Hall Conductivity in the Cosmic Defect and Dislocation Spacetime *

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Influences of topological defect and dislocation on conductivity behavior of charge carriers in external electromagnetic fields are studied. Particularly the quantum Hall effect is investigated in detail. It is found that the nontrivial deformations of spacetime due to topological defect and dislocation produce an electric current at the leading order of perturbation theory. This current then induces a deformation on the Hall conductivity. The corrections on the Hall conductivity depend on the external electric fields, the size of the sample and the momentum of the particle.

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Topological defects, which encode the global geometry structure of the background spacetime, can be realized in various physical systems ranging from the macroscopic to the microscopic scale, and therefore simulated extensive studies on its physical effects.[1–36] For instance, the cosmic string,[1–4] global monopole,[5] and domain wall[6–11] can be formed at the phase transitions of the early universe from symmetric to asymmetric configurations. Even though topological defects have been commonly observed in condensed matter systems, so far there is no direct experimental evidence of the cosmological defects due to their extremely high energies. Therefore it is still attractive to explore the nontrivial geometry structures at the fundamental level.

Various studies on the electromagnetic dynamics of the magnetic and electric dipole moments have shown that both their global and local physical properties can be sensitive to non-trivial geometries.[12–30] However, it is still difficult to observe them experimentally. On the other hand, it is naturally expected that the energy spectra of the charged carriers in various quantum systems can be deformed by the topological defects, and hence can be used to detect the topological properties of the background spacetime. The effects of nontrivial geometries survive in the non-relativistic limit, and have been studied in Refs. [12,13]. Nevertheless, the electromagnetic dynamics of charge carriers in the presence of topological defects were less discussed. It was pointed out that the infinite degeneracy of the Landau levels is lifted by topological disclinations and defects.[31,32] In this Letter, we focus on conductivity behavior of charge carriers under the influences of topological defects and dislocation in the non-relativistic limit. We consider the quantum Hall effect which is one of the most profound phenomena of charge carriers in electromagnetic fields. It is shown that the Hall conductivity receives a correction linearity in the scale parameters of the topological defects, which are in the same order with the corrections on the spin-Hall conductivity.[22,23] However, we can expect that very steady and precise quantized Hall conductivity can provide better experimental sensitivity and can bind on parameters of the models.

We consider the quantum Hall effect in the presence of a topological defect. The dynamics of a relativistic Dirac particle in the curved spacetime is expressed in the generalized covariant form of the Dirac equation[35]

$$[\tilde{\gamma}^{\mu}(x)(p_{\mu}-qA_{\mu}(x)-\Gamma_{\mu}(x))+mc^{2}]\psi(x)=0, \quad (1)$$

where matrix $\Gamma_{\mu}(x)$ is the spinor connection, and $\tilde{\gamma}^{\mu}(x)$ are the elements of coordinate dependent Clifford algebra in the curved spacetime and satisfy the relation $\{\tilde{\gamma}^{\mu}(x),\tilde{\gamma}^{\nu}(x)\}=2g^{\mu\nu}(x)$, with $g^{\mu\nu}(x)$ being the matrix of the spacetime in the presence of a topological defect. In the formalism of vierbein (or tetrad), which allows us to define the spinors in the curved spacetime, the metric has the form[35] $g_{\mu\nu}(x)=e^{\rm a}_{\mu}(x)e^{b}_{\nu}(x)\eta_{ab}$ and the coordinate-dependent Dirac matrix reads $\tilde{\gamma}^{\mu}(x)=\gamma_{\rm a}e^{\rm a}_{\mu}(x)$. The inverse of vierbein is defined by the relations $e^{\rm a}_{\mu}(x)e^{\mu}{}_{b}(x)=\delta^{\rm a}{}_{b}$ and $e^{\mu}_{\rm a}(x)e^{\rm a}_{\nu}(x)=\delta^{\mu}_{\nu}$.

The non-relativistic electromagnetic dynamics for charged Dirac particles in the presence of topological defects has been discussed in Ref. [22] by using the Foldy–Wouthuysen transformation.[37] It was shown that at the non-relativistic limit the Schrödinger equa-

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tion becomes (here we have ignored all the terms involving spin)

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{x}) - q\boldsymbol{K}(\boldsymbol{p},\boldsymbol{x}))^2 + qV(\boldsymbol{x}), \quad (2)$$

where $\boldsymbol{K}$ behaving like an effective vector potential is the correction to the ordinary minimal coupling, and it is as functions of both position and momentum. In the convention of this work it is

$$\boldsymbol{K} = \frac{1}{q}\boldsymbol{\Omega}\cdot(\boldsymbol{p} - q\boldsymbol{A}), \quad (3)$$

where the matrix Ω is the correction to the ordinary verbena

$$\Omega^{\rm a}_{\mu}(x) = e^{\rm a}_{\mu}(x) - \delta^{\rm a}_{\mu}, \quad \Omega^{\mu}_{\rm a}(x) = e^{\mu}_{\rm a}(x) - \delta^{\mu}_{\rm a}. \quad (4)$$

Next, we focus on the influences of the topological defect with line element given by

$$ds^2 = c^2dt^2 - d\rho^2 - \eta^2\rho^2 d\varphi^2 - dz^2, \quad (5)$$

where $\eta = 1 - 4\lambda G/c^2$ is the deficit angle, and λ is the linear mass density of the cosmic string. In general, the deficit angle can assume $\eta > 1$, which corresponds to an anti-conical spacetime with negative curvature. The geometry Eq. (5) corresponds to a conical singularity described by the curvature tensor $R^{\rho,\varphi}_{\rho,\varphi} = \frac{1-\eta}{4\eta}\delta_2(\boldsymbol{r})$. The vierbein for this metric reads

$$e^{\rm a}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\eta\rho\sin\varphi & 0 \\ 0 & \sin\varphi & \eta\rho\cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (6)$$

and

$$e^{\mu}_{\rm a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\frac{\sin\varphi}{\eta\rho} & \frac{\cos\varphi}{\eta\rho} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

The flat spacetime can be recovered for $\eta = 1$. In the system of rectangular coordinates, the corresponding Ω matrix is

$$\Omega^{\rm a}_{\mu} = (1-\eta)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin^2\phi & \sin\phi\cos\phi & 0 \\ 0 & \sin\phi\cos\phi & -\cos^2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

Inserting this into Eq. (3) we can obtain the explicit expression of the effective vector potential

$$\boldsymbol{K} = \frac{\chi}{q}\Big(\frac{qBx^2}{\rho} - \frac{l_z}{\rho}\Big)\boldsymbol{e}_{\phi}, \quad (9)$$

where $\chi = 1 - \eta$. This correction is proportional to the angular momentum of the particle and the external magnetic field. Further, the topological defect affects only the physics on the azimuthal direction. This is due to the scaling of the azimuthal angle in the line element Eq. (5). We will use the perturbation method to calculate the influences of this effective vector potential on the expectation values of the electric currents. In this approximation, the expectation values of the various currents are calculated by using the leading order wave functions which are obtained by ignoring the χ-dependent terms in the full Hamiltonian. The leading order results can be found in the standard textbooks, e.g., in Ref. [38].

The electric current can be obtained by using the Heisenberg equation. For the Hamiltonian (2) we obtain

$$J_x = q\rho_{\rm c}\Big(\frac{p_x}{m} + \chi\frac{qBx^2y}{m\rho^2} - \chi\frac{yl_z + l_zy}{m\rho^2}\Big), \quad (10)$$

$$J_y = q\rho_{\rm c}\Big(\frac{p_y}{m} - \frac{qB}{m}x - \chi\frac{qBx^3}{m\rho^2} + \chi\frac{xl_z + l_zx}{m\rho^2}\Big), \quad (11)$$

$$J_z = q\rho_{\rm c}\frac{p_z}{m}. \quad (12)$$

Then at the leading order the expectation values of these current operators are $\overline{J_i} = \langle n,\ell,k_z|J_i|n,\ell,k_z\rangle$, which have the following forms

$$\overline{J}_x = q\rho_{\rm c}\overline{\Big[\frac{p_x}{m} + \frac{\chi}{m}\big(\frac{2qBx^2y}{\rho^2} - 2\Re\frac{yl_z}{\rho^2}\big)\Big]}, \quad (13)$$

$$\overline{J}_y = q\rho_{\rm c}\overline{\Big[\frac{p_y}{m} - \frac{qB}{m}x - \chi\big(\frac{2qBx^3}{m\rho^2} - 2\Re\frac{xl_z}{m\rho^2}\big)\Big]}, \quad (14)$$

$$\overline{J}_z = q\rho_{\rm c}\frac{\overline{p_z}}{m}. \quad (15)$$

The expectation value of the operator x^2y/ρ^2 can be easily calculated by using the symmetry as follows:

$$\Big\langle k_y\Big|\frac{x^2y}{\rho^2}\Big|k_y\Big\rangle = \frac{1}{L_y}\int_{-a}^{\rm a}dy\frac{x^2y}{\rho^2} = -\frac{1}{L_y}\int_{-a}^{\rm a}dy\frac{x^2y}{\rho^2} = 0, \quad (16)$$

where $a = L_y/2$, and L_y is the length of the sample along the $\hat{y}$ direction. Thus this term does not contribute to the electric current along the $\hat{x}$ direction. The expectation value of the operator yl_z/ρ^2 is slightly complicated. Integrating out the variable y we can obtain

$$\overline{\frac{yl_z}{\rho^2}} = \Big\langle n\Big|\Big(\frac{2x}{L_y}\arctan\Big(\frac{L_y}{2x}\Big) - 1\Big)p_x\Big|n\Big\rangle. \quad (17)$$

Noticing that $p_x|n\rangle$ is purely imaginary,[38] then the expectation value of this operator is also pure imaginary. Thus it also does not give any contribution. The expectation value of the operator x^3/ρ^2 is more complicated. Integrating the variable y we obtain

$$\overline{\frac{x^3}{\rho^2}} = \Big\langle n\Big|\frac{2x^2}{L_y}\arctan\Big(\frac{L_y}{2x}\Big)\Big|n\Big\rangle. \quad (18)$$

Due to the fact that the quantum fluctuation of the position operator is very small for the harmonic oscillator, we can ignore these effects in the order of χ. Then we obtain the approximated expectation value

$$\overline{\frac{x^3}{\rho^2}} \approx \frac{2x_{\rm c}^2}{L_y}\arctan\Big(\frac{L_y}{2x_{\rm c}}\Big). \tag{19}$$

Using the same approximation we can obtain the expectation value of the operator xl_z/ρ^2,

$$\overline{\frac{xl_z}{\rho^2}} = \Big\langle n\Big|\frac{\hbar k_y x^2}{\rho^2}\Big|n\Big\rangle \approx \hbar k_y \frac{2x_{\rm c}}{L_y}\arctan\Big(\frac{L_y}{2x_{\rm c}}\Big). \tag{20}$$

Then collecting all these results we obtain the expectation values of the electric currents

$$\bar{J}_x = 0, \tag{21}$$

$$\bar{J}_y = q\rho_{\rm c}\frac{E}{B}(1+2\chi M(\kappa)), \tag{22}$$

$$\bar{J}_z = q\rho_{\rm c}\frac{\hbar k_z}{m}. \tag{23}$$

Here we define a dimensionless function

$$M(\kappa) = \kappa\arctan\Big(\frac{1}{\kappa}\Big),\ \ \kappa = \frac{2x_{\rm c}}{L_y}. \tag{24}$$

It is as functions of the momentum in the $\hat{y}$-direction k_y, the external fields E and B, center of the harmonic oscillator $x_{\rm c}$ as well as the size of the sample L_y. In terms of the mass density of the topological defect we have $\chi = 4\lambda G/c^2$. Then we have the corrected Hall conductivity

$$\tilde{\sigma}_{\rm H} = \sigma_{\rm H}\Big(1+\frac{8\lambda G}{c^2}M(\kappa)\Big). \tag{25}$$

We study the Hall effect in the presence of topological dislocation. The line element is

$$ds^2 = c^2dt^2 - d\rho^2 - \rho^2 d\varphi^2 - (dz+\xi d\varphi)^2, \tag{26}$$

where ξ is the torsion of the topological dislocation.[33,34] Topological dislocations are much more realistic line defects. They can modify the energy spectrum of electrons moving in a uniform magnetic field. Landau levels in the presence of dislocations have been investigated. The torsion can be identified with the surface density of the Burgers vector in the classical theory of elasticity. The vierbein in this case reads

$$e_\mu^{\rm a}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\rho\sin\varphi & 0 \\ 0 & \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & \xi & 1 \end{pmatrix}, \tag{27}$$

and

$$e_{\rm a}^\mu(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\frac{\sin\varphi}{\rho} & \frac{\cos\varphi}{\rho} & 0 \\ 0 & \frac{\xi}{\rho}\sin\varphi & -\frac{\xi}{\rho}\cos\varphi & 1 \end{pmatrix}. \tag{28}$$

The flat spacetime can be recovered for $\xi = 0$. In the reference frame of the rectangular coordinates, the corresponding Ω matrix is

$$\Omega_\mu^{\rm a} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\xi}{\rho}\sin\phi & \frac{\xi}{\rho}\cos\phi & 0 \end{pmatrix}. \tag{29}$$

Inserting this into Eq. (3) we can obtain the explicit expression of the effective vector potential as follows:

$$\boldsymbol{K} = \frac{\xi}{q}\Big(\frac{l_z}{\rho^2} - \frac{qBx^2}{\rho^2}\Big)\boldsymbol{e}_z. \tag{30}$$

The correction again depends on the angular momentum and the external magnetic field. Furthermore, the components are the same as those obtained. The only difference is that the correction is along the $\hat{z}$ direction rather than the $\hat{\phi}$ direction. This is due to the coupling between ϕ and z in the line element Eq. (26). Again we use the perturbation theory to calculate the expectation values of the electric current operators in this case. Using the Heisenberg equation we obtain

$$J_x = q\rho_{\rm c}\Big(\frac{p_x}{m} + \xi\frac{y}{\rho^2}\frac{p_z}{m}\Big), \tag{31}$$

$$J_y = q\rho_{\rm c}\Big(\frac{p_y}{m} - \frac{qB}{mc}x - \xi\frac{x}{\rho^2}\frac{p_z}{m}\Big), \tag{32}$$

$$J_z = \frac{q\rho_{\rm c}}{m}\Big(p_z - \xi\frac{l_z - qBx^2}{\rho^2}\Big). \tag{33}$$

Then at the leading order the expectation values of the current operators are

$$\overline{J_x} = q\rho_{\rm c}\frac{\xi\hbar k_z}{m}\Big\langle n,k_y\Big|\frac{y}{\rho^2}\Big|n,k_y\Big\rangle, \tag{34}$$

$$\overline{J_y} = q\rho_{\rm c}\Big(\frac{\hbar k_y}{m} - \Big\langle n,k_y\Big|\frac{qB}{mc}x + \frac{\xi\hbar k_z}{m}\frac{x}{\rho^2}\Big|n,k_y\Big\rangle\Big), \tag{35}$$

$$\overline{J_z} = q\rho_{\rm c}\Big(\frac{\hbar k_z}{m} - \xi\Big\langle n,k_y\Big|\frac{l_z - qBx^2}{m\rho^2}\Big|n,k_y\Big\rangle\Big). \tag{36}$$

We have calculated the expectation value of the operator $\overline{y/\rho^2}$, it is zero at the leading order. By using the same approximation, we obtain the expectation values of the operator x/ρ^2, x^2/ρ^2 and l_z/ρ^2,

$$\overline{\frac{x}{\rho^2}} \approx \frac{2}{L_y}\arctan\Big(\frac{L_y}{2x_{\rm c}}\Big), \tag{37}$$

$$\overline{\frac{l_z}{\rho^2}} = \overline{\frac{\hbar k_y x}{\rho^2}} \approx \frac{2\hbar k_y}{L_y}\arctan\Big(\frac{L_y}{2x_{\rm c}}\Big). \tag{38}$$

Then collecting all these results we obtain the expectation values of the electric current operators

$$\overline{J}_x = 0, \tag{39}$$

$$\overline{J}_y = -q\rho_{\rm c}\frac{E}{B}\Big(1 - \frac{2\xi}{L_y}\frac{\hbar k_z}{m}\frac{B}{E}\frac{1}{\kappa}M(\kappa)\Big), \tag{40}$$

$$\overline{J}_z = q\rho_{\rm c}\Big(\frac{\hbar k_z}{m} + \frac{2\xi}{L_y}\frac{\hbar k_y}{m}\frac{1}{\kappa}M(\kappa)\Big). \tag{41}$$

The corrected Hall conductance is

$$\tilde{\sigma}_{\rm H} = \sigma_{\rm H}\Big(1 - \frac{2\xi}{L_y}\frac{\hbar k_z}{m}\frac{B}{E}\frac{1}{\kappa}M(\kappa)\Big). \tag{42}$$

Compared to the results obtained, the corrected Hall conductance depends on the momentum in the $\hat{z}$ direction.

In summary, the influences of topological defect and dislocation on quantum Hall effects have been studied. For the topological defect with line element Eq. (5), the vector potential $\boldsymbol{A}$ receives corrections proportional to the linear density of the cosmic string, angular momentum l_z of the particle and itself. These additional potentials affect only the dynamics along the azimuthal direction. By using the perturbation method we obtain the expectation values of the charge current $\overline{J}_i$ as well as the Hall conductance $\sigma_{\rm H}$. The charge current $\overline{J}_x$ does not receive correction due to the harmonic motion along this direction, and $\overline{J}_z$ also does not influence due to the fact that the line element does not deform the motion of particle along this direction. However, the Hall current $\overline{J}_y$ receives the correction proportional to the linear density of the cosmic string. The correction is always positive and depends on the expectation value of the coordinate $\overline{x} = x_{\rm c}$ as well as the sample size L_y. The dependence is represented by a function $M(\kappa) = \kappa \arctan(1/\kappa)$ with $\kappa = 2x_{\rm c}/L_y$. For a light cosmic string with $\chi = 4\lambda G/c^2 \sim 10^{-9}$, it is very difficult to observe the effects. However, for a cosmic string with heavier mass density, the correction could be observable.

For the topological dislocation with line element Eq. (26), the canonical momentum is deformed and receives a correction proportional to the Ragga vector $\boldsymbol{\xi}$ and the angular momentum l_z of the charge particle. The charge current $\overline{J}_x$ does not receive correction due to the harmonic motion along this direction. There is a correction along the $\hat{z}$ direction, this is a distinct feature from the last case. Even though $\overline{J}_z$ is affected, the effect will be negligible due to the smallness of the Ragga vector compared with the sample size L_y. However, the Hall current $\overline{J}_y$ receives a significant correction due to the enhancement factor B/E, which is of order 10^8 typically. The correction also depends on the momentum along the $\hat{z}$ direction and may be positive and negative depending on the expectation value of the coordinate $\overline{x} = x_{\rm c}$, with $-L_x/2 < x_{\rm c} < L_x/2$. For a Ragga vector of order 1 nm, the correction is of order 10^{-3} that is measurable.

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