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Periodic States in Chaotic Rössler Oscillators with On-Off Coupling *

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It has been revealed that two coupled chaotic Rössler oscillators can transit to a period-5 splay state in an extremely weak coupling region [Zhan et al., Phys. Rev. Lett. 86 (2001) 1510]. Here we show that with further increase of coupling, two other coupling regions exist that may induce the period-4 splay state and period-5 synchronous state. Using an on-off coupling strategy, we find that the coupling regions for inducing period-5 states can be significantly extended and the extending effect is regulated by the match of two time scales: one is of the on-off coupling and the other is of the individual Rössler oscillator. We then compare the sensitivity of the periodic states with the initial conditions of the oscillators. We also analyze the mechanism behind these two new types of periodic states.

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The collective behavior of coupled oscillators has been widely studied, owing to their relevance in a variety of areas such as physics, chemistry, and biology.[1–8] Generally, complete synchronization (CS) is believed to be the strongest type of collective behavior, which implies that the oscillators behave in the same fashion.[9] CS occurs to identical oscillators if their interactions are sufficiently strong, so that it is only a very specific behavior. More common is the collective behavior arising in small coupling, which is far from the condition of CS.

There is much evidence that small coupling does not mean asynchronous or disordered collective behavior. Instead, it may induce rich regular and ordered activities in coupled oscillators. For instance, before CS can occur, two-dimensional coupled chaotic oscillators may spontaneously display spatially mirror symmetric and central symmetric structures.[10] Within a small coupling region, the scale-free networks of chaotic maps can self organize into ordered spatiotemporal patterns, where each chaotic map transits to period-2 state and has an anti-phase with its neighbors.[11] Surprisingly, even when the coupling is extremely weak, the chaotic Rössler oscillators can generate a period-5 splay state, i.e., each oscillator in a ring has an identical period-5 orbit with different while functionally related positions (or phases).[12,13] Recently, such a period-5 splay state induced by extremely weak coupling has been observed for complex networks.[14] One essential point for the mechanism of the period-5 splay state is due to the highest return probability at the unstable period-5 orbit of the single chaotic Rössler oscillator.[12] Therefore, an interesting question is whether other types of period states exist except for the period-5 splay state before CS.

In this Letter, we investigate the dynamical behavior in a system of two chaotic Rössler oscillators with on-off coupling. Unlike time invariant coupling, the on-off coupling is periodically switched on and off, and the on-off ratio is governed by a parameter θ, where $\theta \in [0,1]$. Thus the time invariant coupling can be seen as a special case $\theta = 0$ of the on-off coupling. The on-off type coupling can be found in neuronal ensembles with chemical synapses as well as in moving agents like birds and artificial robots.[15–20] When $\theta = 0$, in addition to the extremely weak coupling region, we show that with further increase of coupling, two other coupling regions indeed exist where the system may transit to the period-4 splay state and period-5 synchronous state, showing different regular trajectory structures to that of the extremely weak coupling region. When $\theta \neq 0$, we find that the coupling regions for observing period-5 states can be effectively extended and the extending effect is regulated by the time scales between the on-off coupling and the individual oscillator. Further, we find that the on-off coupling-induced periodic states show different sensitivities to the initial conditions of the oscillators and are related to a specific coupling configuration.

We consider a system of two identical Rössler oscillators with on-off coupling, whose dynamics is expressed by the equations

$$\begin{aligned} \dot{x}_{1,2} &= -\omega y_{1,2} - z_{1,2} + \varepsilon(t)(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega x_{1,2} + a y_{1,2}, \\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{1}$$

where $a = 0.165$, $f = 0.2$, $c = 10$, and $\omega = 0.99$. Under these parameters, each isolated Rössler oscillator has a chaotic attractor; $\varepsilon(t)$ is the on-off coupling, which is defined as

$$\varepsilon(t) = \begin{cases} 0, & mT \le t \le (m+\theta)T, \\ \varepsilon, & (m+\theta)T < t < (m+1)T, \end{cases} \tag{2}$$

where $m = 0, 1, 2, \ldots$, T is the period of the on-off coupling and $\theta \in [0,1]$ is the on-off ratio. When $\theta = 0$, the on-off coupling is time invariant. In this case, Eq. (1) achieves CS when $\varepsilon \ge \varepsilon_c$, where $\varepsilon_c \approx 0.087$.

We are interested in the transitions from asynchronous chaotic states to periodic states when $\varepsilon < \varepsilon_c$.

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To indicate the transitions, we calculate the largest Lyapunov exponent λ of the system in Eq. (1). The system is believed to be chaotic or periodic depending on whether λ is larger than zero ($\lambda > 0$) or close to it ($\lambda = 0$). In numerical simulations, λ is averaged with $N = 100$ realizations by choosing different initial conditions in Eq. (1). Figure 1(a) shows the evolution of λ with ε for $\theta = 0$, i.e., the case of time invariant coupling. When ε is very weak, λ exhibits a decline in the region of $\varepsilon_1 \in (0.0003, 0.0008)$, consistent with the coupling region observed in Ref. [12]. Interestingly, as ε increases, we find that λ will undergo two other declines, one is around $\varepsilon_2 = 0.014$ and the other is around $\varepsilon_3 = 0.017$. In the former, $\lambda \approx 0$, which means that the system transits to a periodic state again, while in the latter, $\lambda \approx 0.1$, suggesting a slight decline. However, does this mean that the system remains chaotic around $\varepsilon_3 = 0.017$ and what causes the small decline over there?

For this purpose, we introduce a threshold $\lambda_c = 0.0001 \approx 0$ and assume that the system in Eq. (1) is approximately periodic if $\lambda < \lambda_c$. By counting the number n of $\lambda < \lambda_c$ from N realizations, we obtain the probability f for the system transiting from chaotic states to periodic states, where $f = n/N$. Figure 1(b) shows the result of f extracted from Fig. 1(a). It can be seen that f has no peak except for three coupling regions: region I, region II and region III. Those regions correspond to the three declines in Fig. 1(a). Unlike regions I and II, region III has $f \approx 0.2$, indicating that the system is not purely chaotic while occasionally periodic, and therefore leads to the small decline around ε_3. Meanwhile, such small f also suggests that the periodic state in region III is more sensitive to the initial conditions of the oscillators than that in regions I and II. Clearly, f has an advantage of indicating the transitions from chaotic states to periodic states than λ.

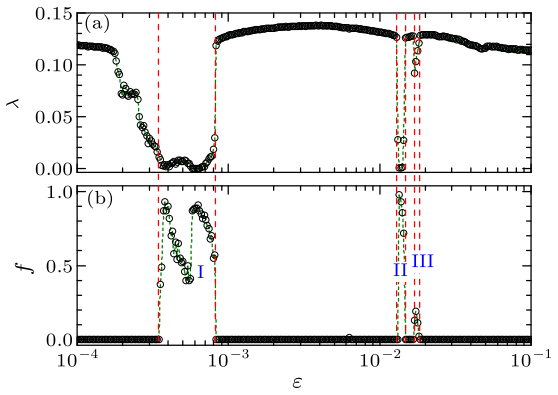


Fig. 1. The mean value of the largest Lyapunov exponent λ versus ε in (a) and the corresponding transition probability f in (b). Dashed lines are the guides for the three coupling regions with $f > 0$.

To compare the periodic states between those three coupling regions, we plot the trajectories of the two oscillators in x–y space, see Fig. 2. As shown in Fig. 2(a), when $\varepsilon = 0.0006$, the two oscillators move on the same period-5 ($T_5 \approx 30.98$) orbit while with different positions/phases (the circles denote the positions of the two oscillators at an arbitrarily chosen time). By plotting $x_2(t)$ versus $x_1(t+\tau)$ in Fig. 2(b), the phase lag in Fig. 2(a) is revealed as $\tau = T_5/5$. An interesting phenomenon is further shown in Fig. 2(c) for $\varepsilon = \varepsilon_2$, at which the orbits of the two oscillators become period-4 ($T_4 \approx 24.85$) and their splay positions satisfy $x_1(t+\tau) = x_2(t)$, where $\tau = T_4/2$ is figured out in Fig. 2(d). As mentioned above, an isolated Rössler oscillator in Eq. (1) has the highest return probability at T_5, which makes a large chance for the oscillators forming period-5 splay orbit at the extremely weak coupling[12] (also see Fig. 7(a)). In contrast, the return probability is rather small at T_4 (see Fig. 7(a)), while Fig. 2(c) implies that a relatively large coupling may stabilize Eq. (1) at the period-4 orbit other than the period-5. As $\varepsilon = \varepsilon_3$, the two oscillators generate the same period-5 orbit as shown in Fig. 2(a), while their phase lag becomes $\tau = 0$, i.e., period-5 synchronous state, see Figs. 2(e) and 2(f).

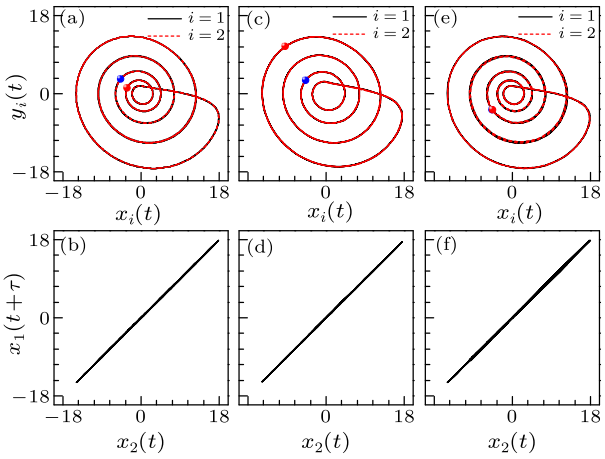


Fig. 2. Upper: the trajectories of the two oscillators on the plane $(x_i(t), y_i(t))$. Lower: the delayed-coordinate plots $x_1(t+\tau)$ versus $x_2(t)$ for different values of coupling. (a, b) $\varepsilon = 0.0006$ and $\tau = T_5/5$, (c, d) $\varepsilon = 0.014$ and $\tau = T_4/2$, (e, f) $\varepsilon = 0.017$ and $\tau = 0$. In the upper panels, the circles denote the positions of the two oscillators at an arbitrarily chosen time.

Next, we let $\theta \neq 0$ and investigate how it affects the transitions from chaotic states to periodic states. We first set $T = T_5$ as the period of the on-off coupling and calculate the transition probability f as shown in Fig. 2(b) for three different θ. Figure 3(a) shows the results. When $\theta = 0.1$, $f > 0$ appears just for an extremely weak coupling region $\varepsilon \in (0.0003, 0.001)$ and beyond it we have $f \approx 0$. Clearly, this region is larger than region I ($\theta = 0$), implying that the weak coupling region for observing periodic states is extended by the on-off coupling. When $\theta = 0.5$, the region of $f > 0$ shifts to $\varepsilon \in (0.0004, 0.002)$ and $\varepsilon \in (0.04, 0.05)$, respectively. However, such shifting and extending effects disappear when θ is large, such as $\theta = 0.8$. The reason for this is that the two oscillators are separate from each other when the coupling is off. From Eq. (2), the un-coupled duration is θT_5 within each

on-off coupling period, thus the effective coupling for each period is $\langle\varepsilon\rangle = \frac{1}{T_5}\int_t^{t+T_5}(1-\theta)\varepsilon(t)dt = (1-\theta)\varepsilon$. When θ is small, the effective coupling $\langle\varepsilon\rangle$ is sufficient to tamp the chaotic behavior of Eq. (1) and thus leads to $f > 0$. When θ is large, $\langle\varepsilon\rangle$ becomes smaller, causing $f \approx 0$ at $\theta = 0.8$. We then set $T = T_4$ and do the same calculations again as T_5, see Fig. 3(b). It shows that $f = 0$ for all three θ, suggesting that the on-off coupling with period T_4 has no effect on inducing the transitions to periodic states.

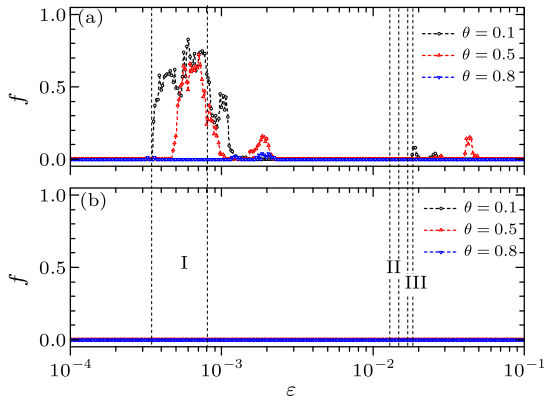


Fig. 3. The transiton probability f as a function of ε for $\theta = 0.1$, 0.5, and 0.8. (a) $T = T_5$ and (b) $T = T_4$. The dashed lines are the guides for the three coupling regions (I, II, and III) of $\theta = 0$.

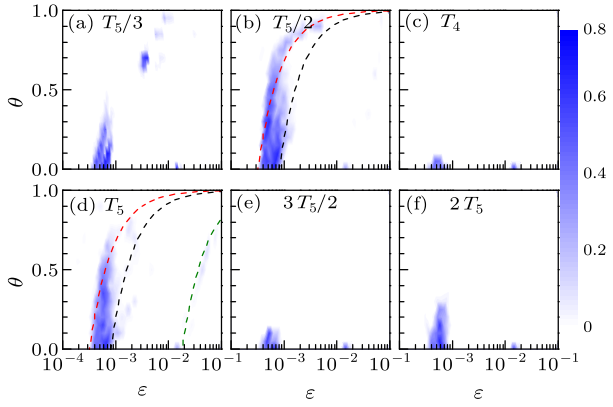


Fig. 4. The transition probability f versus ε and θ for different T. (a) $T = T_5/3$, (b) $T = T_5/2$, (c) $T = T_4 \approx 4T_5/5$, (d) $T = T_5$, (e) $T = 3T_5/2$, and (f) $T = 2T_5$. The dashed lines in (b) and (d) denote $\theta = 1 - 0.0003/\varepsilon$, $\theta = 1 - 0.0008/\varepsilon$, and $\theta = 1 - \varepsilon_3/\varepsilon$.

To give a global view, Fig. 4 plots f versus ε and θ for various on-off coupling periods T. It can be observed that $T = T_5/2$ and $T = T_5$ have large regions of $f > 0$, while too small $T = T_5/3$ or too large $T = 2T_5$ have relatively smaller regions. Moreover, for $T = T_5/2$ and $T = T_5$, the main regions of $f > 0$ are approximately within two boundaries: $\theta = 1 - 0.0003/\varepsilon$ and $\theta = 1 - 0.0008/\varepsilon$, and the narrow region for $T = T_5$ emerges from $\varepsilon \approx 0.017$ approximately satisfying $\theta = 1 - \varepsilon_3/\varepsilon$. These boundaries can be understood as follows. The transitions to periodic states require the effective coupling $\langle\varepsilon\rangle \in (0.0003, 0.0008)$ and $\langle\varepsilon\rangle \approx \varepsilon_3$, which give rise to the three boundaries $\theta \approx 1 - 0.0003/\varepsilon$, $\theta \approx 1 - 0.0008/\varepsilon$, and $\theta \approx 1 - \varepsilon_3/\varepsilon$. As the high transition probability appears at T_5 and $T_5/2$, a matchable time scale between the on-off coupling and the chaotic Rössler oscillator is essential for enhancing the transitions from chaotic states to periodic states.

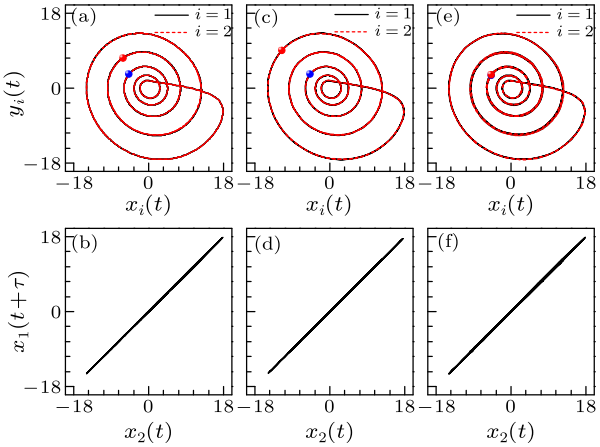


Fig. 5. Upper: the trajectories of the two oscillators on the plane $(x_i(t), y_i(t))$. Lower: the delayed-coordinate plots $x_1(t+\tau)$ versus $x_2(t)$ for different values of coupling. (a, b) $\varepsilon = 0.001$, $\theta = 0.1$, and $\tau = T_5/5$, (c, d) $\varepsilon = 0.002$, $\theta = 0.5$, and $\tau = 2T_5/5$, (e, f) $\varepsilon = 0.045$, $\theta = 0.5$, and $\tau = 0$. In the upper panels, the circles denote the positions of the two oscillators at an arbitrarily chosen time. $T = T_5$ is considered.

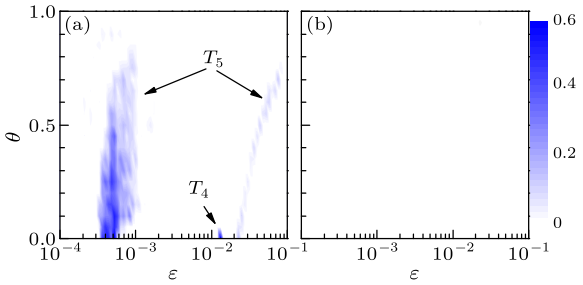


Fig. 6. The transition probability f versus ε and θ for y-coupling (a) and for z-coupling (b). Here $T = T_5$ is used.

Are the periodic states that are induced by the on-off coupling similar to that of $\theta = 0$? In Fig. 5, we give three illustrations by choosing ε outside the three regions (I, II, and III) of $\theta = 0$. As ε is chosen at $\varepsilon = 0.001$, Figs. 5(a) and 5(b) show that the two oscillators move in the same period-5 orbit when $\theta = 0.1$, and their different positions also satisfy $x_1(t+\tau) = x_2(t)$, with a phase lag $\tau = T_5/5$. A similar motion can be observed in Figs. 5(c) and 5(d) for $\varepsilon = 0.002$ and $\theta = 0.5$, while the phase lag is $\tau = 2T_5/5$. When $\varepsilon = 0.045$ and $\theta = 0.5$, the two oscillators achieve $x_1(t+\tau) \approx x_2(t)$ with zero phase lag $\tau = 0$, see Figs. 5(e) and 5(f). In view of them, the on-off coupling ($\theta \neq 0$) can induce the same period-5 orbits as $\theta = 0$.

The above observations motivate us to study

whether the transitions to periodic states can be observed from other coupling configurations. For simplicity, we will consider the system of the two chaotic Rössler oscillators under y-coupling and z-coupling, respectively. In this way, Eq. (1) becomes as follows. For y-coupling: $\dot{x}_{1,2} = -\omega y_{1,2} - z_{1,2}$, $\dot{y}_{1,2} = \omega x_{1,2} + a y_{1,2} + \varepsilon(t)(y_{2,1} - y_{1,2})$, $\dot{z}_{1,2} = f + z_{1,2}(x_{1,2} - c)$, and for z-coupling: $\dot{x}_{1,2} = -\omega y_{1,2} - z_{1,2}$, $\dot{y}_{1,2} = \omega x_{1,2} + a y_{1,2}$, $\dot{z}_{1,2} = f + z_{1,2}(x_{1,2} - c) + \varepsilon(t)(z_{2,1} - z_{1,2})$. Without loss of generality, $T = T_5$ is used as the period of the on-off coupling. In Fig. 6, we show the transition probability f as functions of ε and θ for both the y-coupling and the z-coupling. In the case of y-coupling, Fig. 6(a) displays a similar feature to that of the x-coupling. Our numerical results indicate that there are also three types of periodic states: period-5 splay state ($\tau = kT_5/5, k = 1, 2, 3, 4$), period-4 splay state ($\tau = T_4/2$), and period-5 synchronous state ($\tau = 0$); the intermediate coupling region Ⅱ for period-4 splay state disappears when $\theta \neq 0$. Note that, for y-coupling, the system arrives at CS when $\varepsilon \geq 0.075$ at $\theta = 0$. Thus the above periodic states are self-organized before CS. Why do x-coupling and y-coupling share a similar performance of f? Let $\omega x_{1,2} + \varepsilon(t)(y_{2,1} - y_{1,2}) = \omega X_{1,2}$, we obtain $\dot{X}_{1,2} = -\omega y_{1,2} - z_{1,2} + \varepsilon(t)/\omega[X_{2,1} - X_{1,2} + a(y_{2,1} - y_{1,2})]$. As $\omega \approx 1$ and $a = 0.165$ is small, we obtain $\varepsilon(t)/\omega \approx \varepsilon(t)$ and $|X_{2,1} - X_{1,2}| \gg a|y_{2,1} - y_{1,2}|$. In this way, the system under y-coupling can be simplified to $\dot{X}_{1,2} = -\omega y_{1,2} - z_{1,2} + \varepsilon(t)(X_{2,1} - X_{1,2})$, $\dot{y}_{1,2} = \omega X_{1,2} + a y_{1,2}$, $\dot{z}_{1,2} = f + z_{1,2}(x_{1,2} - c)$, which is identical to Eq. (1) and thus explains the similar results of f between the x-coupling and y-coupling. However, in the case of z-coupling, Fig. 6(b) shows that $f = 0$ for each pair of ε and θ, which is different from the results of the x-coupling and y-coupling. Thus the transition to periodic states depends on the specific coupling configuration.

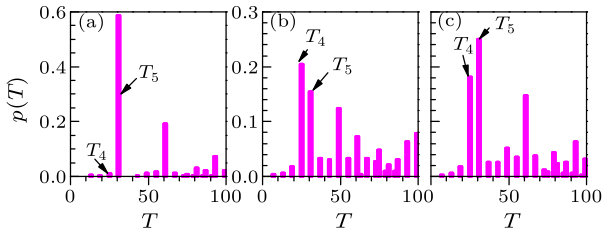


Fig. 7. The return probability $p(T)$ for one of the two oscillators versus T for different ε: (a) $\varepsilon = 0$, (b) $\varepsilon = 0.013$, and (c) $\varepsilon = 0.0165$.

Finally, we analyze the mechanism of the period-4 splay state and period-5 synchronous state. Let us recall the mechanism of the period-5 splay state. For $\varepsilon = 0$ in Eq. (1), each Rössler oscillator has the highest return probability $p(T)$ at the unstable period T_5 orbit, see Fig. 7(a). When $\varepsilon > 0$ and it is extremely weak, it does not considerably alter the single-oscillator orbit. Instead, it stabilizes the oscillators at the period T_5 via mutual resonant interaction.[12] When ε continuously increases, it will significantly change the return probability of the oscillators. For example, when $\varepsilon = 0.013$ close to the coupling region Ⅱ, Fig. 7(b) shows that each oscillator has the highest return probability $p(T)$ at the unstable period T_4 orbit. This means that the oscillators tend to visit the vicinity of the period T_4 orbit. As ε increases to $\varepsilon = 0.014$, such a slight increase in mutual interaction does not considerably disturb the oscillators' orbits, while it may stabilize them at the period T_4 orbit. Thus Figs. 2(c) and 2(d) display the period T_4 splay state. Interestingly, when ε approaches to the coupling region Ⅲ, the highest return probability $p(T)$ returns to the period T_5 again, see the case of $\varepsilon = 0.0165$ in Fig. 7(c). Similarly, a slight increase in ε from 0.0165 to 0.017 may stabilize the oscillators at the period T_5 orbit. As ε is large now, the two oscillators can approximately organize into a synchronous period T_5 state as shown in Figs. 2(e) and 2(f).

In conclusion, we have systematically studied the transitions from chaotic state to periodic states in a system of two coupled chaotic oscillators, and have revealed two new types of periodic states before CS: the period-4 splay state and period-5 synchronous state. In addition, when the coupling between the oscillators is changed from time invariant to periodically switched on-off, we find that the regions for inducing period-5 states can be extended and shifted, improving the probability for observing the periodic states. Since the two coupled oscillators are of the minimum structure of complex systems, our results may provide a clue to understanding the system with a large number of oscillators. On the other hand, to reveal the dynamical patterns in the route to CS is one of the fundamental studies in nonlinear dynamics, such as the patterns of self-organization[5] and the chimera state,[21] our results may enrich the existing knowledge of self-organized periodic behavior before CS.

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