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The Strategic Form of Quantum Prisoners' Dilemma

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In a normal form, prisoners' dilemma (PD) is represented by a payoff matrix showing players' strategies and payoffs. To obtain the distinguishing trait and strategic form of PD, certain constraints are imposed on the elements of its payoff matrix. We quantize PD by a generalized quantization scheme to analyze its strategic behavior in the quantum domain. The game starts with a general entangled state of the form $|\psi\rangle = \cos\frac{\xi}{2}|00\rangle + i\sin\frac{\xi}{2}|11\rangle$ *and the measurement for payoffs is performed in entangled and product bases. We show that for both measurements, there exist respective cutoff values of entanglement of the initial quantum state up to which the strategic form of the game remains intact. Beyond these cutoffs the quantized PD behaves like the chicken game (CG) up to another cutoff value. For the measurement in the entangled basis the dilemma is resolved for* $\sin\xi > 1/7$ *with* $Q \otimes Q$ *as a Nash Equilibrium (NE). However, the quantized game behaves like PD when* $\sin\xi > 1/3$*; whereas in the range* $1/7 < \sin\xi < 1/3$ *it behaves like CG with* $Q \otimes Q$ *as an NE. For the measurement in the product basis the quantized PD behaves like classical PD for* $\sin^2\frac{\xi}{2} < 1/3$ *with* $D \otimes D$ *as an NE. In region* $1/3 < \sin^2\frac{\xi}{2} < 3/7$*, the quantized PD behaves like classical CG with* $C \otimes D$ *and* $D \otimes C$ *as NEs.*

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Game theory deals with a situation where two or more rational players are involved in a strategic contest to maximize their payoffs.[1] The payoff of each player depends on his own strategy and on the strategies adopted by other players.[2] The set of strategies from which unilateral deviation of any player reduces his/ her payoff is called a Nash Equilibrium (NE) of the game.[3] In its normal form a game is represented by a bi-matrix with its elements as payoffs. It is necessary to impose a set of constraints on the elements of the payoff matrix to obtain the strategic form of the game. Take, for example, prisoner dilemma (PD), which is a story of two prisoners, Alice and Bob, who have allegedly committed a crime together. They are being interrogated in separate cells. Each of the prisoners has to decide whether to confess the crime (to defect D) or to deny the crime (to cooperate C) without any communication between them. If both players receive R and U for mutual cooperation and defection, respectively, and a cooperator and a defector engaged in a contest against each other receive S and T, respectively, then the strategic form of PD demands that $T > R > U > S$.[4,5] Due to these constraints, rational reasoning forces each player to defect. As a result DD appears as an NE of the game which is not *Pareto optimal*. This is referred to as the dilemma of this game.

Chicken game (CG) on the other hand depicts a situation in which two players drive their cars straight towards each other. The first to swerve to avoid the collision (to cooperate C) is the loser (chicken) and the one who keeps on driving straight (to defect D) is the winner. By assigning R and U to mutual cooperation and defection, respectively, and S and T to a cooperator and a defector against each other, then the strategic form of CG requires that $T > R > S > U$.[4] As a result there is no dominant strategy and CD, DC appear as an NE. The dilemma of this game is that CC which is *Pareto optimal* is not an NE.

This type of dilemma is resolved by analyzing games in the quantum domain. One of the elegant and foremost steps in this direction is by Eisert *et al.*[6] to remove the dilemma in PD. In this quantization scheme the strategy space of the players is a two-parameter set of 2×2 unitary operators. Starting with maximally entangled initial quantum state the researchers showed that for a suitable quantum strategy the dilemma disappears from the game. The quantum strategy pair $Q \otimes Q$ appears as an NE which is *Pareto optimal*. They also pointed out that the quantum strategy Q always wins over all classical strategies. Eisert *et al.*[6] also showed that $Q \otimes Q$ is a unique NE in CG and is *Pareto optimal*. This quantization scheme has many interesting applications in quantum game theory.[8−21] Later, Marinatto and Weber[22] introduced another interesting and simple scheme for the quantization of non-zero sum games. They gave a Hilbert structure to the strategic spaces of the players. They also used the maximally entangled initial state and allowed the players to play their tactics by applying probabilistic choices of unitary operators. Applying their scheme to the Battle of the Sexes game, they found the strategy for which both the players have equal payoffs. Marinatto and Weber's quantization scheme gave very interesting results while investigating evolutionarily stable strategies (ESS)[11,23,24] and in the analysis of repeated games[25,26] etc. In our earlier work we introduced a generalized quantization scheme that establishes a relation between these two apparently different quantization schemes.[27] Separate set of parameters were identified for which this scheme reduces to that of Eisert *et al.*[7] and Marinatto and Weber[22] quantization schemes.

In this Letter, we address the question of to what

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extent the strategic form of PD remains unaffected if it is quantized by a generalized quantization scheme.[27] Starting with a general entangled state of the form $|\psi\rangle = \cos\frac{\xi}{2}|00\rangle + i\sin\frac{\xi}{2}|11\rangle$ we show that the strategic form of quantized PD depends on entanglement of initial quantum state as well as on the type of measurement basis (entangled or product). For both types of measurements there exist respective cutoff values of entanglement of initial quantum state up to which the strategic form of the game remains intact. Beyond these cutoffs the quantized PD behaves like the chicken game up to another cutoff value.

For the case of PD, if to confess the crime is termed as 'to Defect', the strategy D, and to deny the crime is referred to as 'to Cooperate', the strategy C, then depending upon their decisions the players obtain the payoffs according to the following payoff matrix.

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} C & D \end{array} \\ \text{Alice}\begin{array}{c} C \\ D \end{array} & \left[\begin{array}{cc} (3,3) & (0,5) \\ (5,0) & (1,1) \end{array}\right]. \end{array} \tag{1}$$

It is clear from the above payoff matrix that D is the dominant strategy for both the players. Therefore rational reasoning forces each of them to play D resulting in DD as an NE of PD. From the payoff matrix Eq. (1), we can see that each player gets $(1,1)$ as payoff. However, it is possible for the players to get better payoff of value $(3,3)$ if they would have played CC instead of DD. This is generally known as the dilemma of this game. We can rewrite the payoff matrix Eq. (1) in a general form as

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} C & D \end{array} \\ \text{Alice}\begin{array}{c} C \\ D \end{array} & \left[\begin{array}{cc} (R,R) & (S,T) \\ (T,S) & (U,U) \end{array}\right], \end{array} \tag{2}$$

with

$$T > R > U > S, \tag{3}$$

as constraint on its elements.

In CG, two players Alice and Bob drive their cars straight towards each other. The first to swerve to avoid the collision (the strategy C) is the loser (chicken) and the one who keeps on driving straight (the strategy D) is the winner. The payoff matrix for this game can also be of the form of Eq. (2) but with constraints

$$T > R > S > U. \tag{4}$$

Certainly if both the players cooperate they can avoid a crash and neither of them will be the winner. If one of them steers away (defects D) he will be the loser but will survive, and the opponent will receive the entire honor. If they crash, then the cost of both of them will be higher than the cost of being chicken and the payoff will be lower.[4,28] There is no dominant strategy in this game. The strategy pairs (C, D) and (D, C) are two NEs in this game. The former is preferred by Bob and the latter is preferred by Alice. The dilemma of this game is that CC which is *Pareto optimal* is not the NE of this game.

Next we quantize PD using a generalized quantization scheme for two person non zero sum games.[27] In this quantization scheme an arbiter prepares a two-qubit general entangled state and passes on one qubit to each player. After applying their local unitary operators (strategies) the players return the qubits to the arbiter who then announces the payoffs by performing the measurement with the application of suitable payoff operators depending on the payoff matrix of the game. The payoff operators are Bell-like states which transform to the operators of Eisert *et al.*[6] for maximum entanglement and for zero entanglement they reduce to the payoff operators used by Marinatto and Weber in their quantization scheme.[22] There can be four cases of interest. If both the initial quantum state and payoff operators are in the form of product states, then a classical game will be reproduced. When initial quantum state and the payoff operators are maximally entangled states, then this scheme will transform to the quantization scheme of Eisert *et al.*[6] For maximally entangled initial quantum state and product basis measurement it is reduced to that of the Marinatto and Weber quantization scheme.[22] On the other hand if the game starts with the product state but the measurement for the payoffs is performed in the entangled basis then the payoffs are also quantum mechanical in nature, whereas this feature is absent in both the schemes of Eisert *et al.* and Marinatto and Weber quantization.

For the quantization of PD the classical strategies C (to cooperate) and D (to defect) are assigned two basis vectors $|C\rangle$ and $|D\rangle$, respectively, in a Hilbert space of a two-level system. The state of the game at any instant is a vector in four-dimensional Hilbert space spanned by the basis vectors $|CC\rangle$, $|CD\rangle$, $|DC\rangle$ and $|DD\rangle$. Here the entries in the ket refer to the qubits possessed by Alice and Bob, respectively. Representing that $|C\rangle \to |0\rangle$ and $|D\rangle \to |1\rangle$ lets the initial quantum state of the game be of the form

$$|\psi\rangle = \cos\frac{\xi}{2}|00\rangle + i\sin\frac{\xi}{2}|11\rangle, \tag{5}$$

where ξ is the entanglement parameter. The strategies of the players are represented by unitary operators U_j, given as[27]

$$U_j = \cos\frac{\theta_j}{2}R_j + \sin\frac{\theta_j}{2}C_j, \tag{6}$$

where $j = A$, B and R_j, C_j are the unitary operators defined as

$$\begin{aligned} R_j|0\rangle &= e^{i\phi_j}|0\rangle, \;\; R_j|1\rangle = e^{-i\phi_j}|1\rangle \\ C_j|0\rangle &= -|1\rangle, \;\; C_j|1\rangle = |0\rangle. \end{aligned} \tag{7}$$

After the application of strategies the initial state given by Eq. (5) transforms into

$$\rho_f = (U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger. \tag{8}$$

The payoff operators of Alice and Bob are

$$P^A = 3P_{00} + P_{11} + 5P_{10}$$
$$P^B = 3P_{00} + P_{11} + 5P_{01}, \tag{9}$$

where

$$P_{00} = |\psi_{00}\rangle\langle\psi_{00}|, \quad |\psi_{00}\rangle = \cos\frac{\delta}{2}|00\rangle + i\sin\frac{\delta}{2}|11\rangle, \tag{10a}$$

$$P_{11} = |\psi_{11}\rangle\langle\psi_{11}|, \quad |\psi_{11}\rangle = \cos\frac{\delta}{2}|11\rangle + i\sin\frac{\delta}{2}|00\rangle, \tag{10b}$$

$$P_{10} = |\psi_{10}\rangle\langle\psi_{10}|, \quad |\psi_{10}\rangle = \cos\frac{\delta}{2}|10\rangle - i\sin\frac{\delta}{2}|01\rangle, \tag{10c}$$

$$P_{01} = |\psi_{01}\rangle\langle\psi_{01}|, \quad |\psi_{01}\rangle = \cos\frac{\delta}{2}|01\rangle - i\sin\frac{\delta}{2}|10\rangle, \tag{10d}$$

and $\delta \in [0, \frac{\pi}{2}]$ is the entanglement of measurement basis. These payoff operators reduce to the scheme of Eisert *et al.*[6] for $\delta = \frac{\pi}{2}$, and for $\delta = 0$ these transform to the scheme of Marinatto and Weber.[22] The payoff for player i is calculated by

$$\$_i(\theta_A, \phi_A, \theta_B, \phi_B) = \mathrm{Tr}(P^i \rho_f). \tag{11}$$

Since in a generalized quantization scheme, measurements can be performed in the entangled as well as in the product basis. Therefore we discuss both cases one by one.

Case 1: Entangled measurement

When the measurement is performed in the entangled basis then using Eqs. (1), (5), and (8)–(10), the payoffs of players come out to be

$$\begin{aligned}\$_A(\theta_A, \phi_A, \theta_B, \phi_B) &= [2 + \sin\xi\cos 2(\phi_A + \phi_B)]\cos^2\frac{\theta_A}{2} \\ &\cdot \cos^2\frac{\theta_B}{2} + \frac{5}{2}(1 + \sin\xi\cos 2\phi_B)\sin^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} \\ &+ \frac{5}{2}(1 - \sin\xi\cos 2\phi_A)\cos^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2} \\ &+ (2 - \sin\xi)\sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2} \\ &- \frac{(2 + \sin\xi)}{4}\sin\theta_A\sin\theta_B\sin(\phi_A + \phi_B) \\ &- \frac{5}{4}\sin\theta_A\sin\theta_B\sin(\phi_A - \phi_B),\end{aligned} \tag{12}$$

$$\begin{aligned}\$_B(\theta_A, \phi_A, \theta_B, \phi_B) &= [2 + \sin\xi\cos 2(\phi_A + \phi_B)]\cos^2\frac{\theta_A}{2} \\ &\cdot \cos^2\frac{\theta_B}{2} + \frac{5}{2}(1 + \sin\xi\cos 2\phi_A)\sin^2\frac{\theta_B}{2}\cos^2\frac{\theta_A}{2} \\ &+ \frac{5}{2}(1 - \sin\xi\cos 2\phi_B)\cos^2\frac{\theta_B}{2}\sin^2\frac{\theta_A}{2} \\ &+ (2 - \sin\xi)\sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2} \\ &- \frac{(2 + \sin\xi)}{4}\sin\theta_A\sin\theta_B\sin(\phi_A + \phi_B) \\ &- \frac{5}{4}\sin\theta_A\sin\theta_B\sin(\phi_A - \phi_B).\end{aligned} \tag{13}$$

In this case if the game starts from a maximally entangled state then $Q \otimes Q$ will be the only NE of the game, where Q is the unitary operator $U(\theta, \phi) = U(0, \frac{\pi}{2})$.[6] To see the behavior of $Q \otimes Q$ at other values of entanglement we apply the NE conditions as

$$\begin{aligned}\$_A(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) - \$_A(\theta_A, \phi_A, 0, \frac{\pi}{2}) \geq 0, \\ \$_B(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) - \$_B(0, \frac{\pi}{2}, \theta_B, \phi_B) \geq 0.\end{aligned} \tag{14}$$

With the help of Eqs. (12) and (13) for $i = A, B$, the above inequalities give

$$7\sin\xi + [1 + (2\cos 2\phi_i - 5)\sin\xi]\cos^2\frac{\theta_i}{2} - 1 \geq 0. \tag{15}$$

This inequality is satisfied for $\sin\xi \geq 1/7$. Therefore $Q \otimes Q$ remains the NE for a game that starts with an initial state for which

$$\sin\xi > 1/7. \tag{16}$$

Now we investigate whether the quantum game that we obtained by quantization of PD with $Q \otimes Q$ as the NE possesses the characteristics of PD. Using Eqs. (12) and (13) the elements of the payoff matrix of quantized PD are

$$\begin{aligned}R = 2 + \sin\xi, \ S = \frac{5 - 5\sin\xi}{2}, \\ T = \frac{5 + 5\sin\xi}{2}, \ U = 2 - \sin\xi.\end{aligned} \tag{17}$$

For the above values of payoff elements the constraint Eq. (3) is satisfied if $\sin\xi > 1/3$. It shows that in quantized PD the resolution of the dilemma without affecting its strategic form requires that the entanglement of the initial quantum state must be greater than $\arcsin\frac{1}{3}$. It means that $Q \otimes Q$ is the NE of a quantized PD for all values of entanglement for which $\sin\xi \geq 1/7$. However, it behaves like PD only for $\sin\xi > 1/3$. This is shown in Fig. 1. It can be seen that in the region $1/3 \geq \sin\xi \geq 1/7$ the constraints on payoff elements transforms to

$$T > R > S > U. \tag{18}$$

Comparing with Eq. (4) we can see that for these values of entanglement the quantized PD behaves like CG. It means that when PD is quantized with an initial state of entanglement less than $\arcsin\frac{1}{3}$ then it transforms to CG while $Q \otimes Q$ still remains the NE. When the entanglement is further reduced and $\sin\xi < 1/7$ then the quantum game again changes its form. In this region the payoff matrix elements obey the constraints

$$T > S > R > U, \tag{19}$$

and $Q \otimes D$, $D \otimes Q$ are NEs. For $Q \otimes D$ the NE conditions

$$\begin{aligned}\$_A(Q, D) - \$_A(\theta_A, \phi_A, D) \geq 0, \\ \$_B(Q, D) - \$_B(Q, \theta_B, \phi_B) \geq 0,\end{aligned} \tag{20}$$

become

$$\sin^2\frac{\theta_{\rm A}}{2}+[7-(2-5\cos 2\phi_{\rm A})\cos^2\frac{\theta_{\rm A}}{2}]\sin\xi\geq 0,$$
$$[1-(5-2\cos 2\phi_{\rm B})\sin\xi]\cos^2\frac{\theta_{\rm B}}{2}\geq 0. \quad (21)$$

These inequalities are satisfied for all $\theta's$ and $\phi's$ if $0\leq\sin\xi\leq 1/7$. However at $\sin\xi=0$ two new NEs $C\otimes D$ and $D\otimes C$ also come into play. At this stage we have a game that has four pure strategy NEs. For $C\otimes D$ the NE conditions

$$\$_{\rm A}(C,D)-\$_{\rm A}(\theta_{\rm A},\phi_{\rm A},D)\geq 0,$$
$$\$_{\rm B}(C,D)-\$_{\rm B}(C,\theta_{\rm B},\phi_{\rm B})\geq 0 \quad (22)$$

yield

$$\sin^2\frac{\theta_{\rm A}}{2}-[3+(2-5\cos 2\phi_{\rm A})\cos^2\frac{\theta_{\rm A}}{2}]\sin\xi\geq 0, \quad (23)$$

$$[1+(5-2\cos 2\phi_{\rm B})\sin\xi]\cos^2\frac{\theta_{\rm B}}{2}\geq 0. \quad (24)$$

These inequalities are satisfied for all θ's and ϕ's for $\sin\xi=0$ showing that for zero value of entanglement $C\otimes D$ becomes an NE.

On the other hand it is also obvious from Fig. 1 that there are two points $\sin\xi=1/7$ and $\sin\xi=1/3$ where the constraints on the elements of the payoff matrix are

$$T>R=S>U,\ T>R>S=U, \quad (25)$$

respectively. The former constraint represents the game called compromise dilemma[4] whereas the other constraint represents a game that is also different from PD.

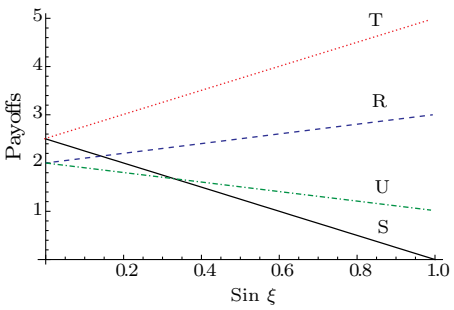


Fig. 1. Payoff elements versus $\sin\xi$ when the measurement is performed in the entangled basis. It shows that the quantized PD behaves like PD for an initial quantum state for which $\sin\xi>1/3$. In the region $1/3\geq\sin\xi\geq 1/7$ the quantized PD behaves like CG and for $\sin\xi<1/7$ the form of the game is again changed.

It proves that when PD starts with a general entangled state of the form (5) and measurement is performed in the entangled basis then it behaves like PD up to a certain cutoff value of entanglement of initial quantum state.

Case 2: Product Measurement

When the measurement is performed in the product basis then using Eqs. (1), (5), (8), (9), and (11), the payoff of player A comes out to be

$$\$_{\rm A}(\theta_{\rm A},\phi_{\rm A},\theta_{\rm B},\phi_{\rm B})=(1+2\cos^2\frac{\xi}{2})\cos^2\frac{\theta_{\rm A}}{2}\cos^2\frac{\theta_{\rm B}}{2}$$
$$+5\cos^2\frac{\xi}{2}\sin^2\frac{\theta_{\rm A}}{2}\cos^2\frac{\theta_{\rm B}}{2}+5\sin^2\frac{\xi}{2}\cos^2\frac{\theta_{\rm A}}{2}\sin^2\frac{\theta_{\rm B}}{2}$$
$$+(1+2\sin^2\frac{\xi}{2})\sin^2\frac{\theta_{\rm A}}{2}\sin^2\frac{\theta_{\rm B}}{2}$$
$$-\frac{1}{4}\sin\xi\sin\theta_{\rm A}\sin\theta_{\rm B}\sin(\phi_{\rm A}+\phi_{\rm B}). \quad (26)$$

The payoffs of player B can be found by replacing $\theta_{\rm A}\to\theta_{\rm B}$ and $\phi_{\rm A}\to\phi_{\rm B}$. For these payoffs $C\otimes C$ is the NE of the game if

$$\$_{\rm A}(C,C)-\$_{\rm A}(\theta_{\rm A},\phi_{\rm A},C)\geq 0. \quad (27)$$

Putting the corresponding values from Eq. (26), we get

$$\sin^2\frac{\theta_{\rm A}}{2}\left(3\sin^2\frac{\xi}{2}-2\right)\geq 0. \quad (28)$$

This inequality is satisfied if $\sin^2\frac{\xi}{2}\geq 2/3$. It shows that $C\otimes C$ is an NE with payoff $3-2\sin^2\frac{\xi}{2}$ for quantized PD that starts with an initial entangled state of the form (5) if $\sin^2\frac{\xi}{2}\geq 2/3$. It is important to note that $C\otimes C$ being an NE does not imply the resolution of the dilemma, because for $\sin^2\frac{\xi}{2}\geq 2/3$ each player could have obtained a better payoff $1+2\sin^2\frac{\xi}{2}$ by playing D instead of C.

For $D\otimes D$ as an NE we have the inequality

$$\$_{\rm A}(D,D)-\$_{\rm A}(\theta_{\rm A},\phi_{\rm A},D)\geq 0, \quad (29)$$

which with the help of Eq. (26) gives

$$\cos^2\frac{\theta_{\rm A}}{2}(1-3\sin^2\frac{\xi}{2})\geq 0. \quad (30)$$

The above inequality is satisfied for $\sin^2\frac{\xi}{2}\leq 1/3$ showing that $D\otimes D$ is an NE for quantized PD if it starts with an initial quantum state with $\sin^2\frac{\xi}{2}\leq 1/3$.

The $D\otimes C$ can be an NE if it satisfies the following NE inequalities

$$\$_{\rm A}(D,C)-\$_{\rm A}(\theta_{\rm A},\phi_{\rm A},C)\geq 0,$$
$$\$_{\rm B}(D,C)-\$_{\rm B}(D,\theta_{\rm B},\phi_{\rm B})\geq 0. \quad (31)$$

With the help of Eq. (26) the above inequalities become

$$\cos^2\frac{\theta_{\rm A}}{2}(2-3\sin^2\frac{\xi}{2})\geq 0,$$
$$\sin^2\frac{\theta_{\rm B}}{2}(3\sin^2\frac{\xi}{2}-1)\geq 0. \quad (32)$$

These inequalities are satisfied if $1/3\leq\sin^2\frac{\xi}{2}\leq 2/3$. Therefore, $D\otimes C$ is an NE of quantized PD which starts with an initial entangled state with $1/3\leq\sin^2\frac{\xi}{2}\leq 2/3$. By similar reasoning it can be proved that $C\otimes D$ is an NE if the entanglement of the initial quantum state is in the range $1/3\leq\sin^2\frac{\xi}{2}\leq 2/3$.

Now we investigate how the strategic form of quantized PD depends upon the entanglement of the initial

state. We find from Eq. (26) that the elements of the payoff matrix in this case are

$$R = 3 - 2\sin^2\frac{\xi}{2},\ S = 5\sin^2\frac{\xi}{2},$$
$$T = 5 - 5\sin^2\frac{\xi}{2},\ U = 1 + 2\sin^2\frac{\xi}{2}. \qquad (33)$$

These elements of the payoff matrix are plotted as a function of $\sin^2\frac{\xi}{2}$ in Fig. 2. The figure shows six regions and each region represents a different game. The constraints (3) required for the game to behave like PD are satisfied in region 1. This region is defined as $0 \le \sin^2\frac{\xi}{2} < 1/3$ with $D \otimes D$ as the NE. In region 2 where $1/3 < \sin^2\frac{\xi}{2} < 3/7$ the payoff matrix elements given in Eq. (33) are transformed into the constraints given in (4). In this region the quantized PD represents classical CG with $C \otimes D$ and $D \otimes C$ as NEs. When the entanglement of the initial state is further increased then the form of the game varies according to Table 1.

Table 1. Different forms of PD for specified range of initial state entanglement when measurement is performed in the product basis.

Region	Entanglement	NE	Game
1	$\sin^2\frac{\xi}{2} < \frac{1}{3}$	$D \otimes D$	Classical PD
2	$\frac{1}{3} < \sin^2\frac{\xi}{2} < \frac{3}{7}$	$C \otimes D$, $D \otimes C$	Classical CG
3	$\frac{3}{7} < \sin^2\frac{\xi}{2} < \frac{1}{2}$	$C \otimes D$, $D \otimes C$	Neither CG nor PD
4	$\frac{1}{2} < \sin^2\frac{\xi}{2} < \frac{4}{7}$	$C \otimes D$, $D \otimes C$	Neither CG nor PD
5	$\frac{4}{7} < \sin^2\frac{\xi}{2} < \frac{2}{3}$	$C \otimes D$, $D \otimes C$	Neither CG nor PD
6	$\sin^2\frac{\xi}{2} > \frac{2}{3}$	$C \otimes C$	Neither CG nor PD

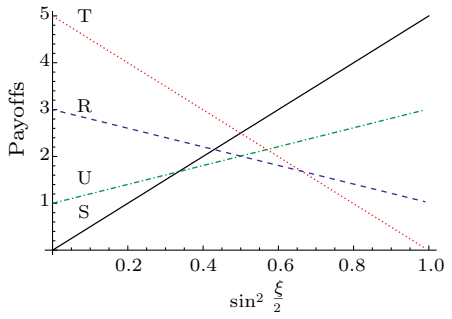


Fig. 2. Payoff elements versus $\sin^2\frac{\xi}{2}$ when the measurement is performed in the product basis. The constraints required for the game to behave like PD are satisfied in the region defined by $0 \le \sin^2\frac{\xi}{2} < 1/3$. In the region $1/3 < \sin^2\frac{\xi}{2} < 3/7$ the quantized PD behaves like classical CG with $C \otimes D$ and $D \otimes C$ as NEs.

Furthermore from Fig. 2, it can be seen that there are points such as $\sin^2\frac{\xi}{2} = 1/3$, $\sin^2\frac{\xi}{2} = 3/7$, $\sin^2\frac{\xi}{2} = 1/2$, $\sin^2\frac{\xi}{2} = 4/7$ and $\sin^2\frac{\xi}{2} = 2/3$ where two or more payoff matrix elements are equal. The form of game at these points can be described as follows:

(1) At $\sin^2\frac{\xi}{2} = 1/3$ the payoff matrix elements are related through the constraints $T > R > S = U$ with $R = 2.3333$, $S = U = 1.6667$, and $T = 3.3333$. Here the game has $C \otimes D$ and $D \otimes C$ as NEs. However, both these NEs are not strict.[29]

(2) At $\sin^2\frac{\xi}{2} = 3/7$ we see that $T > (R = S) > U$ with $R = S = 2.1429$, $T = 2.8571$, $U = 1.8571$ and at this point quantized PD behaves like compromise dilemma. In such a situation it is better to play opposite to the opponent.[4]

(3) At $\sin^2\frac{\xi}{2} = 1/2$ the matrix elements obey the constraints $(R = U) < (S = T)$ where $R = U = 2$, $S = T = 2.5$. Here the game has $C \otimes D$ and $D \otimes C$ as NEs.

(4) At $\sin^2\frac{\xi}{2} = 4/7$ the constraints on the payoff matrix element become $S > (T = U) > R$ and $R = 1.8571$, $S = 2.8571$, $T = U = 2.1429$. The game has $C \otimes D$ and $D \otimes C$ as NEs.

(5) At $\sin^2\frac{\xi}{2} = 2/3$ the constraints take the form $S > U > (R = T)$ where $R = T = 1.6667$, $S = 3.3333$, $U = 2.3333$. This game has $C \otimes D$ and $D \otimes C$ as NEs and both these NEs are not strict.[29]

Note that for all the above cases the quantized game never behaves like PD.

In summary, we have quantized PD by a generalized quantization scheme[27] starting with a general initial entangled state of the form $|\psi\rangle = \cos\frac{\xi}{2}|00\rangle + i\sin\frac{\xi}{2}|11\rangle$. In this scheme the measurements for payoffs can be performed in entangled and product bases. For both types of measurements the strategic form of quantized PD depends upon the entanglement of the initial quantum state. For measurement in the entangled basis when the entanglement of initial quantum state is reduced then beyond a certain level of entanglement the quantized PD behaves like CG with $Q \otimes Q$ as the NE. On further reduction of entanglement the game ceases to behave like CG and transforms into a new game with $Q \otimes D$ and $D \otimes Q$ as NEs. At last for zero entanglement two additional NEs $C \otimes D$ and $D \otimes C$ also appear in resulting a game with four pure strategies NEs. When the measurement is performed in product basis then the quantized PD can be divided in eleven different games with respect to initial state entanglement. In this case for zero entanglement of initial quantum state the game behaves like PD with $D \otimes D$ as an NE. With increasing entanglement of initial state there is a cutoff value beyond which the game behaved like CG with $C \otimes D$ and $D \otimes C$ as NEs. On further increasing there appeared another cutoff value beyond which the quantized PD transformed into a game with $C \otimes C$ as an NE which is not *Pareto optimal*.

The apparent reason for these results is when the players apply their pure strategies (I and σ operators) on a maximally entangled state (Bell state) shared between them then the resulting quantum state is also one of the Bell states. This state overlaps with one of the payoff operators (10) and is orthogonal to the other three operators. Therefore the measurement of payoffs is error free. However, when the entanglement of a shared quantum state is reduced then application of pure strategies transform it into a state which overlaps with two payoff operators (10). The payoffs against the pure strategies (I and σ operators) are transformed into the payoffs corresponding to mixed strategies (linear combination of I and σ). It changes the strategic form of the game. Similarly the case of product measurements can be explained.

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