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State Transition Induced by Self-Steepening and Self Phase-Modulation *

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We present a rational solution for a mixed nonlinear Schrödinger (MNLS) equation. This solution has two free parameters, a and b, representing the contributions of self-steepening and self phase-modulation (SPM) of an associated physical system, respectively. It describes five soliton states: a paired bright-bright soliton, a single soliton, a paired bright-grey soliton, a paired bright-black soliton, and a rogue wave state. We show that the transition among these five states is induced by self-steepening and SPM through tuning the values of a and b. This is a unique and potentially fundamentally important phenomenon in a physical system described by the MNLS equation.

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The mixed nonlinear Schrödinger (MNLS) equation,[1]

$$q_t - iq_{xx} + a(q^*q^2)_x + ibq^*q^2 = 0, \tag{1}$$

has been derived by the modified reductive perturbation method to describe the propagation, for example, of the Alfvén waves with small but finite amplitude along the magnetic field in the cold plasma approximation,[1] which is widely applicable in solar, solar-terrestrial, space and astrophysics.[2,3] It has been shown[4–6] that the MNLS equation also provides an accurate modelling of ultra-short light pulse propagation in optical fibers. In Eq. (1), the complex quantity q represents the magnetic field perturbation in the case of Alfvén waves and the electrical field envelope in the case of waves in optical fibres. The asterisk denotes complex conjugate, a and b are two non-negative constants determined by the unperturbed state, and the subscript x (t) denotes the partial derivative with respect to the spatial coordinate x (time t). The MNLS equation reduces to the Nonlinear Schrödinger (NLS) equation when $a = 0$, and to the Derivative Nonlinear Schrödinger (DNLS) equation when $b = 0$. Because the MNLS equation is applicable even in a case where the lengths of the envelope wave and the carrier wave are comparable,[1] it is more general than the NLS equation in the modelling of waves in optical fibres. The last three terms in Eq. (1) describe the group velocity dispersion (GVD), self-steepening and self phase-modulation (SPM), respectively.

The Lax pair for the MNLS equation provides the mathematical basis for the solvability of this equation by the inverse scattering method and Darboux transformation (DT), the latter is defined by the Wadati–Konno–Ichikawa (WKI) spectral problem and the first non-trivial flow,[7]

$$\begin{aligned}\partial_x\psi &= (-aJ\lambda^2 + Q_1\lambda + Q_0)\psi \equiv U\psi,\\ \partial_t\psi &= (-2a^2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0)\psi\\ &\equiv V\psi,\end{aligned}$$

under the reduction condition $r = -q^*$. Here the complex quantity λ is the eigenvalue (or spectral parameter), and $\psi = (\phi, \varphi)^T$ is the eigenfunction associated with λ. The superscript T denotes transposition. There are also other methods to show the integrability of the MNLS equation and to obtain its exact solutions.[8–11] Various types of solutions to the MNLS equation, including a soliton and a breather, have already been obtained in Refs. [12–21]. The decay of soliton solution for a perturbed MNLS system has been demonstrated numerically.[22] Small perturbations of the MNLS equation have been studied, either by a direct method,[23] or by using the inverse scattering transform.[24,25]

Up to now, all known solutions of the MNLS equation represent rather common nonlinear wave solutions, like solitons and breathers. These are similar to their ancestors (i.e. the NLS and DNLS equations), so they do not describe any specific properties of either the Alfvén waves in magnetized plasmas or ultrashort light pulses in optical fibres. Thus, finding unique phenomena related to the simultaneous effects of self-steepening and SPM that can be described by the explicit analytical solutions of the MNLS equation is a long-standing problem.

In this Letter, we present novel rational solutions of the MNLS equation. These solutions describe five states of the associated systems: a paired bright-bright soliton state, a single soliton state, a paired bright-grey soliton state, a paired bright-dark soliton state, and a rogue wave state. We also show that the transitions among these five states are induced by the self-steepening and SPM through tuning the values of a and b determined by the physical properties of the background state.

By the one-fold DT and Taylor-expansion, according to a similar procedure demonstrated in Refs. [26–30] for constructing the rational rogue wave (RW) of the NLS, DNLS, Hirota and NLS-Maxwell-Bloch

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equations, a novel rational solution of the MNLS equation is given as follows:

$$q^{[1]} = -\exp[i(-x - tb - t + ta)]\frac{r_1 r_2}{r_1^{*2}}, \qquad (2)$$

$$\begin{aligned} r_1 &= X + 1 + 2ia[(3a-2)t - x], \\ r_2 &= X - 3 + 2i[(-6a + 3a^2 + 4b)t - ax], \\ X &= 2(a-b)[(-2b + 3a^2 - 6a + 4)t^2 \\ &\quad + 4(1-a)xt + x^2]. \end{aligned}$$

We omit the tedious calculation of this solution. The validity of this solution has been confirmed by symbolic computation. By letting $x \to \infty$ and $t \to \infty$ it is easy to show that $|q^{[1]}|^2 \to 1$, and $|q^{[1]}|^2(0,0) = 9$. Thus $q^{[1]}$ denotes a rational solution on a nonzero background with a unit height. This is a new and wide class of solutions for the MNLS equation because it can encompass five kinds of solution.

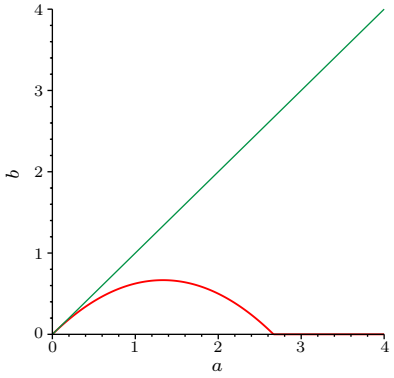

Fig. 1. (Color online) Five regions in the upper-right quadrant on the (a,b)-plane. In each region, $q^{[1]}$ gives a new kind of solution for the MNLS equation. The straight green line is $a = b$, the red curve is defined by $b = a - \frac{3}{8}a^2$ for $a \le \frac{8}{3}$, $b = 0$ for $a \ge \frac{8}{3}$.

We are now in a position to explore the properties of $q^{[1]}$ in more details. The trajectory of $|q^{[1]}|^2$ in the (x,t)-plane is defined by the location of the ridge (or vale) of its profile. In general, a good approximation of the trajectory for $|q^{[1]}|^2$ is given through a simple equation $X + 1 = 0$. By a straightforward but tedious calculation of the stationary point of $|q^{[1]}|^2$ in the (x,t)-plane, we find that $q^{[1]}$ describes five solutions associated with five regions in the upper-right quadrant on the (a,b)-plane (see Fig. 1), as follows:

(I) $(b > a)$. There is only one saddle point of $|q^{[1]}|^2$ at $(0,0)$, and there are two simultaneous trajectories, X_1 and X_2, on the (x,t)-plane. Note that the trajectories are not two straight lines, as would usual in the case of double solitons. If the height of soliton $|q^{[1]}|^2$ is increasing as it evolves along X_1, then it will decrease as it evolves on X_2. The asymptotic height of $|q^{[1]}|^2$ is $H_1 = H_1(a,b)$ as $t = -\infty$, and $H_2 = H_2(a,b)$ as $t = +\infty$ on X_1; $|q^{[1]}|^2$ approaches to H_2 as $t = -\infty$ and to H_2 as $t = +\infty$ on X_2. Thus, when $(a,b) \in$ region I, then $q^{[1]}$ is called a paired bright-bright soliton because $H_2 > H_1 > 1$ and because of the appearance of two peaks (i.e. an increasing peak and a decreasing one). Obviously, there exists an energy exchange between the two bright peaks propagating along X_1 and X_2 in accordance with energy conservation. In particular, the distance between the two peaks is proportional to $\sqrt{\delta_1 t^2 + \delta_2}$ in contrast to a linear function of t for the known two peaks in the case of a double soliton. Here, δ_1 and δ_2 are two real functions of a and b, respectively.

(II) $(b = a)$. $|q^{[1]}|^2$ takes its maximum value 9 when (x,t) is on the line $x = (-2+3a)t$. This line is also the trajectory of $|q^{[1]}|^2$. This is a single soliton solution.

(III) $(a > b > \max[0, a - \frac{3}{8}a^2])$. There is only one extremum of $|q^{[1]}|^2$ at $(0,0)$ on the (x,t)-plane. Most features of the obtained solution are similar to those of region I, except $H_2 > 1 > H_1 > 0$. Thus, in this region, $q^{[1]}$ is called a paired bright-grey soliton. Note that $a - \frac{3}{8}a^2 < 0$ if $a > \frac{8}{3}$.

(IV) $(b = a - \frac{3}{8}a^2,\ a \in [0, \frac{8}{3}])$. There is only one extremum of $|q^{[1]}|^2$ at $(0,0)$ in the (x,t)-plane. The solution resembles that of region III except $H_2 > 1 > H_1 \ge 0$. Thus, in region IV of the (a,b)-plane, $q^{[1]}$ is called a paired bright-dark soliton.

(V) $(0 < b < a - \frac{3}{8}a^2)$. For $|q^{[1]}|^2$, there is only one maximum at (0,0), where $|q^{[1]}|^2 = 9$, and two minima, where $|q^{[1]}|^2 = 0$. The two minima are located at the positions whose coordinates are given by

$$\begin{aligned} x &= \mp \frac{-6a + 3a^2 + 4b}{a-b}\sqrt{\frac{3}{32((a - \frac{3}{8}a^2) - b)}}, \\ t &= \pm \frac{a}{a-b}\sqrt{\frac{3}{32((a - \frac{3}{8}a^2) - b)}}, \end{aligned}$$

at the points in the (x,t)-plane. When (a,b) is region V, $|q^{[1]}|^2$ is localized in both the x and t directions and, thus, $q^{[1]}$ is a rogue wave solution of the MNLS equation.

The main difference between the grey and dark soliton is that the minimum of the solution may or may not reach zero[31] for the grey soliton.

For a physical system modelled by the MNLS equation, each solution presented above gives a particular phase state. Thus, it is rather interesting to observe that in Fig. 1 there exists a state transition induced by tuning the self-steepening and SPM, which can be realized by adjusting the values of a and b. For example, by setting $a = 1/2$ and varying b, the system will evolve consecutively through a paired bright-bright, a single, a paired bright-grey, a paired bright-dark soliton states, and the rogue wave state as b decreases from a value larger than $\frac{1}{2}$ to another value that is smaller than $\frac{13}{32}$. Moreover, for a given $b > b_c(= \frac{2}{3})$, the system passes through the first three (i.e. I–III) states as a increases from *zero* to a sufficiently large value; for $b = b_c$, the system now passes through the first four (i.e. I–IV) states, and it passes through the paired bright-grey soliton state twice; finally, for $b < b_c$, the system passes through all five states and will be in paired bright-grey and paired bright-dark soliton states twice. This newly discovered unique

state transition phenomenon is not described by the rational solutions of the NLS and DNLS equations because they do not describe two different nonlinear effects simultaneously. Thus, we have now solved the long-standing problem mentioned in the introduction.

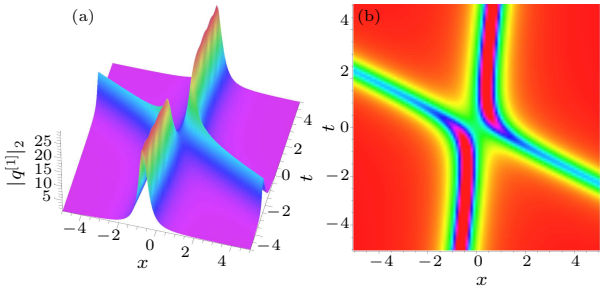

Fig. 2. The profile (a) and density plot (b) for a paired bright-bright soliton.

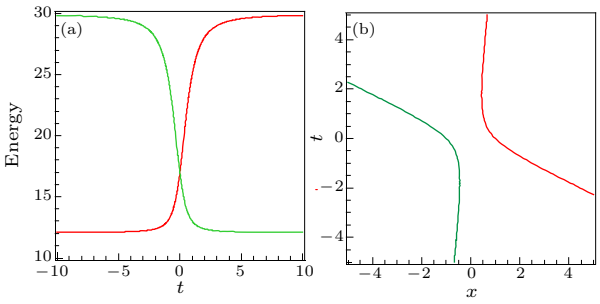

Fig. 3. (a) The energy transmission from a bright soliton (green line) to another one (red line). (b) The trajectories of the solitons, X_2 (green line) and X_1 (red line).

In what follows, we use two methods to visualize function $|q^{[1]}|^2$. The first method consists of plotting the profile of $|q^{[1]}|^2$ which is the graph of this function of two variables in three dimensions. Although the second method is similar to drawing the level lines of this function, it does not use the lines but it uses various colors instead. The figure obtained in this way is called the density plot of the function $|q^{[1]}|^2$. In this plot, each color corresponds to a definite value of $|q^{[1]}|^2$.

To illustrate the general results, we investigate the evolution of the rational solution for a fixed $a = \frac{1}{2}$ and varying b. In Fig. 2 the profile (left panel) and density plot (right panel) of $|q^{[1]}|^2$ are shown for a paired bright-bright soliton with $b = 1$. In Fig. 3 the energy exchange (left panel) for two peaks along the two trajectories (right panel) is plotted. The limit heights in Figs. 2 and 3 are $H_2 = 21+4\sqrt{5} \approx 29.9$ and $H_1 = 21 - 4\sqrt{5} \approx 12.1$. Note that the height of the background in Fig. 2 is equal to 1. On the right panel of Fig. 3, the explicit equations of trajectories X_1 (red line) and X_2 (green line) are $-t + \frac{1}{2}\sqrt{5t^2+4} = x$ and $-t - \frac{1}{2}\sqrt{5t^2+4} = x$, respectively. The distance between the two peaks at a given time is $\sqrt{5t^2+4}$, unlike the case of the two peaks in a double soliton solution. It is straightforward to see in Fig. 3 (left panel) that the energy is transmitted gradually from a bright soliton (green line) moving along X_2 to the other bright soliton (red line) moving along X_1. The reflective symmetry of Fig. 3 (left panel) refers to the energy conservation. By comparing Fig. 2 (right panel) with Fig. 3 (right panel), we can now see that the trajectory gives a very good approximation of the ridge location in the profile of $|q^{[1]}|^2$. Setting $a = b = \frac{1}{2}$, so that (a, b) is in region Ⅱ, we obtain $|q^{[1]}|^2 = \frac{36+(t+2x)^2}{4+(t+2x)^2}$, which is a single soliton with a height 9 and an exact trajectory $t + 2x = 0$. We do not show the profile in this case because it is a standard soliton.

In region Ⅲ, by setting $b = \frac{9}{20}$, we obtain a paired bright-grey soliton. Its profile is plotted in Fig. 4(a). The limit height of the bright soliton is $H_2 = \frac{8-\sqrt{15}}{20-5\sqrt{15}} \approx 6.5$, and the limit height of the grey soliton is $H_1 = \frac{8+\sqrt{15}}{20+5\sqrt{15}} \approx 0.3$. The X_1 (red line) and X_2 (green line) in Fig. 4(b) give good approximation of the trajectories for $|t| > 8$.

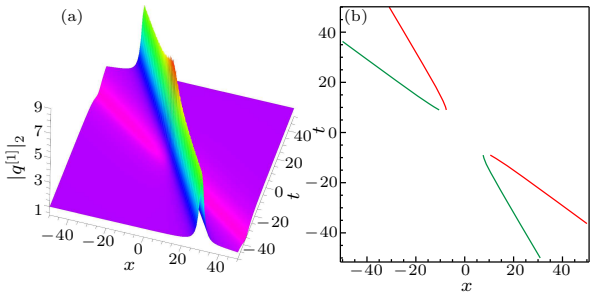

Fig. 4. The paired bright-grey soliton (a) and its trajectories (b).

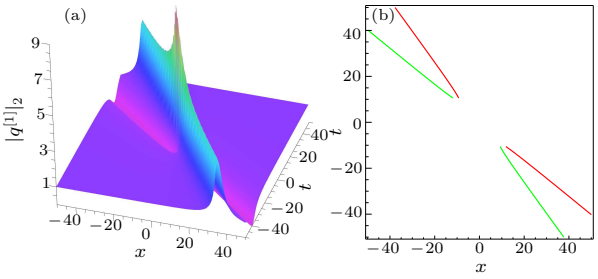

Fig. 5. The paired bright-black soliton (a) and its trajectories (b).

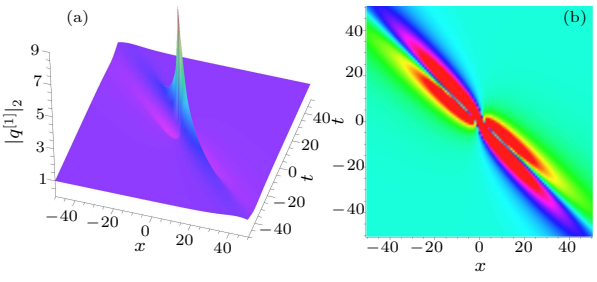

Fig. 6. The first-order rogue wave (a) and its density plot (b).

Let us set $a = \frac{1}{2}$ and $b = a - \frac{3}{8}a^2 \approx 0.41$, the point (a, b) is now in region Ⅳ, and the rational solution describes a bright-dark soliton. The profile of this solutions is shown in Fig. 5(a). The limit heights are $H_2 = 4$ for the bright soliton and $H_1 = 0$ for the dark soliton. The trajectories X_1 and X_2 are shown

in Fig. 5(a) by the red and green lines, respectively. The analysis of the two solutions corresponding to the bright-grey and bright-dark solitons shows that: (i) there is energy exchange between the bright and grey (dark) solitons; and, (ii) the bright soliton is transformed into a grey (dark) one because its energy is lost during the interaction, while the grey (dark) soliton is transformed into a bright one for $t \gg 1$.

Finally, by setting $a = \frac{1}{2}$ and $b = \frac{1}{3}$, we put the point (a, b) in region V. In this case the rational solution describes the first-order rogue wave, which is plotted in Fig. 6.

We have presented a range of new types of rational solution of the MNLS equation that describe the propagation, for example, of Alfvén waves in magnetized plasmas and the femtosecond light pulses in optical fibres. The obtained solutions have two free parameters, a and b, representing the contributions of self-steepening and self phase-modulation. Depending on the values of these parameters, these solutions describe five types of novel solitons corresponding to five states of an associated physical system. These solutions are: a paired bright-bright, single, paired bright-grey, and a paired bright-dark soliton, and a rogue wave. We have found that the state transition among these five states is induced by tuning the effects of self-steepening and SPM. We urge that this novel phenomenon may be observed in laboratory or in magnetized plasma in nature in order to demonstrate an intricate balance between the effects of self-steepening and SPM in an associated physical system. Furthermore, because of the recent discovery of Alfvén waves (see Refs. [32,33], and the references therein) in the magnetized solar atmosphere, it is now of paramount interest to find these novel states and the state transfer in space plasmas, and to establish their connection with the long-standing coronal heating problem.[34–37]

Finally, we briefly discuss the novelty of the solution describing the paired bright-bright soliton. Three of its characteristics (i.e. the existence of two peaks, decreasing or increasing amplitude, and the curved trajectories) are essentially different from the similar characteristics of the recently found solition solutions that include an explode-decay soliton,[38] a two-peak soliton,[39,40] a W-shape soliton,[41] a dark-in-bright soliton,[42,43] a rogue wave[44] and a two-peak rogue wave,[45,46] in addition to the well-known classical soliton, breather, and kink. In particular, the paired bright-bright soliton is not a travelling wave. The non-autonomous solitons[47–50] have properties similar to those of the paired bright-bright soliton but only when they are solutions to the variable coefficient soliton equations. This discussion can also be applied to the paired bright-grey and bright-dark soliton solutions. Lastly, we would like to stress that there is also no doubt of the novelty of our solution $q^{[1]}$ in Eq. (2), although there exists a simple gauge transformation, for example, see Refs. [14,51] between the MNLS equation and the DNLS equation. Because the rational solutions (or rogue wave solutions) in recent works[27,52,53] cannot be mapped to a solution possessing two-peak profiles as $q^{[1]}$ by this transformation.

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