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Generalized Chaos Synchronization of Bidirectional Arrays of Discrete Systems *

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We study constructive generalized synchronization (GS) for bidirectional discrete arrays of difference systems (BDADSs). The result provides a general representation of GS in BDADSs. Numerical simulations are presented to show the effectiveness of the theoretical results.

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Generalized chaos synchronization is one of the most active research topics in nonlinear science. The generalized chaos synchronization phenomenon has the following feature: the trajectories of two different systems converge to each other with respect to a transformation starting from different initial conditions.

Since the pioneering work of Pecora and Carroll,[1] chaos synchronization has been extensively developed.[2−11] Chaos synchronization has found potential applications in secure communication,[1,3,12] chemical reactions,[9] biological science,[13] and information processing.[14] As a few examples, Lü, Yu and Chen[15] investigated the generalized synchronization (GS) of a time-varying complex network and showed that the chaos synchronization of the network is determined by means of the inner coupled link matrix. Wu and Jiao[16] showed that the dynamics of some nonsymmetric coupling networks are not always characterized by the second-largest eigenvalue of its Laplacian matrix.

Based on our previous research on vector GS,[11] and array differential GS,[17] this study establishes a constructive GS theorem for a bidirectional discrete array of difference systems (BDADS), and demonstrates some of the applications of the this theorem.

For each fixed positive integer l, let

$$\boldsymbol{X}^l = (x_{l\,i,j})_{M\times N} \tag{1}$$

be an $M \times N$ matrix. Then,

$$\begin{aligned}\boldsymbol{X} &= (\boldsymbol{X}^1, \boldsymbol{X}^2, \ldots, \boldsymbol{X}^n)^T \\ &\triangleq (x_{l\,i,j})_{n\times M\times N}\end{aligned} \tag{2}$$

is called a space matrix and $x_{l\,i,j}$ is said to be the element of the space matrix $\boldsymbol{X}$. Moreover, $R^{n\times M\times N}$ is called the space of space matrices.

Let $H : \mathbb{R}^{n\times M\times N} \longrightarrow \mathbb{R}^{n\times M\times N}$ be a transform (see Fig. 1). Then, for every $\boldsymbol{X} \in \mathbb{R}^{n\times M\times N}$,

$$\begin{aligned}H(\boldsymbol{X}) &= (h_{l\,i,j}(\boldsymbol{X})),\\ &\quad l = 1, \ldots, n,\\ &\quad i = 1, \ldots, M,\ j = 1, \ldots, N,\end{aligned} \tag{3}$$

or

$$H(\boldsymbol{X}) = (h_{l\,i,j}(\boldsymbol{X}))_{n\times M\times N}. \tag{4}$$

A BDADS consists of two parts:

$$\begin{aligned}x_{1\,i,j}(k+1) &= f_{1\,i,j}(\boldsymbol{X}(k), \boldsymbol{Y}(k)),\\ x_{2\,i,j}(k+1) &= f_{2\,i,j}(\boldsymbol{X}(k), \boldsymbol{Y}(k)),\\ &\cdots\\ x_{m\,i,j}(k+1) &= f_{m\,i,j}(\boldsymbol{X}(k), \boldsymbol{Y}(k)),\\ &\cdots\\ x_{n\,i,j}(k+1) &= f_{n\,i,j}(\boldsymbol{X}(k), \boldsymbol{Y}(k)),\end{aligned} \tag{5}$$

and

$$\begin{aligned}y_{1\,i,j}(k+1) &= g_{1\,i,j}(\boldsymbol{Y}(k), \boldsymbol{X}(k)),\\ y_{2\,i,j}(k+1) &= g_{2\,i,j}(\boldsymbol{Y}(k), \boldsymbol{X}(k)),\\ &\cdots\\ y_{m\,i,j}(k+1) &= g_{m\,i,j}(\boldsymbol{Y}(k), \boldsymbol{X}(k)),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\boldsymbol{X}(k) &= (\boldsymbol{X}^1(k), \boldsymbol{X}^2(k), \ldots, \boldsymbol{X}^n(k))^T\\ &= (x_{l\,i,j}(k))_{n\times M\times N},\end{aligned} \tag{7}$$

$$\begin{aligned}\boldsymbol{Y}(k) &= (\boldsymbol{Y}^1(k), \boldsymbol{Y}^2(k), \ldots, \boldsymbol{Y}^m(k))^T\\ &= (y_{l\,i,j}(k))_{m\times M\times N},\\ &\quad m \le n,\ i = 1, 2, \ldots, M,\ \ j = 1, 2, \ldots, N.\end{aligned} \tag{8}$$

Therefore, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are space matrices. In compact forms, BDADSs (5) and (6) can be written as

$$\boldsymbol{X}(k+1) = F(\boldsymbol{X}(k), \boldsymbol{Y}(k)), \tag{9}$$

$$\boldsymbol{Y}(k+1) = G(\boldsymbol{Y}(k), \boldsymbol{X}(k)), \tag{10}$$

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where

$$\begin{aligned} F(\boldsymbol{X}(k),\boldsymbol{Y}(k)) =& ((f_{1\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & (f_{2\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & \ldots,\\ & (f_{n\,i,j}(\boldsymbol{X}(k).\boldsymbol{Y}(k)))_{M\times N})^T, \end{aligned} \tag{11}$$

$$\begin{aligned} G(\boldsymbol{Y}(k),\boldsymbol{X}(k)) =& ((g_{1\,i,j}(\boldsymbol{Y}(k)\boldsymbol{X}(k)))_{M\times N},\\ & (g_{2\,i,j}(\boldsymbol{Y}(k),\boldsymbol{X}(k)))_{M\times N},\\ & \ldots,\\ & (g_{m\,i,j}(\boldsymbol{Y}(k)\boldsymbol{X}(k)))_{M\times N})^T. \end{aligned} \tag{12}$$

The BDADS defined by Eqs. (9) and (10) are said to be in GS with respect to a transformation $H : \mathbb{R}^{m\times M\times N} \longrightarrow \mathbb{R}^{m\times M\times N}$ (see Fig. 1), if there exists an open subset $B \subset \mathbb{R}^{n\times M\times N} \times \mathbb{R}^{m\times M\times N}$, such that for any trajectory $(\boldsymbol{X}(k),\boldsymbol{Y}(k))$ of systems (9) and (10) under the initial condition $(\boldsymbol{X}(0),\boldsymbol{Y}(0)) \in B$, the following property holds,

$$\lim_{k\to+\infty} \|H(\boldsymbol{X}_m(k)) - \boldsymbol{Y}(k)\| = 0, \tag{13}$$

where

$$\begin{aligned} \boldsymbol{X}_m(k) &= (\boldsymbol{X}^1(k),\ldots,\boldsymbol{X}^m(k))^T\\ &= (x_{l\,i,j}(k))_{m\times M\times N}, \end{aligned} \tag{14}$$

$$\begin{aligned} \boldsymbol{Y}(k) &= (\boldsymbol{Y}^1(k),\ldots,\boldsymbol{Y}^m(k))^T\\ &= (y_{l\,i,j}(k))_{m\times M\times N}. \end{aligned} \tag{15}$$

Remark 1: On a finite-dimensional vector space, all norms $\|\ \|$ are equivalent. Consequently, it is not necessary to specify the norm used in Eq. (13).

Remark 2: Condition (13) implies that under any initial condition $\boldsymbol{Y}(0) = H(\boldsymbol{X}_m(0))$, one has $H(\boldsymbol{X}_m(k)) = \boldsymbol{Y}(k)$.

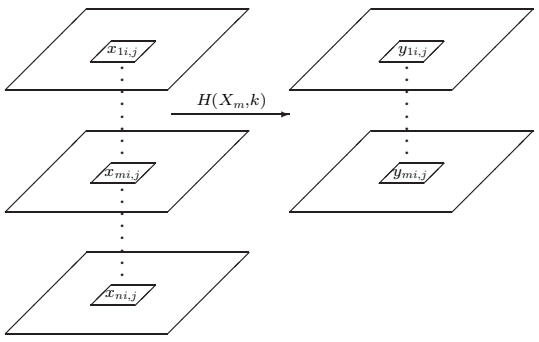

Fig. 1. Transformation H, which is a mapping that maps a subset in the space of space matrix $\mathbb{R}^{m\times M\times N}$ into the space of space matrices $\mathbb{R}^{m\times M\times N}$.

A fundamental GS problem on the BDADS defined by Eqs.,(9) and (10) is whether the BDADS is in GS with respect to an invertible transformation and what the general form of the BDADS is. The following theorem answers this question.

Let $\boldsymbol{X}(k)$, $\boldsymbol{Y}(k)$, $F(\boldsymbol{X}(k),\ \boldsymbol{Y}(k))$ and $G(\boldsymbol{Y}(k),\ \boldsymbol{X}(k))$ be defined by Eqs. (7), (8), (11) and (12), respectively. Suppose that $H : B \subset \mathbb{R}^{m\times M\times N} \longrightarrow \mathbb{R}^{m\times M\times N}$ defined by

$$H((x_{l\,i,j}(k))_{m\times M\times N}) = (y_{l\,i,j}(k))_{m\times M\times N} \tag{16}$$

is an invertible transformation. Then, the two systems (9) and (10) are in GS via the transformation $\boldsymbol{Y}(k) = H(\boldsymbol{X}(k))$, if and only if the function $G(\boldsymbol{Y}(k),\boldsymbol{X}(k))$ given in Eq. (12) has the form

$$\begin{aligned} G(\boldsymbol{Y}(k),\boldsymbol{X}(k)) =& H(F_m(\boldsymbol{X}(k)),\boldsymbol{Y}(k))\\ & - Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)), \end{aligned} \tag{17}$$

where F_m is the first m space matrices in F, that is,

$$\begin{aligned} F_m(\boldsymbol{X}(k),\boldsymbol{Y}(k)) =& ((f_{1\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & (f_{2\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & \ldots,\\ & (f_{m\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N})^T, \end{aligned} \tag{18}$$

and the function

$$\begin{aligned} Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)) =& ((q_{1\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & (q_{2\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N},\\ & \ldots,\\ & (q_{m\,i,j}(\boldsymbol{X}(k),\boldsymbol{Y}(k)))_{M\times N})^T \end{aligned}$$

guarantees that the zero solution of the error equation in the following is asymptotically stable on the open set B,

$$\begin{aligned} \boldsymbol{e}(k+1) &= H(\boldsymbol{X}_m(k+1)) - \boldsymbol{Y}(k+1)\\ &= Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)). \end{aligned} \tag{19}$$

Proof: Denote

$$G(\boldsymbol{Y}(k),\boldsymbol{X}(k)) - H(F_m(\boldsymbol{X}(k))) = -Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)). \tag{20}$$

Then, one has Eq. (19), i.e.,

$$\begin{aligned} \boldsymbol{e}(k+1) &= H(\boldsymbol{X}_m(k+1)) - \boldsymbol{Y}(k+1)\\ &= Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)). \end{aligned} \tag{21}$$

Therefore, the two dynamic systems (9) and (10) are in GS with respect to the transform H if and only if the function $Q(\boldsymbol{X}(k),\boldsymbol{Y}(k))$ makes the zero solution of the error equation (19) be asymptotically stable. This completes the proof.

Remark 3: Theorem 1 provides a general framework for constructing two GS discrete-time arrays of dynamic systems. In fact, $Q(\boldsymbol{X}(k),\boldsymbol{Y}(k))$ can be easily constructed. For example, take $Q(\boldsymbol{X}(k),\boldsymbol{Y}(k)) = \lim_{k\to\infty} \lambda\ \boldsymbol{e}(k)$, where $\lambda < 1$. Then

$$\lim_{k\to+\infty} \|\boldsymbol{e}(k)\| \le \lim_{k\to+\infty} |\lambda|^k \|\boldsymbol{e}(0\| \to 0.$$

Now, we show an application of Theorem 1 on the chaotic Lorenz equation.[18] We consider a bidirectional Lorenz equation CNN with five state variables.

First we select a continuously differentiable and invertible transformation $H : \mathbb{R}^{2\times21\times21} \to \mathbb{R}^{2\times21\times21}$ as follows. For each fixed (i, j), let

$$\begin{aligned} H(\boldsymbol{X}_2) &= [h_1(x_{1\,i,j}, x_{2\,i,j}), h_2(x_{1\,i,j}, x_{2\,i,j})]^T \\ &= [\tanh(x_{1\,i,j}) + \tanh(x_{2\,i,j}), \\ &\quad \tanh(x_{1\,i,j}) + 2\tanh(x_{2\,i,j})]^T. \end{aligned} \tag{22}$$

Here, $\tanh(x)$ is the hyperbolic tangent of the variable x. Let

$$\begin{aligned} &x_{1\,i,j}(k+1) = x_{1\,i,j}(k)x_{2\,i,j}(k) - x_{3\,i,j}(k) + py_{1\,i,j}(k), \\ &x_{2\,i,j}(k+1) = x_{1\,i,j}(k), \\ &x_{3\,i,j}(k+1) = x_{2\,i,j}(k) + D\nabla^2(x_{3\,i,j}), \\ &\quad i, j = 1, \ldots, 51, \end{aligned} \tag{23}$$

where $D = p = 0.001$ and $\nabla^2(x_{3\,i,j}) = x_{3i+1,j} + x_{3i-1,j} + x_{3i,j+1} + x_{3i,j-1} - 4x_{3i,j}$.

In a compact form, the BDADS (23) can be written as

$$\boldsymbol{X}(k+1) = F(\boldsymbol{X}(k), \boldsymbol{Y}(k)). \tag{24}$$

Let

$$\boldsymbol{Y}(k+1) = G(\boldsymbol{X}(k), \boldsymbol{Y}(k)), \tag{25}$$

where

$$\begin{aligned} G(\boldsymbol{X}, \boldsymbol{Y}) &= (G_{l\,i,j}(\boldsymbol{Y}(k), \boldsymbol{X}(k)))_{2\times21\times21} \\ &= H(F_2(\boldsymbol{X}, \boldsymbol{Y})) + (q_{l\,i,j}(\boldsymbol{Y}, \boldsymbol{X}))_{2\times51\times51}. \end{aligned} \tag{26}$$

$$q_{l\,i,j}(\boldsymbol{Y}, \boldsymbol{X}) = (h_{l\,i,j}(\boldsymbol{X}_2) - y_{l\,i,j}). \tag{27}$$

Hence, from Theorem 1, we can realize that systems (24) and (25) are in GS with respect to the H defined by Eq. (22).

Now we select the following initial conditions

$$\begin{aligned} (x_{l\,i,j}(0))_{51\times51} &= \boldsymbol{X}_0(l) + 0.002(\mathrm{rand}(51, 51) - 0.5, \\ l &= 1, 2, 3, \end{aligned} \tag{28}$$

$$\begin{aligned} (y_{l\,i,j}(0))_{51\times51} &= (x_{l\,i,j}(0))_{51\times51} \\ &\quad + 0.002(\mathrm{rand}(51, 51) - 0.5, \\ l &= 1, 2. \end{aligned} \tag{29}$$

where $\boldsymbol{X}_0 = [0.5, 0.5, -1]$, $\mathrm{rand}(51, 51)$ is a Matlab command which returns a 51×51 matrix containing pseudo-random values drawn from a uniform distribution on the unit interval.

The chaotic orbits of some components $x_{l\,i,j}$ of the state variable $\boldsymbol{X}$ of the first 2500 iterations are shown in Figs. 2(a)–2(d). In Fig. 2(a) $l = 1, 2, 3$ and $(i, j) = (11, 10)$. In Fig. 2(b), $l = 1, 2, 3$ and $(i, j) = (11, 11)$. In Fig. 2(c), $l = 1, 2, 3$ and $(i, j) = (11, 12)$. In Fig. 2(d), $l = 1, 2, 3$ and $(i, j) = (12, 11)$.

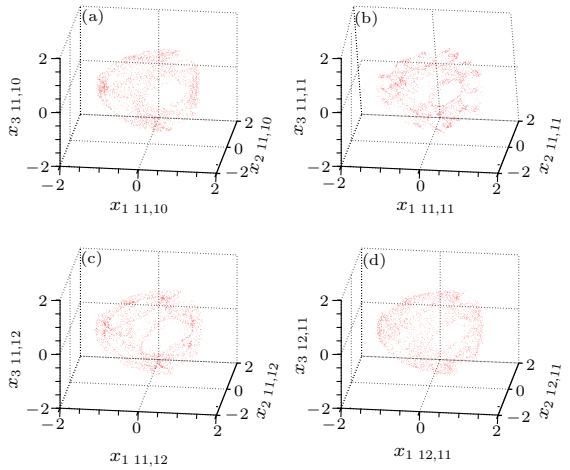

Fig. 2. The chaotic trajectories of the components of the state variables $\boldsymbol{X}(k)$: (a) $x_{1\,12,11}$, $x_{2\,12,11}$, $x_{3\,12,11}$; (b) $x_{1\,12,10}$, $x_{2\,12,10}$, $x_{3\,12,10}$; (c) $x_{1\,12,12}$, $x_{2\,12,12}$, $x_{3\,12,12}$; (d) $x_{1\,13,11}$, $x_{2\,13,11}$, $x_{3\,13,11}$.

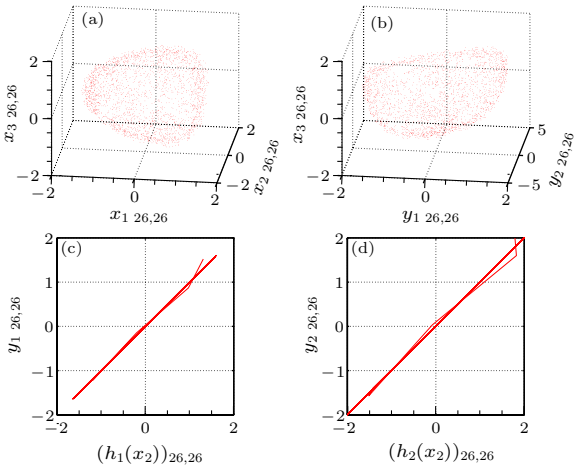

Fig. 3. (a) The trajectories of variables $x_{1\,26,26}$, $x_{2\,26,26}$ and $x_{3\,26,26}$; (b) the trajectories of variables $y_{1\,26,26}$, $y_{2\,26,26}$, and $y_{3\,26,26}$; (c) $(h_1(\boldsymbol{X}_2))_{26,26}$ and $y_{1\,26,26}$ are in synchronization; (d) $(h_2(\boldsymbol{X}_2))_{26,26}$ and $y_{2\,26,26}$ are in synchronization.

The first 2500 iterations of the chaotic orbits of variables $x_{1\,26,26}$, $x_{2\,26,26}$ and $x_{3\,26,26}$ are shown in Fig. 3(a). The first 2500 iterations of the chaotic trajectories of variables $y_{1\,26,26}, y_{2\,26,26}$, and $x_{3\,26,26}$ are shown in Fig. 3(b). It is difficult to recognize that variables $x_{1\,26,26}$, $x_{2\,26,26}$ and $y_{1\,26,26}$, $y_{2\,26,26}$ are in GS.

Figures 3(c) and 3(d) show that $\boldsymbol{X}_2$ and $\boldsymbol{Y}$ are rapidly in GS, although the initial conditions (28) and (29) have perturbations.

The evolution of chaotic orbits of the state vari-

ables $x_{1\,26,26}, x_{2\,26,26}$, $x_{3\,26,26}$, $y_{1\,26,26}, y_{2\,26,26}$ for the first 2500 iterations are shown in Fig. 4.

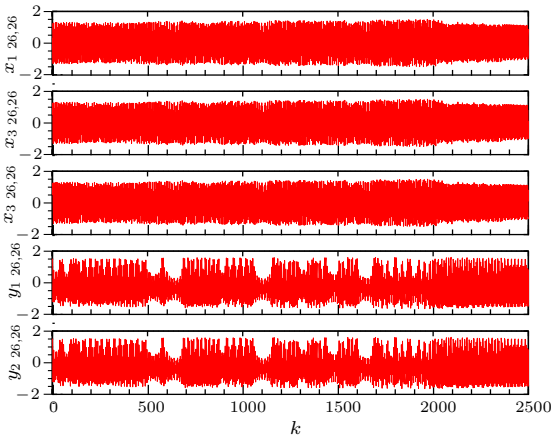

Fig. 4. The evolution of state variables: $k - x_{1\,26,26}$, $k - x_{2\,26,26}$, $k - x_{1\,26,26}$, $k - x_{3\,26,26}$, $k - y_{1\,26,26}$, and $k - y_{2\,26,26}$.

We can observe that the random perturbed initial patterns were revolting irregularly and that the GS relationships between $(x_{l\,26,26})$ and $(y_{l\,26,26}), l = 1, 2$ cannot be clearly seen.

The main result presented here is a constructive GS theorem, which describes general forms of a BDADS to be in GS via a prescribed invertible transformation $H : \mathbb{R}^{m\times M\times N} \longrightarrow \mathbb{R}^{m\times M\times N}$. The theorem gives the equivalent classes, such that any two systems in the equivalent class can be in GS.

The new approach in this study is to use the GS theorems to construct the GS of BDADS rather than using the Lyapunov exponent technique, as those reported in literature. Generally speaking, the Lyapunov exponents of BDADS are very difficult to calculate. Theorem 1 in this study generalizes our previous work on the GS for vector or array equations,[11,17,22] which provides convenient tools for constructing the GS of arrays of equation systems. Finally, some numerical simulations show the complex dynamical behaviors of the CNN GS systems. In summary, the GCS theorem may be helpful for understanding GCS phenomena of practical systems, and could provide new tools for various purposes.

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