A Short Note on a Differential-Difference Gauge Transformation and a New Spectral Problem *

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We show that a class of spectral problems are related to the spectral problem of the Volterra lattice through a gauge transformation. The transformation is given. We hope that our discussion can draw attention to the study of gauge transformation theory of differential-difference integrable systems.

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The well-known Volterra lattice (or the Langmuir lattice) is

$$v_{n,t} = v_n(v_{n+1} - v_{n-1}), \tag{1}$$

which is related to a Lax pair

$$\phi_{n+1} = M_n\phi_n,\ M_n = \begin{pmatrix} 0 & 1 \\ v_n & \lambda \end{pmatrix},\ \phi_n = (\phi_{1,n}, \phi_{2,n})^{\mathrm{T}}, \tag{2a}$$

$$\phi_{n,t} = N_n\phi_n,\ N_n = \begin{pmatrix} \lambda^2 + v_{n-1} - 1 & -\lambda \\ -\lambda v_n & v_n - 1 \end{pmatrix}, \tag{2b}$$

or

$$\phi_{2,n+1} - v_n\phi_{2,n-1} = \lambda\phi_{2,n}, \tag{3a}$$

$$(\phi_{2,n})_t = -\lambda v_n\phi_{2,n-1} + (v_n - 1)\phi_{2,n}. \tag{3b}$$

In Eq. (1) v_n denotes the function $v(n,t)$ and tends to 1 as $|n| \to +\infty$. Starting from the spectral problem Eq. (2a), one can derive the Volterra hierarchy

$$(\ln v_n)_{t_m} = R^{m-1}(v_{n+1} - v_{n-1}),\ m = 1, 2, \ldots, \tag{4a}$$

where the recursion operator R is

$$R = (E+1)(E^{-1}v_n - v_nE)(E-1)^{-1}, \tag{4b}$$

in which the shift operator E is defined by $E^i f_n = f_{n+i}$ for $i \in \mathbb{Z}$. This hierarchy is related to the Korteweg–de Vries hierarchy through suitable continuum limits. Integrabilities of the Volterra hierarchy have been well studied (see Ref. [1] and the references therein).

In a recent work,[2] the following spectral problem is studied,

$$\psi_{n+1} = U_n\psi_n,\ U_n = \begin{pmatrix} 0 & s_n^2 \\ 1/s_n & \lambda/s_n \end{pmatrix},\ \psi_n = (\psi_{1,n}, \psi_{2,n})^{\mathrm{T}}. \tag{5}$$

The related hierarchy derived in Ref. [2] (Eq. (15) in Ref. [2]) can be written as

$$(\ln s_n^3)_{t_m} = \widetilde{R}^{m-1}(s_{n-1}^3 - s_{n+1}^3),\ m = 1, 2, \ldots, \tag{6a}$$

where the recursion operator $\widetilde{R}$ is

$$\widetilde{R} = (E+1)(E^{-1}s_n^3 - s_n^3E)(E-1)^{-1}. \tag{6b}$$

We now introduce an invertible gauge transformation

$$\phi_n = T_n\psi_n, \tag{7a}$$

where

$$T_n = Q_n\begin{pmatrix} 1/v_n & 0 \\ 0 & 1 \end{pmatrix},\ Q_n = \prod_{j=-\infty}^{n-1} s_j,\ v_n = s_{n-1}^{l+1}, \tag{7b}$$

and l is an arbitrary real number while $l \neq -1$. It then follows from Eq. (2a) that ψ_n defined in Eq. (7) satisfies a spectral problem

$$\psi_{n+1} = \widetilde{U}_n\psi_n,\ \widetilde{U}_n = \begin{pmatrix} 0 & s_n^l \\ 1/s_n & \lambda/s_n \end{pmatrix}, (l \neq -1). \tag{8}$$

When $l = 2$ Eq. (8) yields the so-called new spectral problem Eq. (5) considered in Ref. [2]. In addition to the above gauge equivalence, we would also like to point out that if we suppose

$$\psi_{n,t_m} = V_m\psi_n,\ V_m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}, \tag{9}$$

the hierarchy (4a) (with $v_n = s_{n-1}^{l+1}$) can be derived from, as well as expressed by, the compatibility of Eqs. (8) and (9), i.e.,

$$\widetilde{U}_{n,t_m} = (EV_m)\widetilde{U}_n - \widetilde{U}_nV_m, \tag{10}$$

while only when $l = 1$ the trace of V_m is zero, which is different from the result of Ref. [2].

The hierarchy Eq. (10) is the same as the one derived from Eq. (2a). In fact, if we suppose the time part corresponding to Eq. (2a) to be

$$\phi_{n,t_m} = N_m\phi_n, \tag{11}$$

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then the Volterra hierarchy (4a) will be equivalent to the compatibility of Eqs. (2a) and (11), i.e.,

$$M_{n,t_m} = (EN_m)M_n - M_nN_m. \tag{12}$$

In the light of the gauge transformation Eq. (7), we first have the following relations

$$M_n = T_{n+1}\widetilde{U}_nT_n,\ \ N_m = T_{n,t_m}T_n^{-1} + T_nV_mT_n^{-1}.$$

Then, we substitute these relations into Eq. (12) and it turns out that

$$\begin{aligned} &M_{n,t_m} - (EN_m)M_n + M_nN_m \\ =&T_{n+1}(\widetilde{U}_{n,t_m} - (EV_m)\widetilde{U}_n + \widetilde{U}_nV_m)T_n^{-1}, \end{aligned}$$

which means that the two hierarchies (12) and (10) are the same.

Finally, we end this work with the following comment. It is well known that gauge transformation theory of continuous integrable systems has been well established (for description one can refer to Ref. [3] and chapters 4 and 7 in Ref. [4]). However, in the discrete case, due to the lack of differentiation and integration, the analogue theory is still absent and even examples of gauge-equivalent semi-discrete systems are rare. In addition to the example that we give in the present study and the one in Ref. [5], one more example is that the following spectral problem,

$$\psi_{n+1} = \widetilde{S}_n\psi_n,\ \ \widetilde{S}_n = \begin{pmatrix} \lambda^2 + u_n & \lambda \\ \lambda v_n & 0 \end{pmatrix}, \tag{13}$$

which was respectively studied in Refs. [6,7], is related to the well-known α-relativistic Toda spectral problem (see Ref. [1])

$$\phi_{n+1} = S_n\phi_n,\ \ S_n = \begin{pmatrix} \eta(1+\alpha b_n) - 1/\eta & \alpha a_{n-1} \\ -\alpha & 0 \end{pmatrix}, \tag{14}$$

by a gauge transformation

$$\begin{aligned} \psi_n =& T_n\phi_n,\ \ T_n = \lambda^n \begin{pmatrix} 1 & 0 \\ 0 & -v_{n-1}/\alpha \end{pmatrix}, \\ \lambda =& -\frac{1}{\eta},\ \ u_n = -(1+\alpha b_n),\ \ v_n = -\alpha^2 a_n, \end{aligned} \tag{15}$$

where α is a parameter. We hope that our discussion and examples presented in this work could draw attention to the gauge transformation theory of differential-difference integrable systems.

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