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Quantum Search Algorithm Based on Multi-Phase *

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The success probability of the Grover quantum search algorithm decreases quickly when the fraction of target items exceeds 1/4, where the phase plays a significant role. Therefore, we use multiple phases to complement each other. We obtain three useful properties and an important theorem of the success probability and design a systematic solution of the optimal phases for an arbitrary number of phases. Based on these results, we finally propose a multi-phase quantum search algorithm whose success probability rises with the increase of the number of phases with just a single iteration, and it tends to be 100% when the fraction of target items is over a lower limit.

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The Grover algorithm[1,2] has been of great interest since it was proposed in 1996. When applied to an unstructured database of size N with only one target item, the Grover algorithm can obtain the target item within $O(\sqrt{N})$ which achieves quadratic speedup. However, when the fraction of target items increases from 1/4 to 1/2, the algorithm's success probability decreases rapidly and reaches its minimum at 50%.

A great deal of effort has been expended to analyze and/or generalize the original Grover algorithm,[3−9] where the phase research plays an important role. The arbitrary phase quantum search algorithm was studied in Refs. [3,4], by extending the algorithm phases to be arbitrary. Researchers focused on the success probability of finding a target item, and obtained a number of algorithms. In particular, a quantum search algorithm with full probability was shown in Ref. [5], with $O(\sqrt{N/M})$ original Grover transforms and an additional arbitrary-phase transform. However, the phase of this algorithm varies along with the number of target items, which needs to be known.

Trying to let the algorithm phase be fixed, researchers have carried out much work on the fixed phase algorithm.[10,11] In Ref. [10], Li *et al.* proposed a fixed-phase quantum search algorithm with more flexible behavior, yielding a success probability of 99.38% with the phase fixed at 0.1π. In Ref. [11], Younes presented a fixed-phase search algorithm, yielding the minimum success probability at 99.58% with the phase fixed at 1.91684π. By researching the relationship among the success probability, the phases, the number of target items and the number of iterations comprehensively, Zhong *et al.* obtained a quantum search algorithm based on the phase 1.018 with the success probability larger than 93.43%.[12]

To obtain a high success probability and algorithm efficiency, the phase matching conditions have been well researched.[4,13−15] Long *et al.* showed in Ref. [14] that several other phase conditions were special cases of their condition. Based on the condition proposed in Ref. [15], Toyama *et al.* designed the multiple Grover operation, and presented a multiphase matching for the Grover algorithm.[16] Unlike the one Grover operation, the multiple Grover operation involves multiple pairs of phases, which operates pair by pair. The required iteration of this algorithm equals the number of pairs. This algorithm discussed one-, two-, and six-pair-phase special cases, and showed that the six-pair-phase operation can yield a success probability over 99.8% when the fraction of target items exceeds 0.1.

On the research of multiple phases, Zhong designed a two-phase algorithm for the multi-object search problem with phases 1.8 and 1.4.[17] By using phases in a different and selectable way, the algorithm yields a success probability over 97.42%, with a single iteration. However, the general multi-phase search algorithm with arbitrary n phases is not pointed out in Ref. [17], which is confined to two phases. The known public literature shows that the general multi-phase search problem is not well researched yet. Thus inspired by the ideas of multi-phase in Refs. [16,17], focused on the success probability, we have conducted some research on the arbitrary-phase Grover algorithm and the multi-phase Grover algorithm.

In this Letter, firstly we analyze the properties of the success probability function of the arbitrary-phase Grover algorithm. Then for the multi-phase algorithm with an arbitrary phase number, we study the relationship between the minimum success probability and the phases and the specific mathematical relationship among the optimal phases and put forward a systematic solution of the optimal phases. Based on these results, a multi-phase quantum search algorithm is designed finally, which yields almost full

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success probability with a single iteration. In our algorithm, phases are selectively used depending on the fraction of targets, which is different from the way in Ref. [16].

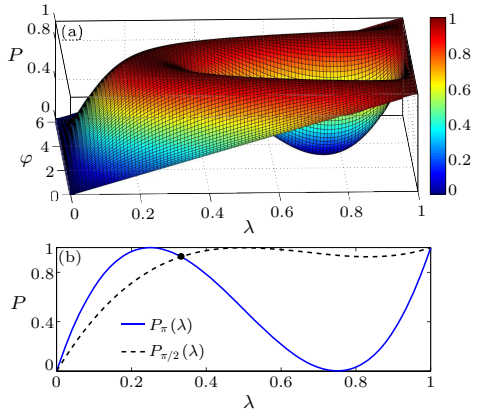


Fig. 1. The relationship among P, φ and λ.

We summarize and analyze the arbitrary-phase Grover search algorithm as follows. The first step is to prepare the equal superposition state $|\psi\rangle$. Here $|\alpha\rangle = \frac{1}{\sqrt{M}} \sum_{x'=0}^{M-1} |x'\rangle, |\beta\rangle = \frac{1}{\sqrt{N-M}} \sum_{x''=0}^{N-M-1} |x''\rangle$, where x' and x'' represent the target item and the non-target item, respectively. Applying $H^{\otimes n}$ on $|0\rangle^{\otimes n}$, we obtain

$$\begin{aligned} |\psi\rangle &= H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \\ &= \sqrt{\frac{M}{N}} \cdot |\alpha\rangle + \sqrt{\frac{N-M}{N}} \cdot |\beta\rangle, \end{aligned} \tag{1}$$

where H is the Walsh–Hadamard transform. Denoting the fraction of target items as λ, we obtain $\lambda = \frac{M}{N}$, and $|\psi\rangle = \sqrt{\lambda}|\alpha\rangle + \sqrt{1-\lambda}|\beta\rangle$. If we adopt

$$S_{f,\phi} = I - (1 - e^{i\phi})|t\rangle\langle t|, \tag{2}$$

$$S_{0,\varphi} = I - (1 - e^{i\varphi})|0\rangle\langle 0|, \tag{3}$$

where ϕ and φ are the phases of the algorithm. Then the Grover transform can be described as $G = -HS_{0,\varphi}HS_{f,\phi}$. Applying G on $|\psi\rangle$, we obtain $G|\psi\rangle = a_1|\alpha\rangle + b_1|\beta\rangle$, with

$$\begin{aligned} a_1 &= \sqrt{\lambda}[(1 - e^{i\varphi}) - e^{i\phi} - (1 - e^{i\varphi})(1 - e^{i\phi})\lambda], \\ b_1 &= \sqrt{1-\lambda}[-e^{i\varphi} - (1 - e^{i\varphi})(1 - e^{i\phi})\lambda]. \end{aligned} \tag{4}$$

Based on the phase matching condition in Ref. [14], $\phi = \varphi$. Let $P = P(\lambda, \varphi)$ represent the corresponding success probability, when the target items fraction is λ and the value of the phase is φ. Considering the situation of one iteration, then $P = |a_1|^2$, and we obtain

$$\begin{aligned} P(\lambda, \varphi) = \lambda \cdot [&4(1-\lambda)^2 - 4(\lambda - \lambda^2)\cos^2\varphi \\ &+ 4(3\lambda - 2\lambda^2 - 1)\cos\varphi + 1]. \end{aligned} \tag{5}$$

The relationship among P, φ and λ is shown in Fig. 1.

Figure 1(a) describes the φ and λ dependence of P, from which we can see that, when $\lambda \geq \frac{1}{4}$, a certain φ exists which makes P close to 1.

Given the value of φ, P is a function of λ, denoted as $P_\varphi(\lambda)$. Figure 1(b) shows the graph of $P_{\frac{\pi}{2}}(\lambda)$ and $P_\pi(\lambda)$. We can obtain that the minimum of $P_\pi(\lambda)$ over the range $0 \leq \lambda < 1$ is 0 and the maximum of $P_\pi(\lambda)$ is 1. After mathematical analysis on $P_\varphi(\lambda)$, we obtain its three properties as follows: firstly, function $P_\varphi(\lambda)$ has a local maximum point and a local minimum point. Let λ_φ^{max} and λ_φ^{min} represent the maximum point and the minimum point, respectively, and make the derivative of $P_\varphi(\lambda)$ equal to 0, we obtain

$$\lambda_\varphi^{\max} = \frac{-1}{2t-2}, \ \lambda_\varphi^{\min} = \frac{4t-5}{6t-6}, \tag{6}$$

where $t = \cos\varphi$.

Secondly, function $P_{\varphi_1}(\lambda)$ and $P_{\varphi_2}(\lambda)$ have one and only one intersection over the range $0 < \lambda < 1$. Let a represent the intersection and solve the equation $P_{\varphi_1}(\lambda) = P_{\varphi_2}(\lambda)$, we obtain

$$a = \frac{1}{2 - \cos\varphi_1 - \cos\varphi_2}. \tag{7}$$

Thirdly, $P_\varphi(\lambda_1)$ and $P_\varphi(\lambda_2)$ $(\lambda_1 < \lambda_2)$ are equal to each other if and only if

$$\begin{aligned} \lambda_1 = &-\frac{1}{2(t-1)^2}[\lambda_2(t-1)^2 + 3t - 2t^2 - 2 + (1-t) \\ &\cdot \sqrt{(1-\lambda_2)(1-t)(3\lambda_2 - 3\lambda_2 t - t - 1)}], \end{aligned} \tag{8}$$

where $t = \cos\varphi$.

The first property describes the characteristics of extreme points, which gives a rough profile of $P_\varphi(\lambda)$. The second and third properties obtain Eqs. (7) and (8) as the solution of $P_{\varphi_1}(\lambda) = P_{\varphi_2}(\lambda)$ and $P_\varphi(\lambda_1) = P_\varphi(\lambda_2)$, respectively. These three properties play important roles in our algorithm.

When $\varphi = \pi$, $P_\pi(\lambda)$ reaches a minimum 0 at $\lambda = \frac{3}{4}$. When λ falls within the scope of $[\frac{1}{3}, 1]$, the algorithm in Ref. [17] promotes the minimum of P to 97.42% by using two different phases. The algorithm consists of two parts.

When $\frac{1}{3} \leq \lambda \leq \frac{1}{2}$, we set $\varphi = 1.8$ in Eq. (3) to apply G transform. When $\frac{1}{2} < \lambda \leq 1$, we set $\varphi = 1.4$. Here $\lambda = \frac{1}{2}$ is the separation point of two parts. We obtain

$$\begin{aligned} &\min_{\frac{1}{3} \leq \lambda < 1} \{P_\varphi(\lambda)\} \\ &= \min\Big\{ \min_{\frac{1}{3} \leq \lambda \leq \frac{1}{2}} \{P_{1.8}(\lambda)\}, \min_{\frac{1}{2} < \lambda < 1} \{P_{1.4}(\lambda)\} \Big\}. \end{aligned}$$

In this way, $\varphi = 1.8$ and $\varphi = 1.4$ complement each other's advantages and improve the algorithm.

However, after mathematical analysis, we find that making the intersection a of $P_{1.8}(\lambda)$ and $P_{1.4}(\lambda)$ be the separation point is optimal to enlarge $\min\{P_\varphi(\lambda)\}$, which can be seen from Fig. 1(b). Based on Eq. (7), $a = 0.4861$. We obtain $\min\{P_\varphi(\lambda)\} = 97.43\%$ with $\lambda = \frac{1}{2}$ replaced by $\lambda = a$. If we change $\varphi = 1.8$ and $\varphi = 1.4$ to other phases, $\min\{P_\varphi(\lambda)\}$ varies.

Then there is a problem of how to find the optimal values of φ to enlarge $\min\{P_\varphi(\lambda)\}$, when we use two phases or three phases, or a general case with n phases.

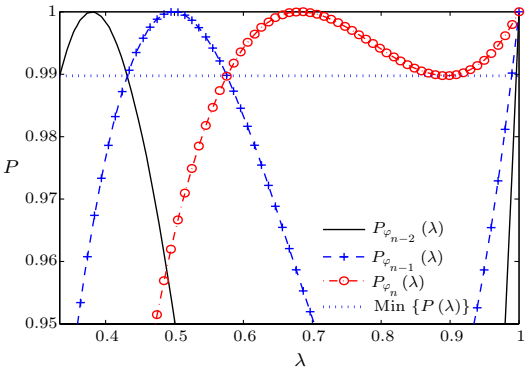


Fig. 2. The profile of $P_{\varphi_{n-2}}(\lambda), P_{\varphi_{n-1}}(\lambda)$ and $P_{\varphi_n}(\lambda)$.

Suppose that we use n values of φ, and assume that the value precision is 10^{-3}, then the value space of φ is over $2\pi^n \cdot 10^{3n}$. Thus, it is infeasible and inaccurate to find out the optimal values by exhausting the values. However, after mathematical analysis, we find out the relationship among the optimal values. It is described in the following theorem, which reduces the value space sharply.

Suppose that the optimal values of φ are $\varphi_1, \varphi_2, \ldots, \varphi_n$, where $0 \le \varphi_i < 2\pi$. Let a_i represent the intersection of $P_{\varphi_i}(\lambda)$ and $P_{\varphi_{i+1}}(\lambda)$. Then

$$P_{\varphi_1}(a_1) = P_{\varphi_2}(a_1) = P_{\varphi_2}(a_2) = \cdots = P_{\varphi_{n-1}}(a_{n-2}) = P_{\varphi_{n-1}}(a_{n-1}) = P_{\varphi_n}(a_{n-1}) = P_{\varphi_n}(\lambda_{\varphi_n}^{\min}), \tag{9}$$

where $\lambda_{\varphi_n}^{\min}$ is the minimum point of function $P_{\varphi_n}(\lambda)$. From Eq. (9), we can further obtain

$$\varphi_{n-1} = \arccos\left[\frac{-(\cos^2\varphi_n - 4\cos\varphi_n + 1)}{\cos\varphi_n + 1}\right], \tag{10}$$

$$a_{i-1} = \frac{-1}{2(t_i-1)^2}[a_i(t_i-1)^2 + 3t_i - 2t_i^2 - 2 + (1-t_i) \cdot \sqrt{(1-a_i)(1-t_i)(3a_i - 3a_i t_i - t_i - 1)}], \tag{11}$$

where $t_i = \cos\varphi_i$.

We can prove the theorem as follows. Based on the properties of $P_\varphi(\lambda)$, we obtain the profile of $P_{\varphi_{n-2}}(\lambda), P_{\varphi_{n-1}}(\lambda)$ and $P_{\varphi_n}(\lambda)$ as shown in Fig. 2.

From Fig. 2, we can find

$$\min_{\lambda_{\varphi_{n-1}}^{\max} \le \lambda \le \lambda_{\varphi_n}^{\max}} \{P_\varphi(\lambda)\} = P_{\varphi_n}(a_{n-1})$$

$$\min_{\lambda_{\varphi_n}^{\max} < \lambda < 1} \{P_\varphi(\lambda)\} = P_{\varphi_n}(\lambda_{\varphi_n}^{\min})$$

$$\min_{\lambda_{\varphi_{n-1}}^{\max} \le \lambda < 1} \{P_\varphi(\lambda)\} = \min\{P_{\varphi_n}(a_{n-1}), P_{\varphi_n}(\lambda_{\varphi_n}^{\min})\}.$$

When $a_{n-1} < \lambda_{\varphi_n}^{\min}$, $P_{\varphi_n}(a_{n-1})$ increases monotonically about φ_n, while $P_{\varphi_n}(\lambda_{\varphi_n}^{\min})$ decreases monotonically about φ_n on the contrary. Thus $P_{\varphi_n}(a_{n-1}) = P_{\varphi_n}(\lambda_{\varphi_n}^{\min})$ is necessarily used to enlarge $\min\{P_\varphi(\lambda)\}$.

We can also prove that $P_{\varphi_{n-1}}(a_{n-2}) = P_{\varphi_{n-1}}(a_{n-1})$. Given that φ_{n-2}, φ_{n-1} exists meeting the above equation. In this situation, we denote $\min\{P_\varphi(\lambda)\}$ as P_0. Assume that another phase φ'_{n-1} does not meet the above equation, as is described in Fig. 3.

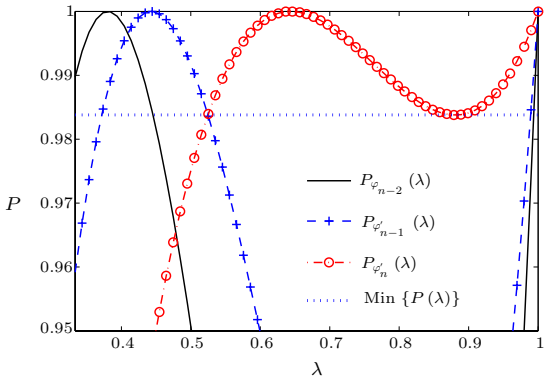


Fig. 3. The profile of $P_{\varphi_{n-2}}(\lambda), P_{\varphi'_{n-1}}(\lambda)$ and $P_{\varphi'_n}(\lambda)$.

If $P_{\varphi'_{n-1}}(a_{n-2}) < P_0$, then $\min\{P_\varphi(\lambda)\} < P_0$. If $P_{\varphi'_{n-1}}(a_{n-2}) > P_0$, based on the properties, we know $\varphi'_{n-1} > \varphi_{n-1}$, and $P_{\varphi'_{n-1}}(a_{n-1}) < P_0$. Thus $\min\{P_\varphi(\lambda)\} < P_0$. Thus $P_{\varphi_{n-1}}(a_{n-2}) = P_{\varphi_{n-1}}(a_{n-1})$ is necessary. Similarly, we can obtain $P_{\varphi_{i-1}(a_{i-2})} = P_{\varphi_{i-1}(a_{i-1})}$, where $3 \le i \le n$. Finally Eq. (9) is proved. Equation (10) can be obtained by solving $P_{\varphi_n}(a_{n-1}) = P_{\varphi_n}(\lambda_{\varphi_n}^{\min})$, with the help of Eqs. (6) and (7). With the help of Eq. (8), Eq. (11) can be obtained by solving $P_{\varphi_{i-1}}(a_{i-2}) = P_{\varphi_{i-1}}(a_{i-1})$.

Based on the above theorem, we can obtain a systematic solution of the optimal phases $\{\varphi_i\}$. Equation (10) is the expression of φ_{n-1} with respect to φ_n, and a_{n-1} can be calculated by Eq. (7). This process can be summarized as follows:

$$\varphi_n \overset{\text{Eq. (10)}}{\rightarrow} \varphi_{n-1} \overset{\text{Eq. (7)}}{\rightarrow} a_{n-1}. \tag{12}$$

For Eq. (11), when $i = n-1$, it becomes the expression of a_{n-2} with respect to φ_n. Based on Eq. (7), we obtain the expression of φ_{i-2}

$$\varphi_{i-2} = \arccos(2 - 1/a_{i-2} - \cos\varphi_{i-1}). \tag{13}$$

Finally, the calculation process is depicted as follows:

$$\varphi_n \overset{\text{Eq. (11)}}{\rightarrow} a_{n-2} \overset{\text{Eq. (13)}}{\rightarrow} \varphi_{n-2} \rightarrow \cdots \rightarrow \varphi_2 \overset{\text{Eq. (11)}}{\rightarrow} a_1 \overset{\text{Eq. (13)}}{\rightarrow} \varphi_1. \tag{14}$$

Based on Eq. (9), we obtain that when $\lambda_{\varphi_1^{max}} \le \lambda < 1$, $\min\{P_\varphi(\lambda)\}$ is $P_{\varphi_n}(\lambda_{\varphi_n}^{\min})$. Let c represent the minimum value of λ we are interested in. Then when $c \le \lambda < \lambda_{\varphi_1^{\max}}$, $\min\{P_\varphi(\lambda)\}$ is $P_{\varphi_1}(c)$. Thus

$$\min_{c \le \lambda < 1} \{P_\varphi(\lambda)\} = \min\{P_{\varphi_n}(\lambda_{\varphi_n}^{\min}), P_{\varphi_1}(c)\}.$$

Equations (12) and (14) show that all φ_i including φ_1 can be calculated by φ_n. Thus $\min_{c \le \lambda < 1} \{P_\varphi(\lambda)\}$ is determined by φ_n. We search the scope $0 \le \varphi_n < 2\pi$ and

can obtain

$$\max_{\varphi_n}\{\min_{c\le\lambda<1}\{P_\varphi(\lambda)\}\}, \tag{15}$$

and the relevant optimal $\{\varphi_i\}$. The complexity is just related to the phase number and the search precision.

Suppose that we use n phases $\varphi_1, \varphi_2, \ldots, \varphi_n$, where $0 \le \varphi_i < 2\pi$. Let c represent the minimum value of λ we are interested in. Then we propose our multi-phase search algorithm as follows.

Step 1: find the optimal $\{\varphi_i\}$ by Eq. (15).

Step 2: for a given λ, choose the value of φ in the following way. If $\lambda < a_1$, $\varphi = \varphi_1$. If $a_i \le \lambda < a_{i+1}$, $\varphi = \varphi_i$. If $\lambda \ge a_{n-1}$, $\varphi = \varphi_n$.

Step 3: apply a G transform once, with φ in step 2.

When $n \le 4$, $c = 1/3$ or $c = 1/4$, the algorithm results $\min\{P_\varphi(\lambda)\}$ and $\{\varphi_i\}$ are shown in Table 1.

Table 1. The algorithm results of $\min\{P_\varphi(\lambda)\}$ and $\{\varphi_i\}$.

n	$c = 1/3$		$c = 1/4$	
	$\min\{P_\varphi(\lambda)\}$	$\{\varphi_i\}$	$\min\{P_\varphi(\lambda)\}$	$\{\varphi_i\}$
1	92.59%	$\pi/2$	86.22%	1.7141
2	97.83%	1.8025, 1.3789	95.93%	2.1340, 1.4653
3	98.98%	1.8903, 1.5738, 1.3007	98.07%	2.3179, 1.7584, 1.3651
4	99.40%	1.9371,1.6820 1.4629,1.2565	98.88%	2.4273,1.9346 1.5957,1.3089

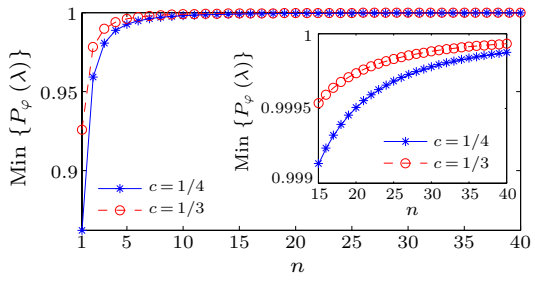


Fig. 4. The relationship between $\min P_\varphi(\lambda)$ and n.

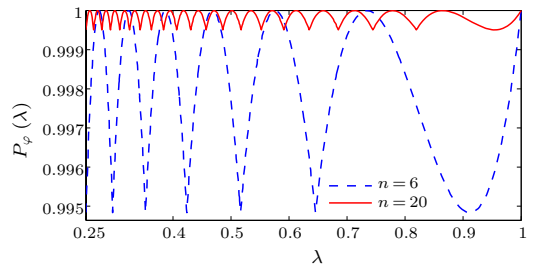


Fig. 5. The relationship between $P_\varphi(\lambda)$ and λ, when $c = 1/4$, $n = 6$ and $n = 20$.

In the complexity analysis of our proposed multi-phase search algorithm, the iteration number of the G transform is 1. Here $\min\{P_\varphi(\lambda)\}$ is relevant to phase number n and λ's lower limit c. Figure 4 shows the relationship between $\min\{P_\varphi(\lambda)\}$ and n, when $c = 1/3$ and $c = 1/4$, from which we can see that P tends to be 100% with the increase of n, when $c \ge 1/4$. In particular, when $c = 1/4$, $n = 6$ and 20, the relationship between $P_\varphi(\lambda)$ and λ is described in Fig. 5.

Compared with other existing algorithms, our proposed multi-phase algorithm is valuable.

When $c = 1/3$ and $n = 1$, the optimal value of φ in our algorithm is $\pi/2$, equivalent to the result of Ref. [15]. Compared with the original Grover algorithm, $\min\{P_\varphi(\lambda)\}$ is larger ($92.59\% > 50\%$) and the number of iteration is smaller. When $c = 1/3$ and $n = 2$, our algorithm is the two-phase Grover algorithm with $\varphi_1 = 1.8025$ and $\varphi_2 = 1.3789$. Compared with the two-phase algorithm in Ref. [17], $\min\{P_\varphi(\lambda)\}$ is promoted from 97.42% to 97.83%. For the general n and $c \ge 1/4$, $\min\{P_\varphi(\lambda)\}$ can be larger than any real number lower than 1. For example, $\min\{P_\varphi(\lambda)\}$ of our algorithm with $c = 1/3$ and $n = 5$ is 99.61% larger than that of Ref. [11].

In summary, we have proposed a multi-phase quantum search algorithm with a single iteration. For the search problem whose target items are not unique, we find that phases can complement each other's advantages and promote the algorithm result finally. Thus we have studied the arbitrary-phase Grover algorithm with multiple phases. We obtain three useful properties of the success probability function and an important theorem which depicts the exact necessary condition that the optimal multiple phases meet. These lead to a systematic solution of the optimal phases, for the arbitrary phase number. Given λ and its lower limit c, our algorithm can find the target item with nearly full probability through one iteration, when $c \ge 1/4$.

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