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Heun Functions Describing Fermions Evolving in Parallel Electric and Magnetic Fields

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The present work deals with the behavior of fermions moving in a static magnetic induction and a time-harmonic electric field, both oriented along Oz. For the ultra-relativistic particles described by a Heun double confluent equation, we derive the corresponding wave functions and the conserved current density components.

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The theory of particles moving in static or time-periodic external fields has a long history, with a wide range of applications, worked out in the original QED, further extended to non-Abelian theories,[1,2] up to the more exotic processes occurring in the presence of gravity.[3] As the theory of pair creation, belonging to Schwinger[4] who in 1951 derived the formula of the corresponding probability, requires extremely high electric fields, beyond technological capabilities, only processes under extreme conditions such as the ones encountered in high-energy astrophysical objects have been considered as plausible candidates. However, a few years ago, Ringwald mentioned some prospects to observe the Schwinger mechanism in terrestrial facilities.[5] When the produced (charged) particle and its antiparticle are described by normalizable and asymptotically stable modes, one can employ the canonical quantization. Even now, there are extensive studies dedicated to this procedure where both Klein–Gordon and Dirac equations are worked out, in the presence of different configurations of static background fields, to derive the positive and negative energy states, to write down the in- and out-modes and to identify the creation and annihilation operators.[6] Once one gains a deep understanding on how particles are produced in high intense fields, the whole recipe can be applied to more up-to-date situations as for example to hadronic production from a flux tube,[7] which is based on the appearance of a strong electric field just after two nuclei collide and pass through each other.[8]

Inspired by the Glasma model,[8] within a dynamical view of pair creation in pre-equilibrium stages of ultra-relativistic heavy-ion collisions, the Schwinger mechanism has been recently revisited by Tanji.[6] Based on quantum field theory, he has dealt with gluon production[9] and with quark pair creation in the so-called color flux tube,[10] where a longitudinal color magnetic field coexists collinear to the electric one, making the evolution of the fermionic system faster.

In a more involved theoretical study of particle creation, one should take the constant and homogeneous electric field E_0 as an initial state and should go further by taking into account the back reaction, that is the electromagnetic field which is generated by particles and antiparticles originally produced by E_0.[10] Thus after being created, accelerated particles generate a positive conduction current, which decreases the (original) electric field according to Maxwell's law

$$j_z = -\frac{\partial E_z}{\partial t},$$

where j_z can be computed as the conserved current density associated with the $U(1)$ symmetry of the Lagrangian. As a result, the particles start to decelerate until they stop. The whole process is repeated (the plasma oscillation) and the oscillating current can be seen as a result of the harmonic behavior of an electric field of the form $E_z(t) = E_0 \cos(\omega_0 t)$, where the frequency ω_0 is directly related to the pair creation probability per unit volume and time.[4]

The aim of the present work is to extend our previous analysis on the behavior of relativistic fermions in parallel static electric and magnetic fields, oriented along Oz,[11] with particular emphasis on aspects that, in our view, have not been underlined elsewhere.

Configurations with electric and magnetic fields parallel to each other as a result of the rearrangement of the magnetic topology are assumed to exist in the so-called magnetic reconnection.[12] In this physical process that takes place in a highly conducting medium, the magnetic energy is converted into particle acceleration. The resulting electric field aligned to the magnetic one is not permitted in standard fluid formulations, where two distinct field lines remain separate and are not allowed to mix. However, based on observations, it seems that this phenomena might be responsible for explosions known as gigantic flares, routinely observed in Earth's magnetosphere.

We start with the Dirac equation describing relativistic fermions of mass m_0,

$$[\gamma^i D_i + m_0]\,\Psi = 0, \tag{1}$$

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where $D_i = \partial_i - iqA_i$ are the $U(1)$-gauge derivatives, and writing the spinor field Ψ as

$$\Psi = [\gamma^j D_j - m_0]\psi, \tag{2}$$

we can come to the squared equation

$$[\gamma^i D_i + m_0][\gamma^j D_j - m_0]\psi = 0. \tag{3}$$

With the 4-potential components

$$\begin{aligned} A_2 &= A_y(x) = B_0 x, \\ A_3 &= A_z(t) = -\frac{E_0}{\omega_0}\sin(\omega_0 t), \end{aligned} \tag{4}$$

so that $E_z = E_0\cos(\omega_0 t)$, the explicit form of Eq. (3) reads

$$\begin{aligned} &[\partial_x^2 + (\partial_y - iqB_0 x)^2 + (\partial_z + iq\frac{E_0}{\omega_0}\sin(\omega_0 t))^2 \\ &- \partial_t^2 - iqB_0\gamma^1\gamma^2 + iqE_0\gamma^4\gamma^3\cos(\omega_0 t) - m_0^2]\psi = 0. \end{aligned} \tag{5}$$

In the Dirac representation, where

$$\gamma^\mu = -i\beta\alpha^\mu, \quad \gamma^4 = -i\beta, \quad \mu = \overline{1,3}, \tag{6}$$

with

$$\beta = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix}, \quad \alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \tag{7}$$

the calculations greatly simplify since the matrices $\Gamma_3 \equiv -i\gamma^1\gamma^2$ and $\Gamma_4 \equiv \gamma^3\gamma^4$ are commuting and they have the same eigenvectors

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{8}$$

corresponding to the eigenvalues $\lambda_{1,2} = \pm 1$.

With the variables separation

$$\psi_{1,2} = \eta_{1,2}(x)\varphi_{1,2}(t)\exp[i(p_y y + p_z z)]w_{1,2}, \tag{9}$$

Eq. (5) splits into the following system of decoupled equations

$$\frac{d^2\eta_{1,2}}{dx^2} + [A \pm qB_0 - (qB_0 x - p_y)^2]\eta_{1,2} = 0, \tag{10a}$$

$$\begin{aligned} &\frac{d^2\varphi_{1,2}}{dt^2} + \Big[A \pm iqE_0\cos(\omega_0 t) + m_0^2 \\ &+ \Big(q\frac{E_0}{\omega_0}\sin(\omega_0 t) + p_z\Big)^2\Big]\varphi_{1,2} = 0, \end{aligned} \tag{10b}$$

where the $+$ and $-$ signs correspond to ψ_1 and ψ_2, respectively.

With the new variable

$$\sigma = \frac{1}{\sqrt{qB_0}}(qB_0 x - p_y), \tag{11}$$

the two equations in (10), which have bounded solutions for $A = 2nqB_0$, convert into the ones satisfied by the Hermite associated functions $\eta_n(\sigma)$ and $\eta_{n-1}(\sigma)$, where

$$\eta_n(\sigma) = C_n e^{-\sigma^2/2}H_n(\sigma), \tag{12}$$

with the normalization constant

$$C_n = \sqrt{\frac{1}{2\pi}}(qB_0)^{1/4}(2^n n!\sqrt{\pi})^{-1/2}. \tag{13}$$

In a previous work,[11] we have focused on a particular case, by neglecting the coupling between p_z and qE_0 and the term in $(qE_0)^2$ and we have arrived to a Mathieu-type equation. As a result of parametric resonance, the wave function is experiencing (unnormalizable) unstable modes, which alternate with the stable ones, and this behavior can be associated with the particle production phenomena in magnetars.[13] Now, let us turn to a more involved case by solving the full form of Eq. (10b) which, in terms of the dimensionless quantities

$$\Omega^2 \equiv \frac{2nqB_0 + m_0^2 + p_z^2}{\omega_0^2}, \quad \mathcal{E} \equiv \frac{qE_0}{\omega_0^2}, \quad P_z \equiv \frac{p_z}{\omega_0}, \tag{14}$$

and the new variable $\zeta = \omega_0 t/2$, read

$$\begin{aligned} &\frac{d^2\varphi_{1,2}}{d\zeta^2} + 4[\Omega^2 \pm i\mathcal{E}\cos(2\zeta) + 2\mathcal{E}P_z\sin(2\zeta) \\ &+ \mathcal{E}^2\sin^2(2\zeta)]\varphi_{1,2} = 0. \end{aligned} \tag{15}$$

For the plus sign in Eq. (15), corresponding to λ_1, the wave function φ_1 can be taken as

$$\varphi_1 = \exp[\mathcal{E}\sin(2\zeta)]u_1(\zeta), \tag{16}$$

where the new function u_1 satisfies the differential equation

$$\begin{aligned} &\frac{d^2u_1}{d\zeta^2} + 4\mathcal{E}\cos(2\zeta)\frac{du_1}{d\zeta} + 4[\Omega^2 + \mathcal{E}^2 + i\mathcal{E}\cos(2\zeta) \\ &+ \mathcal{E}(2P_z - 1)\sin(2\zeta)]u_1 = 0. \end{aligned} \tag{17}$$

Firstly, we change the variable ζ to $s = \tan\zeta$ and, to obtain a by-the-book form, we must turn to the purely imaginary variable $\chi = i/s$. One may notice that the corresponding equation

$$\begin{aligned} &\frac{d^2u_1}{d\chi^2} - \frac{1}{(\chi^2-1)^2}[4i\mathcal{E}(\chi^2+1) + 2\chi - 2\chi^3]\frac{du_1}{d\chi} \\ &+ \frac{4}{(\chi^2-1)^3}[\Omega^2 + \mathcal{E}^2 - i\mathcal{E} - 2i\mathcal{E}\chi(2P_z - 1) \\ &- (\Omega^2 + \mathcal{E}^2 + i\mathcal{E})\chi^2]u_1 = 0 \end{aligned} \tag{18}$$

is Heun's double confluent equation[14]

$$\begin{aligned} &\frac{d^2y}{dz^2} - \frac{1}{(z^2-1)^2}[\alpha + 2z + \alpha z^2 - 2z^3]\frac{dy}{dz} \\ &+ \frac{1}{(z^2-1)^3}[\delta + (2\alpha + \gamma)z + \beta z^2]y = 0. \end{aligned} \tag{19}$$

Thus the function u_1 is Heun's double confluent function of parameters

$$\alpha = 4i\mathcal{E}, \quad \beta_1 = -4[\Omega^2 + \mathcal{E}^2 + i\mathcal{E}],$$
$$\gamma = -16i\mathcal{E}P_z, \quad \delta_1 = 4[\Omega^2 + \mathcal{E}^2 - i\mathcal{E}], \quad (20)$$

thus the function φ_1 defined in Eq. (16) is

$$\varphi_1 = \exp[\mathcal{E}\sin(2\zeta)]HeunD[\alpha, \beta_1, \gamma, \delta_1; i\cot\zeta]. \quad (21)$$

By employing the same procedure, the function φ_2 reads

$$\varphi_2 = \exp[\mathcal{E}\sin(2\zeta)]HeunD[\alpha, \beta_2, \gamma, \delta_2; i\cot\zeta], \quad (22)$$

with the parameters

$$\alpha = 4i\mathcal{E}, \quad \beta_2 = -4[\Omega^2 + \mathcal{E}^2 - i\mathcal{E}],$$
$$\gamma = -16i\mathcal{E}P_z, \quad \delta_2 = 4[\Omega^2 + \mathcal{E}^2 + i\mathcal{E}]. \quad (23)$$

Now, one has all the essentials to write down the wave functions $\Psi_{1,2}$ as

$$\Psi_1 = \frac{1}{\sqrt{2m_0}} \cdot \begin{bmatrix} \left[p_z - m_0 - i\omega_0\mathcal{E}e^{i\omega_0 t} - \frac{i\omega_0}{2}\frac{u_1'}{u_1}\right]\eta_n \\ i\sqrt{2nqB_0}\eta_{n-1} \\ \left[-p_z - m_0 + i\omega_0\mathcal{E}e^{i\omega_0 t} + \frac{i\omega_0}{2}\frac{u_1'}{u_1}\right]\eta_n \\ -i\sqrt{2nqB_0}\eta_{n-1} \end{bmatrix} \cdot \exp[\mathcal{E}\sin(\omega_0 t)]\exp[i(p_y y + p_z z)]u_1, \quad (24)$$

and

$$\Psi_2 = \frac{1}{\sqrt{2m_0}} \cdot \begin{bmatrix} i\sqrt{2nqB_0}\eta_n \\ \left[-p_z - m_0 - i\omega_0\mathcal{E}e^{-i\omega_0 t} - \frac{i\omega_0}{2}\frac{u_2'}{u_2}\right]\eta_{n-1} \\ -i\sqrt{2nqB_0}\eta_n \\ \left[p_z - m_0 + i\omega_0\mathcal{E}e^{-i\omega_0 t} + \frac{i\omega_0}{2}\frac{u_2'}{u_2}\right]\eta_{n-1} \end{bmatrix} \cdot \exp[\mathcal{E}\sin(\omega_0 t)]\exp[i(p_y y + p_z z)]u_2, \quad (25)$$

where the prime denotes the derivatives of the functions u_A (with $A = 1, 2$), with respect to $\zeta = \omega_0 t/2$ and we have used the $2m_0$-normalization convention which holds for massive fermions. However, a clear estimation of the proper constant is difficult to carry out due to the fact that, in the case of Heun's functions, there are unsolved problems related to their normalization procedure and computation of their derivatives.

As in relativistic quantum mechanics, the current density components and the charge density are defined as

$$j_\mu = q\,\Psi^\dagger\alpha^\mu\,\Psi, \quad \rho = q\,\Psi^\dagger\,\Psi. \quad (26)$$

Using the expressions of Ψ_1 and Ψ_2 given in Eqs. (24) and (25), one is able to write the components j_x and j_y as

$$j_x^A = \frac{2q}{m_0}\sqrt{2nqB_0}\eta_n\eta_{n-1}\exp[2\mathcal{E}\sin(\omega_0 t)] \times \{\omega_0\mathcal{E}\cos(\omega_0 t)|u_A|^2 + \frac{\omega_0}{2}[\mathrm{Re}(u_A)\mathrm{Re}(u_A') + \mathrm{Im}(u_A)\mathrm{Im}(u_A')]\}, \quad (27)$$

and

$$j_y^A = \frac{2q}{m_0}\sqrt{2nqB_0}\eta_n\eta_{n-1}\exp[\mathcal{E}\sin(\omega_0 t)]\{[-p_z - \omega_0\mathcal{E}\sin(\omega_0 t)]|u_A|^2 \pm \frac{\omega_0}{2}[\mathrm{Re}(u_A')\mathrm{Im}(u_A) - \mathrm{Re}(u_A)\mathrm{Im}(u_A')]\}. \quad (28)$$

In the case of j_z, by keeping only the contributions which survive in the superposition $j_z = j_z^1 + j_z^2$ and using $\int \eta_i\eta_k = \delta_{ik}$, we obtain

$$j_z = \frac{2q}{m_0}\{\omega_0(p_z + \omega_0\mathcal{E}\sin(\omega_0 t))[\mathrm{Re}(u_1')\mathrm{Im}(u_1) - \mathrm{Re}(u_1)\mathrm{Im}(u_1')] - qE_0\cos(\omega_0 t)[\mathrm{Re}(u_1')\mathrm{Re}(u_1) + \mathrm{Im}(u_1)\mathrm{Im}(u_1')]\}\exp[2\mathcal{E}\sin(\omega_0 t)]. \quad (29)$$

On the r.h.s. of Eq. (29), one may notice a (classic) current created by the fast moving color charge densities, in the electric and magnetic fields parallel to the collision beam axis, Oz,

$$j_z^{\mathrm{classic}} = \frac{2q}{m_0}\omega_0 p_z[\mathrm{Re}(u_1')\mathrm{Im}(u_1) - \mathrm{Re}(u_1)\mathrm{Im}(u_1')]\exp[2\mathcal{E}\sin(2\zeta)], \quad (30)$$

the induction current

$$j_z^{\mathrm{ind}} = \frac{2q}{m_0}[qE_0\sin(\omega_0 t)][\mathrm{Re}(u_1')\mathrm{Im}(u_1) - \mathrm{Re}(u_1)\mathrm{Im}(u_1')]\exp[2\mathcal{E}\sin(2\zeta)], \quad (31)$$

and an Ohmic-type current

$$j_z^{\mathrm{Ohm}} = \frac{2q}{m_0}[qE_0\cos(\omega_0 t)][\mathrm{Re}(u_1')\mathrm{Re}(u_1) + \mathrm{Im}(u_1)\mathrm{Im}(u_1')]\exp[2\mathcal{E}\sin(\omega_0 t)]. \quad (32)$$

Finally, by using the series expansion

$$u = 1 + \frac{\beta\zeta^2}{2} + \mathcal{O}(\zeta^3),$$

to first order in ζ, the expressions Eqs. (30)–(32) can be put into the physical form

$$j_z^{\mathrm{classic}} \approx \frac{4q}{m_0}p_z(qE_0 t), \quad j_z^{\mathrm{ind}} \approx \frac{4q}{m_0}(qE_0 t)^2,$$

and

$$j_z^{\mathrm{Ohm}} \approx \frac{4q}{m_0\omega_0}[qE_0\cos(\omega_0 t)]W^2 t,$$

where

$$W^2 \equiv 2nqB_0 + m_0^2 + p_z^2 + \left(\frac{qE_0}{\omega_0}\right)^2,$$

the above expressions being valid for $\omega_0 t \ll 1$.

In summary, we have extended our previous analysis,[11] by considering the time-harmonic field $E_z = E_0 \cos(\omega_0 t)$. This would appear as a back reaction, namely as a result of the oscillating current generated by the particles and antiparticles originally produced by E_0. For ultra-relativistic particles moving in harmonic (in time) external fields, in the approximation $qE_0/\omega_0 \ll p_z$, both the Klein–Gordon and Dirac equations are leading to a Mathieu-type equation.[11,15] One can find ranges of the model parameters where parametric resonances occur, the wave function is exponentially growing[15] and the canonical quantization is not applicable. In the more involved case investigated in this work, when the full Dirac equation is considered, one ends up with a Heun-type equation, whose solutions are the Heun double confluent functions. Eqs. (24) and (25) of spinors have allowed us to compute the conserved current density components, where we have identified the usual contributions, namely the Ohmic one and the one corresponding to the (classical) movement of real particles.

Finally, we would like to mention that there is renewed interest in constructing closed-form analytical solutions to equations describing the quantum world, and several approaches have been formulated.[15–19]

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