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Motion of a Nonrelativistic Quantum Particle in Non-commutative Phase Space

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The equation governing the motion of a quantum particle is considered in nonrelativistic non-commutative phase space. For this aim, we first study new Poisson brackets in non-commutative phase space and obtain the modified equations of motion. Next, using novel transformations, we solve the equation of motion and report the exact analytical solutions.

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The ingenious idea of non-commutative (NC) spacetime geometry goes back to Schrödinger and Heisenberg. Peierls, Pauli and Snyder were the next ones who took the theory some steps forward. In particular, Snyder used the NC structure of spacetime to introduce a small length scale cutoff in field theory without breaking Lorentz invariance.[1] Soon afterwards, Yang generalized Snyder's work by working on a quantized spacetime.[2] Neumann contributed to the theory by introducing the non-commutative geometry. This was the first step to formulate the Neumann algebras and also the NC geometry.[3] A great achievement in the mathematical basis of the theory was the works of Connes[4] and Woronowicz,[5] where a differential structure was introduced for the NC framework.[6] In recent years, there has been increasing interest for studies in non-commutative space (NCS)[7−10] and non-commutative phase space (NCPS), with a variety of problems revisited in this formulation.[11−14] Until now, various important problems of quantum mechanics have been analyzed in this formulation including quantum Hall effect,[15,16] Landau problem[17−19] and DKP oscillator.[20] In this work, we are going to consider the motion of a nonrelativistic quantum particle in the field of a harmonic oscillator interaction in NCPS.

In the NC formalism, the position coordinates satisfy

$$[x_\mu, x_\upsilon] = i\theta_{\mu\upsilon}, \tag{1}$$

where $\theta_{\mu\upsilon}$ is an anti-symmetric tensor and is of space dimension $(L)^2$.[21] The commutation relations in NCS are[22]

$$\begin{aligned}[x^{i(\mathrm{Nc})}, x^{j(\mathrm{Nc})}] &= i\theta^{ij},\\ [x^{i(\mathrm{Nc})}, p^{j(\mathrm{Nc})}] &= i\hbar\delta^{ij},\\ [p^{i(\mathrm{Nc})}, p^{j(\mathrm{Nc})}] &= 0.\end{aligned} \tag{2}$$

The commutation relations in NCPS appear as[21]

$$\begin{aligned}[x^{i(\mathrm{Nc})}, x^{j(\mathrm{Nc})}] &= i\theta^{ij},\\ [x^{i(\mathrm{Nc})}, p^{j(\mathrm{Nc})}] &= i\hbar_{\mathrm{eff}}\delta^{ij},\\ [p^{i(\mathrm{Nc})}, p^{j(\mathrm{Nc})}] &= i\eta^{ij},\end{aligned} \tag{3}$$

with the effective Planck's constant

$$\hbar_{\mathrm{eff}} = 1 + \frac{\eta\theta}{4\hbar}, \tag{4}$$

where $\theta^{ij} = \varepsilon^{ijk}\theta_k$ and $\eta^{ij} = \varepsilon^{ijk}\eta_k$ are real antisymmetric matrix elements and represent the NC parameters. The linear transformations between commutative and NC variables are defined as

$$x_i^{(\mathrm{Nc})} = x_i^{(\mathrm{c})} - \frac{1}{2}\theta_{ij}p_j^{(\mathrm{c})}, \tag{5}$$

$$p_i^{(\mathrm{Nc})} = p_i^{(\mathrm{c})} + \frac{1}{2}\eta_{ij}x_j^{(\mathrm{c})}. \tag{6}$$

The general form of the Poisson brackets in NCPS formulation can be written as[22,23]

$$\begin{aligned}\{A, B\} = &\left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i}\right)\\ &+ \theta_{ij}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_j} + \eta_{ij}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_j}.\end{aligned} \tag{7}$$

We consider a two-dimensional Hamiltonian in the form

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + V(x_1, x_2), \tag{8}$$

where p_1, p_2 and x_1, x_2 represent the components of momentum and space along the x and y directions, respectively. The equations of motion corresponding to this Hamiltonian are given by

$$\dot{x}_i = \{x_i, H\} = \frac{p_i}{m} + \theta_{ij}\frac{\partial V}{\partial x_j}, \tag{9}$$

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial V}{\partial x_i} + \frac{1}{m}\eta_{ij}p_j. \tag{10}$$

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The combination of Eqs. (9) and (10) results in

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} + \eta_{ij}\dot{x}_j + m\theta_{ij}\frac{d}{dt}\left(\frac{\partial V}{\partial x_j}\right),\ i = 1, 2, \quad (11)$$

which is the new second law of Newton. For a free particle in two dimensions, we have

$$H^{(\mathrm{Nc})} = \frac{p_1^{2(\mathrm{Nc})}}{2m} + \frac{p_2^{2(\mathrm{Nc})}}{2m}. \quad (12)$$

Substituting Eq. (6) into Eq. (12) and considering $L = x \times P$, we have

$$H_{\mathrm{f}}^{(\mathrm{Nc})} = [p_{1\mathrm{f}}^2 + p_{2\mathrm{f}}^2 + \frac{1}{4}\eta^2(x_{1\mathrm{f}}^2 + x_{2\mathrm{f}}^2) + \eta(x_{2\mathrm{f}}p_{1\mathrm{f}} - x_{1\mathrm{f}}p_{2\mathrm{f}})]/(2M). \quad (13)$$

Introducing

$$\tilde{\omega}^2 = \frac{1}{4M}\eta^2, \quad (14)$$

the Hamiltonian (13) is alternatively written as

$$H_{\mathrm{f}}^{(\mathrm{Nc})} = \frac{p_{1\mathrm{f}}^2 + p_{2\mathrm{f}}^2}{2M} + \frac{1}{2}M\tilde{\omega}^2(x_{1\mathrm{f}}^2 + x_{2\mathrm{f}}^2) + \frac{-\eta(-x_{2\mathrm{f}}p_{1\mathrm{f}} + x_{1\mathrm{f}}p_{2\mathrm{f}})}{2M}. \quad (15)$$

The index f indicates the free particle. The latter shows that the free particle example in NCPS resembles the ordinary harmonic oscillator. Now considering the eigenstate as $|n, m\rangle$, the Z-component of the angular momentum satisfies

$$L_z|n, m\rangle = m\hbar|n, m\rangle. \quad (16)$$

According to the eigenvalue equation $H^{(\mathrm{Nc})}$, we have

$$H_{\mathrm{f}}^{(\mathrm{Nc})}|N_1^{(\mathrm{Nc})}, N_2^{(\mathrm{Nc})}\rangle = E_{\mathrm{f}}^{(\mathrm{Nc})}|N_1^{(\mathrm{Nc})}, N_2^{(\mathrm{Nc})}\rangle. \quad (17)$$

Using Eqs. (16) and (17), we find

$$E_{\mathrm{f(total)}}^{(\mathrm{Nc})} = (n_1 + n_2 + 1)\hbar\tilde{\omega} - \frac{\eta}{2M}m\hbar. \quad (18)$$

Now we obtain the equation of motion of Hamiltonian (15). We first introduce the classical Poisson brackets as[24]

$$[A(q,p), B(q,p)]_{\mathrm{classical}} \equiv \sum_s \left(\frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s}\right). \quad (19)$$

By using Eq. (19), the equations of motion are given by

$$\dot{x}_{1\mathrm{f}} = \{x_{1\mathrm{f}}, H_{\mathrm{f}}^{(\mathrm{Nc})}\} = \frac{\partial x_{1\mathrm{f}}}{\partial x_{1\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial p_{1\mathrm{f}}} - \frac{\partial x_{1\mathrm{f}}}{\partial p_1 f}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial x_{1\mathrm{f}}},$$
$$\dot{p}_{1\mathrm{f}} = \{p_{1\mathrm{f}}, H_{\mathrm{f}}^{(\mathrm{Nc})}\} = \frac{\partial p_{1\mathrm{f}}}{\partial x_{1\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial p_{1\mathrm{f}}} - \frac{\partial p_{1\mathrm{f}}}{\partial p_{1\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial x_{1\mathrm{f}}}, \quad (20a)$$

$$\dot{x}_{2\mathrm{f}} = \{x_{2\mathrm{f}}, H_{\mathrm{f}}^{(\mathrm{Nc})}\} = \frac{\partial x_{2\mathrm{f}}}{\partial x_{2\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial p_{2\mathrm{f}}} - \frac{\partial x_{2\mathrm{f}}}{\partial p_{2\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial x_{2\mathrm{f}}},$$
$$\dot{p}_{2\mathrm{f}} = \{p_{2\mathrm{f}}, H_{\mathrm{f}}^{(\mathrm{Nc})}\} = \frac{\partial p_{2\mathrm{f}}}{\partial x_{2\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial p_{2\mathrm{f}}} - \frac{\partial p_{2\mathrm{f}}}{\partial p_{2\mathrm{f}}}\frac{\partial H_{\mathrm{f}}^{(\mathrm{Nc})}}{\partial x_{2\mathrm{f}}}, \quad (20b)$$

which give

$$\dot{x}_{1\mathrm{f}} = \frac{p_{1\mathrm{f}}}{M} + \frac{\eta x_{2\mathrm{f}}}{2M},\quad \dot{p}_{1\mathrm{f}} = -M\tilde{\omega}^2 x_{1\mathrm{f}} + \frac{\eta p_{2\mathrm{f}}}{2M}, \quad (21a)$$
$$\dot{x}_{2\mathrm{f}} = \frac{p_{2\mathrm{f}}}{M} - \frac{\eta x_{1\mathrm{f}}}{2M},\quad \dot{p}_{2\mathrm{f}} = -M\tilde{\omega}^2 x_{2\mathrm{f}} - \frac{\eta p_{1\mathrm{f}}}{2M}. \quad (21b)$$

The combination of Eqs. (21a) and (21b) results in

$$\ddot{x}_{1\mathrm{f}} = \beta x_{1\mathrm{f}} - \alpha\dot{x}_{2\mathrm{f}}, \quad (22a)$$
$$\ddot{x}_{2\mathrm{f}} = \beta x_{2\mathrm{f}} + \alpha\dot{x}_{1\mathrm{f}}, \quad (22b)$$

where

$$\alpha = -\frac{\eta}{M},\quad \beta = -\tilde{\omega}^2 - \frac{\eta^2}{4M^2}. \quad (23)$$

Introducing

$$x_{1\mathrm{f}}(t) = \varphi_{1\mathrm{f}}(t),$$
$$x_{2\mathrm{f}}(t) = \varphi_{2\mathrm{f}}(t), \quad (24a)$$

and combining Eqs. (22a), (22b) and (24a), we have

$$\dot{\varphi}_{1\mathrm{f}}(t) = \varphi_{3\mathrm{f}}(t),$$
$$\dot{\varphi}_{2\mathrm{f}}(t) = \varphi_{4\mathrm{f}}(t),$$
$$\dot{\varphi}_{3\mathrm{f}}(t) = \beta\varphi_{1\mathrm{f}}(t) - \alpha\varphi_{4\mathrm{f}}(t),$$
$$\dot{\varphi}_{4\mathrm{f}}(t) = \beta\varphi_{2\mathrm{f}}(t) + \alpha\varphi_{3\mathrm{f}}(t). \quad (24b)$$

The above equations can be alternatively written in the matrix form

$$\begin{pmatrix}\dot{\varphi}_{1\mathrm{f}}\\ \dot{\varphi}_{2\mathrm{f}}\\ \dot{\varphi}_{3\mathrm{f}}\\ \dot{\varphi}_{4\mathrm{f}}\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \beta & 0 & 0 & -\alpha\\ 0 & \beta & \alpha & 0\end{pmatrix}\begin{pmatrix}\varphi_{1\mathrm{f}}\\ \varphi_{2\mathrm{f}}\\ \varphi_{3\mathrm{f}}\\ \varphi_{4\mathrm{f}}\end{pmatrix}, \quad (25)$$

which possesses the following four eigenvalues

$$\lambda_{1\mathrm{f}} = \frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{2\mathrm{f}} = -\frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{3\mathrm{f}} = \frac{1}{2}\sqrt{4\beta - 2\alpha^2 - 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{4\mathrm{f}} = -\frac{1}{2}\sqrt{4\beta - 2\alpha^2 - 2\sqrt{-4\alpha^2\beta + \alpha^4}}. \quad (26)$$

Choosing λ_1, Eq. (24b) yields the set

$$A_{\mathrm{f}}\lambda_{1\mathrm{f}} - C_{\mathrm{f}} = 0,$$
$$B_{\mathrm{f}}\lambda_{1\mathrm{f}} - D_{\mathrm{f}} = 0,$$
$$C_{\mathrm{f}}\lambda_{1\mathrm{f}} - \beta A_{\mathrm{f}} + \alpha D_{\mathrm{f}} = 0,$$
$$D_{\mathrm{f}}\lambda_{1\mathrm{f}} - \beta B_{\mathrm{f}} - \alpha C_{\mathrm{f}} = 0. \quad (27)$$

Considering $C_{\rm f} = 1$, other unknown coefficients appear as

$$A_{\rm f} = \frac{1}{\lambda_{1\rm f}},\ B_{\rm f} = \frac{1}{\alpha}\Big(\frac{\beta - \lambda_{1\rm f}^2}{\lambda_{1\rm f}^2}\Big),\ D_{\rm f} = \frac{1}{\alpha}\Big(\frac{\beta - \lambda_{1\rm f}^2}{\lambda_{1\rm f}}\Big). \tag{28}$$

Thus we can conclude that the solution to the equation of motion for a free particle in the NCPS is

$$\begin{aligned} \varphi_{1\rm f} &= A_{\rm f} e^{\lambda_{1\rm f} t},\quad \varphi_{2\rm f} = B_{\rm f} e^{\lambda_{1\rm f} t}, \\ \varphi_{3\rm f} &= C_{\rm f} e^{\lambda_{1\rm f} t},\quad \varphi_{4\rm f} = D_{\rm f} e^{\lambda_{1\rm f} t}. \end{aligned} \tag{29}$$

The results obtained for λ_1 can be repeated for λ_2, λ_3, and λ_4. The harmonic oscillator model plays an essential role in classical quantum mechanics and NC quantum mechanics[22] and has been the subject of many important studies.[25] Let us consider now a harmonic oscillator characterized by the two-dimensional Hamiltonian

$$H_{\rm osc}^{\rm (Nc)} = \frac{1}{2m}(p_{1\rm osc}^{\rm (Nc)2} + p_{2\rm osc}^{\rm (Nc)2}) + V(x_{1\rm osc}^{\rm (Nc)}, x_{2\rm osc}^{\rm (Nc)}), \tag{30}$$

where the subscript osc indicates the Harmonic oscillator and

$$V(x_{1\rm osc}^{\rm (Nc)}, x_{2\rm osc}^{\rm (Nc)}) = \frac{1}{2}k\sum_{i=1}^{2} x_{i\rm osc}^{\rm (Nc)2}. \tag{31}$$

Substituting Eq. (31) into Eq. (11), we have

$$m\ddot{x}_{i\rm osc} = -kx_{i\rm osc} + \eta_{ij}\dot{x}_{j\rm osc} + m\theta_{ij}k\dot{x}_{j\rm osc}, \tag{32}$$

which is obviously the new second law of Newton for the harmonic oscillator. Let us now consider the Hamiltonian of a harmonic oscillator in NCPS.[22] Substitution of Eqs. (5) and (6) into Eq. (30) brings the Hamiltonian into the form

$$\begin{aligned} H_{\rm osc}^{\rm (Nc)} = \frac{1}{2m}\Big(&p_{1\rm osc}^2 + \frac{1}{2}\eta p_{1\rm osc}x_{2\rm osc} + \frac{1}{2}\eta x_{2\rm osc}p_{1\rm osc} \\ &+ \frac{1}{4}\eta^2 x_{2\rm osc}^2 + p_{2\rm osc}^2 - \frac{1}{2}p_{2\rm osc}\eta x_{1\rm osc} \\ &- \frac{1}{2}\eta x_{1\rm osc}p_{2\rm osc} + \frac{1}{4}\eta^2 x_{1\rm osc}^2\Big) + \frac{1}{2}k\Big(x_{1\rm osc}^2 \\ &- x_{1\rm osc}\theta\frac{p_{2\rm osc}}{2} - \frac{1}{2}\theta p_{2\rm osc}x_{1\rm osc} + \frac{1}{4}\theta^2 p_{2\rm osc}^2 \\ &+ x_{2\rm osc}^2 + \frac{1}{2}x_{2\rm osc}\theta p_{1\rm osc} \\ &+ \frac{1}{2}\theta p_{1\rm osc}x_{2\rm osc} + \frac{1}{4}\theta^2 p_{1\rm osc}^2\Big), \end{aligned} \tag{33}$$

or

$$\begin{aligned} H_{\rm osc}^{\rm (Nc)} = \frac{1}{2m}&(p_{1\rm osc}^2 + p_{2\rm osc}^2 + \eta x_{2\rm osc}p_{1\rm osc} \\ &- \eta x_{1\rm osc}p_{2\rm osc}) + \frac{1}{2}k(x_{1\rm osc}^2 + x_{2\rm osc}^2 \\ &+ \theta x_{2\rm osc}p_{1\rm osc} - \theta x_{1\rm osc}p_{2\rm osc}). \end{aligned} \tag{34}$$

Using Eqs. (19) and (20), the equations of motion for the harmonic oscillator are given by

$$\dot{x}_{1\rm osc} = \{x_{1\rm osc}, H_{\rm osc}^{\rm (Nc)}\} = \frac{p_{1\rm osc}}{m} + \frac{\eta x_{2\rm osc}}{2m} + \frac{k\theta x_{2\rm osc}}{2}, \tag{35}$$

$$\begin{aligned} \dot{p}_{1\rm osc} &= \{p_{1\rm osc}, H_{\rm osc}^{\rm (Nc)}\} = \frac{\eta p_{2\rm osc}}{2m} - kx_{1\rm osc} + \frac{k\theta p_{2\rm osc}}{2} \\ &= -kx_{1\rm osc} + \frac{\eta\dot{x}_{2\rm osc}}{2} + \frac{k\theta m\dot{x}_{2\rm osc}}{2}. \end{aligned} \tag{36}$$

Combination of Eqs. (35) and (36) results in

$$m\ddot{x}_{1\rm osc} = -kx_{1\rm osc} + \eta\dot{x}_{2\rm osc} + k\theta m\dot{x}_{2\rm osc}. \tag{37}$$

In the same way, we can write for the second component

$$\begin{aligned} \dot{x}_{2\rm osc} &= \{x_{2\rm osc}, H_{\rm osc}^{\rm (Nc)}\} \\ &= \frac{p_{2\rm osc}}{m} - \frac{\eta x_{1\rm osc}}{2m} - \frac{k\theta x_{1\rm osc}}{2}, \\ \dot{p}_{2\rm osc} &= \{p_{2\rm osc}, H_{\rm osc}^{\rm (Nc)}\} \\ &= -kx_{2\rm osc} - \frac{\eta\dot{x}_{1\rm osc}}{2} - \frac{k\theta m\dot{x}_{1\rm osc}}{2}, \end{aligned} \tag{38}$$

which gives

$$m\ddot{x}_{2\rm osc} = -kx_{2\rm osc} - \eta\dot{x}_{1\rm osc} - k\theta m\dot{x}_{1\rm osc}. \tag{39}$$

The obtained Eqs. (37) and (39) are the equations of motion governing the harmonic oscillator problem in NCPS.

We arrange Eqs. (37) and (39) as

$$\ddot{x}_{1\rm osc} = -\frac{k}{m}x_{1\rm osc} + \Big(\frac{\eta + k\theta m}{m}\Big)\dot{x}_{2\rm osc}, \tag{40}$$

$$\ddot{x}_{2\rm osc} = -\frac{k}{m}x_{2\rm osc} - \Big(\frac{\eta + k\theta m}{m}\Big)\dot{x}_{1\rm osc}. \tag{41}$$

Introducing the variables

$$a = -\frac{k}{m},\quad b = \frac{\eta + \theta km}{m} \tag{42}$$

and functions

$$x_{1\rm osc}(t) = \varphi_{1\rm osc}(t),\quad x_{2\rm osc}(t) = \varphi_{2\rm osc}(t), \tag{43a}$$

the combination of Eqs. (40), (41) and (43a) results in

$$\begin{aligned} \dot{\varphi}_{1\rm osc}(t) &= \varphi_{3\rm osc}(t), \\ \dot{\varphi}_{2\rm osc}(t) &= \varphi_{4\rm osc}(t), \\ \dot{\varphi}_{3\rm osc}(t) &= \beta\varphi_{1\rm osc}(t) - \alpha\varphi_{4\rm osc}(t), \\ \dot{\varphi}_{4\rm osc}(t) &= \beta\varphi_{2\rm osc}(t) + \alpha\varphi_{3\rm osc}(t). \end{aligned} \tag{43b}$$

The present set of equations can be alternatively written in the matrix form

$$\begin{pmatrix} \dot{\varphi}_{1\rm osc} \\ \dot{\varphi}_{2\rm osc} \\ \dot{\varphi}_{3\rm osc} \\ \dot{\varphi}_{4\rm osc} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & -\alpha \\ 0 & \beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} \varphi_{1\rm osc} \\ \varphi_{2\rm osc} \\ \varphi_{3\rm osc} \\ \varphi_{4\rm osc} \end{pmatrix}, \tag{44}$$

which possesses the eigenfunctions

$$\lambda_{1\text{osc}} = \frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{2\text{osc}} = -\frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{3\text{osc}} = \frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}},$$
$$\lambda_{4\text{osc}} = -\frac{1}{2}\sqrt{4\beta - 2\alpha^2 + 2\sqrt{-4\alpha^2\beta + \alpha^4}}. \quad (45)$$

Let us consider λ_1. Eq. (43b) yields the set

$$\begin{aligned} &A_{\text{osc}}\lambda_{1\text{osc}} - C_{\text{osc}} = 0,\\ &B_{\text{osc}}\lambda_{1\text{osc}} - D_{\text{osc}} = 0,\\ &C_{\text{osc}}\lambda_{1\text{osc}} - \beta A_{\text{osc}} + \alpha D_{\text{osc}} = 0,\\ &D_{\text{osc}}\lambda_{1\text{osc}} - \beta B_{\text{osc}} - \alpha C_{\text{osc}} = 0. \end{aligned} \quad (46)$$

Considering $C_{\text{osc}} = 1$, other unknown coefficients appear as

$$A_{\text{osc}} = \frac{1}{\lambda_{1\text{osc}}},\ B_{\text{osc}} = \frac{1}{\alpha}\Big(\frac{\beta - \lambda_{1\text{osc}}^2}{\lambda_{1\text{osc}}^2}\Big),$$
$$D_{\text{osc}} = \frac{1}{\alpha}\Big(\frac{\beta - \lambda_{1\text{osc}}^2}{\lambda_{1\text{osc}}}\Big). \quad (47)$$

Thus we can conclude that the solution of motion equation for a harmonic oscillator is

$$\varphi_{1\text{osc}} = A_{\text{osc}}e^{\lambda_{1\text{osc}}t},\ \ \varphi_{2\text{osc}} = B_{\text{osc}}e^{\lambda_{1\text{osc}}t},$$
$$\varphi_{3\text{osc}} = C_{\text{osc}}e^{\lambda_{1\text{osc}}t},\ \ \varphi_{4\text{osc}} = D_{\text{osc}}e^{\lambda_{1\text{osc}}t}. \quad (48)$$

The results obtained for λ_1 can be repeated for λ_2, λ_3 and λ_4.

In summary, we have considered new Poisson brackets in non-commutative phase space and have developed the modified Newton's second law for harmonic oscillator and free particle cases. We also find the solution for the obtained equation of motion. We hope that our study may open new insight into possible further research in the area.

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