A Chain of Type Ⅱ and Its Exact Solutions *

Yu Zhang(张宇), Ru-Guang Zhou(周汝光)**

School of Mathematics and Statistics, Jiangsu Normal University Xuzhou 221116

(Received 30 August 2016)

The exact solutions of a chain of type Ⅱ are investigated. The chain of type Ⅱ is first transformed to an integrable differential-difference equation, which has the Kaup–Newell spectral problem as its continuous spatial spectral problem and a Darboux transformation of the Kaup–Newell equation as its discrete temporal spectral problem. Then, with these spectral problems, a Darboux transformation of the transformed equation is constructed. Finally, as an application of the Darboux transformation, an exact solution of the transformed equation and thus the chain of type Ⅱ are presented.

PACS: 02.30.Ik, 45.05.+x, 11.10.Lm DOI: 10.1088/0256-307X/33/11/110203

In 1997, Adler *et al.* constructed and classified the 'integrable' chains $q_{n,xx} = \Phi(q_n, q_{n\pm1}, z_n, z_{n\pm1})$, $z_n = q_{n,x}$, and proposed the generic integrable chains[1,2]

$$\begin{aligned}\ddot{y}_n =& r(\dot{y}_n)(\dot{y}_{n+1}f(y_{n+1}-y_n)\\ &-\dot{y}_{n-1}f(y_n-y_{n-1})+g(y_{n+1}-y_n)\\ &-g(y_n-y_{n-1})),\end{aligned} \tag{1}$$

where dots stand for derivatives with respective to variable x, and r, f and g are some smooth functions. The chains with $f=0$ (in the absence of a magnetic field) are called chains of type I and the chains with $f\neq 0$ are called the chains of type Ⅱ.[2] The chains of type I whose representative is the Toda chain are remarkable and have been well investigated.[3,4] However, except for the relativistic Toda chain,[5] very few type Ⅱ chains have been reported or investigated.

In this Letter, we consider the following type Ⅱ chain[6]

$$\ddot{u}_n = \frac{1}{2}\dot{u}_n^2(\dot{u}_{n+1}e^{u_n-u_{n+1}} - \dot{u}_{n-1}e^{u_{n-1}-u_n}), \tag{2}$$

where $u_n = u(x,n)$. Like all of the type Ⅱ chains, chain (2) is a Lagrange dynamic system. Indeed, chain (2) is equivalent to the Euler–Lagrange equation

$$\frac{\delta S}{\delta u_n} = 0,$$

for the functional

$$\mathcal{S} = \int dx \sum_{n\in Z} S(u_{n-1}, u_n, u_{n,x}),$$

with

$$S(u_{n-1}, u_n, u_{n,x}) = -2\ln u_{n,x} + u_{n,x}\exp(u_{n-1}-u_n).$$

To solve Eq. (2), the following transformation is crucial

$$b(x,n) = e^{u_n}, \quad a(x,n) = -\frac{1}{2}u_{n,x},$$

which reduces Eq. (2) to

$$\begin{aligned}a_{n,x} =& 2a_n^2\Big(\frac{b_n}{b_{n+1}}a_{n+1} - \frac{b_{n-1}}{b_n}a_{n-1}\Big),\\ b_{n,x} =& -2a_nb_n.\end{aligned} \tag{3}$$

Equation (3) is a Bäcklund transformation of the Kaup–Newell hierarchy of equations and is the first member of integrable hierarchy of differential-difference equations.[6] Actually, under transformation $(a_n, b_n) \to (\frac{1}{a_n}, \frac{1}{b_n})$, Eq. (3) may be transformed to the equation obtained by Cao *et al.*,[7] up to a coefficient. In the following we construct exact solutions to Eq. (2) via a Darboux transformation of Eq. (3).

It has been shown[6] that Eq. (3) is a compatible condition of the spectral problem

$$\varphi_{n+1} = U_n\varphi_n, U_n = \begin{pmatrix}\frac{\lambda^2}{a_n}+1 & \frac{\lambda}{b_n}\\ \lambda b_n & 0\end{pmatrix}, \tag{4}$$

and the Kaup–Newell spectral problem

$$\varphi_{n,x} = V_n\varphi_n, V_n = \begin{pmatrix}-\lambda^2 & -2\lambda p_n\\ -2\lambda q_n & \lambda^2\end{pmatrix}, \tag{5}$$

where $p_n = \frac{a_n}{b_n}$, and $q_n = a_{n-1}b_{n-1}$. Actually, Eq. (4) is one of the Darboux transformations of the Kaup–Newell hierarchy of equations.[6]

The so-called Darboux transformation of Eq. (3) is an invertible linear transformation[8]

$$\widetilde{\varphi}_n = T_n\varphi_n, \tag{6}$$

which transforms spectral problems Eqs. (4) and (5) into $\widetilde{\varphi}_{n+1} = \widetilde{U}_n\widetilde{\varphi}_n$ and $\widetilde{\varphi}_{n,x} = \widetilde{V}_n\widetilde{\varphi}_n$, respectively, where

$$\widetilde{U}_n = T_{n+1}U_n(T_n)^{-1}, \ \widetilde{V}_n = (T_{n,x}+T_nV_n)(T_n)^{-1}.$$

*Supported by the National Natural Science Foundation of China under Grant Nos 11271168 and 11671177, the Priority Academic Program Development of Jiangsu Higher Education Institutions, and the Innovation Project of the Graduate Students in Jiangsu Normal University.

**Corresponding author. Email: zhouruguang@jsnu.edu.cn

© 2016 Chinese Physical Society and IOP Publishing Ltd

We assume

$$T_n = \begin{pmatrix} A_n\lambda^2+1 & B_n\lambda \\ C_n\lambda & D_n\lambda^2+1 \end{pmatrix}, \tag{7}$$

where A_n, B_n, C_n and D_n are undetermined functions of variables x and n.

Let $y_n = (y_n^1, y_n^2)^T$, $z_n = (z_n^1, z_n^2)^T$ be two linear independent solutions to Eqs. (4) and (5) with potentials a_n, b_n and set

$$\Gamma_n = \begin{pmatrix} y_n^1 & z_n^1 \\ y_n^2 & z_n^2 \end{pmatrix}.$$

Hence we have

$$\widetilde{\Gamma}_n = T_n\Gamma$$

and

$$\widetilde{\Gamma}_n = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $M_{11} = (A_n\lambda^2+1)y_n^1 + \lambda B_n y_n^2$, $M_{12} = (A_n\lambda^2+1)z_n^1 + \lambda B_n z_n^2$, $M_{21} = \lambda C_n y_n^1 + (D_n\lambda^2+1)y_n^2$, and $M_{22} = \lambda C_n z_n^1 + (\lambda D_n^2+1)z_n^2$. To construct the Darboux matrix T_n, we assume that $\pm\lambda_1$, $\pm\lambda_2$ are four roots of $\mathrm{Det}(T_n) = 0$. Therefore there exist two nonzero constants γ_1 and γ_2 such that

$$\begin{cases} & (A_n\lambda_i^2+1)y_n^1 + B_n\lambda_i y_n^2 \\ = & \gamma_i((A_n\lambda_i^2+1)z_n^1 + B_n\lambda_i z_n^2), \\ & C_n\lambda_i y_n^1 + (D_n\lambda_i^2+1)y_n^2 \\ = & \gamma_i(C_n\lambda_i z_n^1 + (D_n\lambda_i^2+1)z_n^2), \\ & i = 1, 2. \end{cases} \tag{8}$$

Solving Eq. (8) leads to

$$A_n = \frac{\lambda_1\sigma_1(n) - \lambda_2\sigma_2(n)}{\lambda_1\lambda_2(\lambda_1\sigma_2(n) - \lambda_2\sigma_1(n))}, \tag{9}$$

$$B_n = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1\lambda_2(\lambda_1\sigma_2(n) - \lambda_2\sigma_1(n))}, \tag{10}$$

$$C_n = \frac{(\lambda_2^2 - \lambda_1^2)\sigma_1(n)\sigma_2(n)}{\lambda_1\lambda_2(\lambda_1\sigma_1(n) - \lambda_2\sigma_2(n))}, \tag{11}$$

$$D_n = \frac{\lambda_1\sigma_2(n) - \lambda_2\sigma_1(n)}{\lambda_1\lambda_2(\lambda_1\sigma_1(n) - \lambda_2\sigma_2(n))}, \tag{12}$$

where

$$\sigma_i(n) = \frac{y_n^2 - \gamma_i z_n^2}{y_n^1 - \gamma_i z_n^1}, \quad i = 1, 2. \tag{13}$$

It is easy to verify that

$$\mathrm{Det}(T_n) = \lambda_1^{-2}\lambda_2^{-2}(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2), \tag{14}$$

and from Eq. (13)

$$\sigma_i(n+1) = \frac{u_i(n)}{v_i(n)}, \quad i = 1, 2, \tag{15}$$

where $u_i(n) = \lambda_i B_n$, $v_i(n) = 1 + \frac{\lambda_i^2}{A_n} + \frac{\lambda_i\sigma_i(n)}{B_n}$.

Furthermore, from Eqs. (9)–(12) and (15) we can obtain

$$A_{n+1} = \frac{\lambda_1 u_1(n)v_2(n) - \lambda_2 u_2(n)v_1(n)}{\lambda_1\lambda_2(\lambda_1 u_2(n)v_1(n) - \lambda_2 u_1(n)v_2(n))}, \tag{16}$$

$$B_{n+1} = \frac{(\lambda_2^2 - \lambda_1^2)v_1(n)v_2(n)}{\lambda_1\lambda_2(\lambda_1 u_2(n)v_1(n) - \lambda_2 u_1(n)v_2(n))}, \tag{17}$$

$$C_{n+1} = \frac{(\lambda_1^2 - \lambda_2^2)u_1(n)u_2(n)}{\lambda_1\lambda_2(\lambda_2 u_2(n)v_1(n) - \lambda_1 u_1(n)v_2(n))}, \tag{18}$$

$$D_{n+1} = \frac{\lambda_1 u_2(n)v_1(n) - \lambda_2 u_1(n)v_2(n)}{\lambda_1\lambda_2(\lambda_1 u_1(n)v_2(n) - \lambda_2 u_2(n)v_1(n))}. \tag{19}$$

In addition, through tedious calculations and from Eqs. (9)–(12), we have the following identities

$$A_{n+1}(a_nA_n - b_nB_n) - a_nA_n(1 - b_nB_n)D_n = 0, \tag{20}$$

$$a_n(1 - b_nB_n)D_{n+1} - a_nA_n + b_nB_n = 0, \tag{21}$$

$$C_{n+1}(1 - b_nB_n) - b_n^2B_n = 0. \tag{22}$$

Proposition 1: the matrix $\widetilde{U}_n$ has the same form as U_n, in which the old potentials a_n and b_n are mapped into new potentials $\widetilde{a}_n$ and $\widetilde{b}_n$ according to

$$\begin{cases} \widetilde{a}_n = & \frac{A_n}{A_{n+1}}a_n, \\ \widetilde{b}_n = & \frac{b_n}{1 - b_nB_n}. \end{cases} \tag{23}$$

Proof: let $T_n^{-1} = \frac{T_n^*}{\mathrm{Det}(T_n)}$ and

$$T_{n+1}U_nT_n^* = \begin{pmatrix} h_{11}(\lambda, n) & h_{12}(\lambda, n) \\ h_{21}(\lambda, n) & h_{22}(\lambda, n) \end{pmatrix},$$

with

$$\begin{aligned} h_{11}(\lambda, n) = {} & \frac{A_{n+1}D_n}{a_n}\lambda^6 + \Big(A_{n+1}D_n + \frac{D_n}{a_n} \\ & + b_nB_{n+1}d_n + \frac{A_{n+1}}{a_n} - \frac{A_{n+1}C_n}{b_n}\Big)\lambda^4 \\ & + \Big(A_{n+1} + \frac{1}{a_n} + D_n - \frac{C_n}{b_n} \\ & + b_nB_{n+1}\Big)\lambda^2 + 1, \\ h_{12}(\lambda, n) = {} & A_{n+1}\Big(\frac{A_n}{b_n} - \frac{B_n}{a_n}\Big)\lambda^5 + \Big(A_{n+1}b_n \\ & - A_{n+1}B_n + \frac{A_n}{b_n} - \frac{B_n}{a_n} - b_nB_nB_{n+1}\Big)\lambda^3 \\ & + (b_n - B_n)\lambda, \\ h_{21}(\lambda, n) = {} & \Big(\frac{C_{n+1}D_n}{a_n} + b_nD_nD_{n+1}\Big)\lambda^5 \\ & + \Big[C_{n+1}\Big(\frac{1}{a_n} + D_n - \frac{C_n}{b_n}\Big) \\ & + b_nD_{n+1} + b_nD_n\Big]\lambda^3 \\ & + (C_{n+1} + b_n)\lambda, \\ h_{22}(\lambda, n) = {} & \Big(\frac{C_{n+1}A_n}{b_n} - \frac{C_{n+1}B_n}{a_n} - b_nB_nD_{n+1}\Big)\lambda^4 \\ & + [C_{n+1}(b_n - B_n) - b_nB_n]\lambda^2. \end{aligned}$$

It is not difficult to find that $h_{11}(\lambda, n)$ is a 6th-order polynomial in λ, $h_{12}(\lambda, n)$ and $h_{21}(\lambda, n)$ are 5th-order

polynomials in λ, and $h_{22}(\lambda,n)$ is a 4th-order polynomial in λ. Through a tedious but direct computation, we can obtain $h_{lj}(\pm\lambda_i,n)=0$, $i,j,l=1,2$. Therefore, we have

$$T_{n+1}U_nT_n^*=\mathrm{Det}(T_n)\tau_n, \tag{24}$$

with

$$\tau_n=\begin{pmatrix}\tau_{11}^{(2)}\lambda^2+\tau_{11}^{(1)}\lambda+\tau_{11}^{(0)} & \tau_{12}^{(1)}\lambda+\tau_{12}^{(0)}\\ \tau_{21}^{(1)}\lambda+\tau_{21}^{(0)} & \tau_{22}^{(0)}\end{pmatrix},$$

where $\tau_{11}^{(2)}$, $\tau_{11}^{(1)}$, $\tau_{12}^{(1)}$, $\tau_{21}^{(1)}$, $\tau_{jl}^{(0)}$, $j,l=1,2$, all are independent of λ. Hence we obtain

$$T_{n+1}U_n=\tau_nT_n.$$

Equating the coefficients of λ^i, $i=0,1,2$ in Eq. (24), we have

$$\tau_{11}^{(0)}=1,\ \tau_{12}^{(0)}=\tau_{21}^{(0)}=\tau_{22}^{(0)}=\tau_{11}^{(1)}=0, \tag{25}$$

$$\tau_{12}^{(1)}=\frac{1-b_nB_n}{b_n}=\frac{1}{\widetilde{b}_n},\ \tau_{21}^{(1)}=\frac{C_{n+1}}{b_nB_n},$$

$$\tau_{11}^{(2)}=\frac{A_{n+1}}{a_nA_n}=\frac{1}{\widetilde{a}_n}. \tag{26}$$

The proof is completed.

Proposition 2: the matrix $\widetilde{V}_n$ has the same form as V_n, in which the old potentials p_n and q_n are mapped into new potentials $\widetilde{p}_n$ and $\widetilde{q}_n$ according to

$$\begin{cases}\widetilde{p}_n=&\frac{A_np_n-B_n}{D_n},\\ \widetilde{q}_n=&\frac{C_n+D_nq_n}{A_n}.\end{cases} \tag{27}$$

Proof: let

$$(T_{n,x}+T_nV_n)T_n^*=\begin{pmatrix}k_{11}(\lambda,n) & k_{12}(\lambda,n)\\ k_{21}(\lambda,n) & k_{22}(\lambda,n)\end{pmatrix}.$$

It is easy to obtain

$$\begin{aligned}k_{11}(\lambda,n)=&-A_nD_n\lambda^6-[A_n+(1+2B_nq_n\\&-A_{n,x})D_n+(B_n-2A_np_n)C_n]\lambda^4\\&+[(2p_n-B_{n,x})C_n\\&-(1+2B_nq_n-A_{n,x})]\lambda^2,\\ k_{12}(\lambda,n)=&[A_nB_n+A_n(B_n-2A_np_n)]\lambda^5+[(1\\&+2B_nq_n-A_{n,x})B_n+(B_n-2A_np_n)\\&-A_n(2p_n-B_{n,x})]\lambda^3-(2p_n-B_{n,x})\lambda,\\ k_{21}(\lambda,n)=&[(-C_n-2B_nq_n)D_n-B_nC_n]\lambda^5\\&+[(-C_n-2B_nq_n)+(C_{n,x}-2q)D_n\\&-(1+B_{n,x}-2C_np_n)C_n]\lambda^3\\&+(C_{n,x}-2q_n)\lambda,\\ k_{22}(\lambda,n)=&A_nB_n\lambda^6+[B_n(C_n+2B_nq_n)+B_n\\&+A_n(1+B_n-2C_np_n)]\lambda^4\\&+[(1+B_{n,x}-2C_np_n)\\&-B_n(C_{n,x}-2q_n)]\lambda^2.\end{aligned}$$

It is easy to see that $k_{11}(\lambda,n)$ and $k_{22}(\lambda,n)$ are 6th-order polynomials in λ, $k_{12}(\lambda,n)$ and $k_{21}(\lambda,n)$ are 5th-order polynomials in λ.

Through a lengthy but direct calculation, we can obtain

$$k_{lj}(\pm\lambda_i,n)=0,\quad i,j,l=1,2.$$

Therefore, we may assume

$$(T_{n,x}+T_nV_n)T_n^*=\mathrm{Det}(T_n)\rho_n, \tag{28}$$

with

$$\rho_n=\begin{pmatrix}\rho_{11}^{(2)}\lambda^2+\rho_{11}^{(0)} & \rho_{12}^{(1)}\lambda\\ \rho_{21}^{(1)}\lambda & \rho_{22}^{(2)}\lambda^2+\rho_{22}^{(0)}\end{pmatrix},$$

where $\rho_{11}^{(2)}$, $\rho_{11}^{(0)}$, $\rho_{12}^{(1)}$, $\rho_{21}^{(1)}$, $\rho_{22}^{(2)}$ and $\rho_{22}^{(0)}$ are all independent of λ. From Eq. (28), we obtain

$$T_{n,x}+T_nV_n=\rho_nT_n. \tag{29}$$

Comparing the coefficients of λ^j, $j=0,1,2$, in Eq. (29), we obtain

$$\rho_{11}^{(0)}=\rho_{22}^{(0)}=0,\ \rho_{11}^{(2)}=-\rho_{22}^{(2)}=-1, \tag{30}$$

$$\rho_{12}^{(1)}=2\frac{B_n-A_np_n}{D_n},\ \rho_{21}^{(1)}=-2\frac{C_n+D_nq_n}{A_n}. \tag{31}$$

The proof is completed.

Finally, we can verify that Eq. (27) is consistent with Eq. (23) in terms of relations (20)–(22). Therefore, Eq. (23) is a Darboux transformation of Eq. (3).

Next we construct an exact solution to Eq. (3) as an application of the Darboux transformation (23). First of all, it is easy to see that Eq. (3) has a simple solution $a_n=1$, $b_n=\exp(-2x)$ or $p_n=q_n^{-1}=\exp(2x)$. Substituting them into Eqs. (4) and (5), we obtain

$$\varphi_{n+1}=\begin{pmatrix}\lambda^2+1 & \lambda\exp(2x)\\ \lambda\exp(-2x) & 0\end{pmatrix}\varphi_n,$$

$$\varphi_{n,x}=\begin{pmatrix}-\lambda^2 & -2\lambda\exp(2x)\\ -2\lambda\exp(-2x) & \lambda^2\end{pmatrix}\varphi_n.$$

Solving the above two equations, we obtain two real linear independent solutions

$$\begin{aligned}\varphi_n=&\Big(\frac{\lambda^2+1+\Delta(\lambda)}{2}\Big)^n\\&\cdot\exp[(1-\Delta(\lambda))x]\begin{pmatrix}2\lambda^2\exp(2x)\\ \Delta(\lambda)-\lambda^2-1\end{pmatrix},\\ \psi_n=&\Big(\frac{\lambda^2+1-\Delta(\lambda)}{2}\Big)^n\\&\cdot\exp[(1+\Delta(\lambda))x]\begin{pmatrix}2\lambda^2\exp(2x)\\ -\Delta(\lambda)-\lambda^2-1\end{pmatrix},\end{aligned}$$

where $\lambda\in\mathbb{R}-\{0\}$, and $\Delta(\lambda)=\sqrt{\lambda^4+6\lambda^2+1}$.

From Eq. (13), we have

$$\begin{aligned}\sigma_i\equiv\sigma_i(n)=&\frac{\Delta(\lambda_i)-\lambda_i^2-1}{2\lambda_i}\\&+\gamma_i\frac{\Delta(\lambda_i)+\lambda_i^2+1}{2\lambda_i\xi_i^n\exp[-2\Delta(\lambda_i)x]},\end{aligned}$$

where $\xi_i = -(\frac{\lambda_i^2+1+\sqrt{\lambda_i^4+6\lambda_i^2+1}}{2\lambda_i})^2$, and $\Delta(\lambda_i) = \sqrt{\lambda_i^4+6\lambda_i^2+1}$, $(i=1,2)$.

Finally we obtain a new solution to Eq. (3) as follows:

$$\begin{aligned}\widetilde{a}_n =& \{\lambda_1\lambda_2(\lambda_1\sigma_1-\lambda_2\sigma_2)[\lambda_1^2-\lambda_2^2+(\lambda_1\sigma_1\\ &-\lambda_2\sigma_2)\exp(2x)]\}/\{(\lambda_1\sigma_2-\lambda_2\sigma_1)[\lambda_1^2\\ &-\lambda_2^2+\lambda_1\lambda_2(\lambda_1\sigma_2-\lambda_2\sigma_1)\exp(2x)]\}, \quad (32)\end{aligned}$$

$$\widetilde{b}_n = \frac{\lambda_1\lambda_2(\lambda_1\sigma_2-\lambda_2\sigma_1)}{\lambda_1\lambda_2(\lambda_1\sigma_2-\lambda_2\sigma_1)\exp(2x)+\lambda_1^2-\lambda_2^2}. \quad (33)$$

Hence Eq. (2) has the following exact solution

$$\begin{aligned}\tilde{u}_n &= \ln b_n\\ &= \ln\frac{\lambda_1\lambda_2(\lambda_1\sigma_2-\lambda_2\sigma_1)}{\lambda_1\lambda_2(\lambda_1\sigma_2-\lambda_2\sigma_1)\exp(2x)+\lambda_1^2-\lambda_2^2}.\end{aligned}$$

References

[1] Adler V E and Shabat A B 1997 *Theor. Math. Phys.* **111** 647

[2] Adler V E and Shabat A B 1997 *Theor. Math. Phys.* **112** 935

[3] Toda M 1981 *Theory of Nonlinear Lattices* (New York: Springer)

[4] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel: Birkhäuser)

[5] Ruijsenaars S N M 1990 *Commun. Math. Phys.* **133** 217

[6] Zhou R G and Chen J 2015 *Commun. Theor. Phys.* **63** 1

[7] Cao C W and Yang X 2008 *J. Phys.* A **41** 025203

[8] Gu C H, Hu H S and Zhou Z X 2005 *Darboux Transformation in Integrable Systems* (Dordrecht: Springer)