The matrix ε is

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (23)$$

In this case, we have $\lambda_1 = 1$, and $\mu\nu \leq \hbar^2$. This is consistent with the result in Ref. [24].

Let us consider the following special type of $2n$-dimensional generalized NCPS, the coordinate operators $\hat{x}_i$ and $\hat{p}_i$ satisfy the following extended Heisenberg algebra

$$\begin{aligned} &[\hat{x}_i, \hat{p}_j] = i\delta_{ij}\hbar, \\ &[\hat{x}_1, \hat{x}_2] = [\hat{x}_2, \hat{x}_3] = \ldots = [\hat{x}_n, \hat{x}_1] = i\mu, \\ &[\hat{p}_1, \hat{p}_2] = [\hat{p}_2, \hat{p}_3] = \ldots = [\hat{p}_n, \hat{p}_1] = i\nu, \\ &(n \geq 3, \ i, j = 1, \ldots, n) \end{aligned} \quad (24)$$

and others vanish. In this case we have[22]

$$\det M = F_n(\theta^2), \ F_n(x) = 1 + \sum_{i=1}^{k} (-1)^i c_i^{(n)} x^i, \quad (25)$$

where

$$c_1^{(n)} = n, \ c_m^{(n)} = \frac{n}{m!} \prod_{i=1}^{m-1} (n - 2m + i), \ (2 \leq m \leq k), \quad (26)$$

and $k = [\frac{n-1}{2}]$ is the largest integer not larger than $\frac{n-1}{2}$.

Let $0 < r_1^{(n)} \leq r_2^{(n)} \leq \ldots \leq r_k^{(n)}$ be the k positive roots of the equation $F_n(x) = 0$. Since $F_n(x) = \prod_{i=1}^{k}(1 - \lambda_i^2 x)$, we have $r_i^{(n)} = 1/\lambda_i^2$. It is easy to see that there must be $r_1^{(n)} < 1/2$ by the Viète's formulas. From Eq. (21) we have the following restriction relation of the noncommutative parameters in this type of $2n$-dimensional generalized NCPS,

$$\mu\nu \leq r_1^{(n)}\hbar^2 < \frac{\hbar^2}{2}, \ (n \geq 3). \quad (27)$$

For example, we have

$$\begin{cases} \mu\nu \leq \frac{1}{3}\hbar^2, & (n = 3), \\ \mu\nu \leq \frac{1}{4}\hbar^2, & (n = 4), \\ \mu\nu \leq \frac{2}{5+\sqrt{5}}\hbar^2, & (n = 5). \end{cases} \quad (28)$$

Thus we have obtained the explicit constraint relations of the noncommutative parameters in some types of simple higher-dimensional generalized NCPS. In general, these constraint relations of the noncommutative parameters surely depend on the structure matrices ε, and the results of different types of NCPS with different dimensions are different.

In our opinion, the properties of the noncommutative parameters μ and ν reflect the intrinsic essence of NCPS, which should be independent of the concrete physical model on which it is based. Although our results are obtained by virtue of the special kind of deformed boson operators and coherent state representations, these results can provide some boundary conditions of the noncommutative parameters of some types of NCPS. Thus we think these restriction relations of the noncommutative parameters are non-trivial, and they may help us to study the structure of higher-dimensional generalized NCPS. Also, the coherent state representations which we have formulated may be used to solve other physical problems in higher-dimensional NCPS.

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