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Numerical Computation of the Tau Approximation for the Delayed Burgers Equation

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We investigate an efficient extension of the operational Tau method for solving the delayed Burgers equation(DBE) arising in physical problems. This extension gives a useful numerical algorithm for the DBE including linear and nonlinear terms. The orthogonality of the Laguerre polynomials as the basis function is the main characteristic behind the method to decrease the volume of computations and runtime of the method. Numerical results are also presented for some experiments to demonstrate the usefulness and accuracy of the proposed algorithm.

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Delay differential equations were originally introduced as a phenomenological model of physiological diseases, physics, chemical technology, population dynamics, optimal control, circuit analysis, simulation of mechanical systems, biotechnology, economics and vibrating masses attached to an elastic bar.[1−4] The delayed Burgers equation(DBE) is very important in materials science and modeling.[5] A generalized form of the time-delayed Burgers–Fisher equation is defined as[5]

$$\tau\frac{\partial^2 u}{\partial t^2}+(1-\tau f_u)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-ru^s\frac{\partial u}{\partial x}+f(u),\quad f(u)=qu(1-u), \tag{1}$$

and is subject to the conditions

$$u(x,0)=f(x),\quad \frac{\partial u(x,0)}{\partial t}=g(x), \tag{2}$$

where τ, r and q are any real numbers and $s\in N$. If we set $\tau=q=0$ and $r=s=1$, then the classic Burger equation will be obtained.[5]

In this Letter, we consider the operational Tau method (OTM) for solving the DBE. The OTM was first presented by Ortiz and Samara[6] in 1981, for the numerical solution of nonlinear ordinary differential equations under some supplementary conditions based on the standard Tau method.[7] The OTM has been extensively applied for numerical solutions of various problems in different fields. This method is applicable to a wide class of ordinary differential equation(ODEs); partial differential equation(PDEs); integral equation(IEs) and integro-differential equation(IDEs).[6−12] The OTM involves the expansion of the solution in a suitable orthogonal basis set, and the reduction of the differential equation to a system of algebraic equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. Furthermore, the differential parts appearing in the DBE are replaced by their operational matrix representation. The orthogonality of the bases used in the proposed method reduces the computations of the resolvent algorithm. In continuation, some notations and preliminary concepts of the OTM are given. In the OTM we are seeking a polynomial solution to approximate the problem in the defined domain. A function $u(x,t)$ of two independent variables may be expanded as

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{j,i}\phi_i(x)\phi_j(t)=\boldsymbol{\Phi}^T(t)\boldsymbol{U}\boldsymbol{\Phi}(x), \tag{3}$$

where

$$\boldsymbol{U}=[u_{i,j}]_{i,j=0}^{\infty},\ \boldsymbol{\Phi}(x)=[\phi_0(x),\phi_1(x),\ldots]^T,\ \boldsymbol{\Phi}(t)=[\phi_0(t),\phi_1(t),\ldots]^T,$$

where $\{\phi_i(x)\}_{i=0}^{\infty}$ and $\{\phi_j(t)\}_{j=0}^{\infty}$ are the sets of orthogonal polynomial bases defined on one-dimensional space. In this study, we use orthogonal Laguerre polynomials. We can suppose

$$\Phi(x)=\boldsymbol{\Phi}\boldsymbol{X}_x,\quad \boldsymbol{\Phi}=[\phi_{i,j}]_{i,j=0}^{\infty},\quad \boldsymbol{X}_x=[1,x,x^2,\ldots]^T,$$
$$\Phi(t)=\boldsymbol{\Phi}\boldsymbol{X}_t,\quad \boldsymbol{\Phi}=[\phi_{i,j}]_{i,j=0}^{\infty},$$

where $\boldsymbol{X}_x$ and $\boldsymbol{X}_t$ are called the standard bases, and $\boldsymbol{\Phi}$ is called the coefficient matrix of the desired orthogonal polynomials.

Orthogonal functions have received considerable attention in dealing with various problems. Their most important property is reducing the computations and converting the problem to a system of algebraic equations. The suitable orthogonal polynomial for solving DBE (1) is the well-known Laguerre polynomials. They are defined as follows:

$$\phi_0(x)=1,\quad \phi_1(x)=1-x,\quad x\in[0,\infty),$$
$$\phi_i(x)=\left(\frac{2i-1-x}{i}\right)\phi_{i-1}(x)-\left(\frac{i-1}{i}\right)\phi_{i-2}(x),\ i\geq 2.$$

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They are orthogonal on the interval $[0,\infty)$ with respect to the weight function $w(x)=e^{-x}$ and satisfy

$$\int_0^\infty w(x)\phi_m(x)\phi_n(x)dx=\begin{cases}1, & m=n,\\ 0, & m\neq n.\end{cases}$$

Now, we consider the following lemma.

Lemma 1: Suppose that $\boldsymbol{X}_x=[1,x,x^2,\ldots]^T$, then we have

$$D^r(\boldsymbol{X}_x)=\frac{d^r}{dx^r}\boldsymbol{X}_x=\boldsymbol{M}^r\boldsymbol{X}_x,\quad r=0,1,2,\ldots,$$
$$x^s\boldsymbol{X}_x=\boldsymbol{N}^s\boldsymbol{X}_x,\quad s=0,1,2,\ldots,$$

where $\boldsymbol{M}$ and $\boldsymbol{N}$ are infinite matrices with only nonzero elements $\boldsymbol{M}_{i+1,i}=i+1$, $\boldsymbol{N}_{i,i+1}=1$, $i=0,1,2,\ldots$.

Proof: See Ref. [10].

To show applications of the OTM, we consider Eq. (1) for $q=0$ as follows:

$$\tau\frac{\partial^2u}{\partial t^2}+\frac{\partial u}{\partial t}+ru^s\frac{\partial u}{\partial x}-\frac{\partial^2u}{\partial x^2}=0,\quad x\in\mathbb{R},\ t>0,\ (4)$$

subject to the initial conditions

$$u(x,0)=f(x),\quad \frac{\partial u(x,0)}{\partial t}=g(x).$$

Now, the aim is to write Eq. (4) in an operational form.

Firstly, we intend to compute $\frac{d}{dx}(\Phi(x))$ and $\frac{d}{dt}(\Phi(t))$. We can write

$$\begin{aligned}\frac{d}{dx}(\Phi(x))&=\frac{d}{dx}(\boldsymbol{\Phi}\boldsymbol{X}_x)=\boldsymbol{\Phi}\frac{d}{dx}(\boldsymbol{X}_x)=\boldsymbol{\Phi}\boldsymbol{M}\boldsymbol{X}_x\\&=\boldsymbol{\Phi}\boldsymbol{M}\boldsymbol{\Phi}^{-1}\boldsymbol{\Phi}\boldsymbol{X}_x=\tilde{\boldsymbol{M}}\Phi(x),\\ \tilde{\boldsymbol{M}}&=\boldsymbol{\Phi}\boldsymbol{M}\boldsymbol{\Phi}^{-1}.\end{aligned}$$

In the same manner, we have

$$\begin{aligned}\frac{d}{dt}(\Phi(t))&=\frac{d}{dt}(\boldsymbol{\Phi}\boldsymbol{X}_t)=\boldsymbol{\Phi}\frac{d}{dt}(\boldsymbol{X}_t)=\boldsymbol{\Phi}\boldsymbol{M}\boldsymbol{X}_t\\&=\boldsymbol{\Phi}\boldsymbol{M}\boldsymbol{\Phi}^{-1}\boldsymbol{\Phi}\boldsymbol{X}_t=\tilde{\boldsymbol{M}}\Phi(t).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)&=\frac{\partial}{\partial t}(\Phi^T(t)\boldsymbol{U}\Phi(x))=\frac{\partial}{\partial t}(\Phi^T(t))\boldsymbol{U}\Phi(x)\\&=[\frac{\partial}{\partial t}(\Phi(t))]^T\boldsymbol{U}\Phi(x)=[\tilde{\boldsymbol{M}}\Phi(t)]^T\boldsymbol{U}\Phi(x)\\&=\Phi^T(t)\tilde{\boldsymbol{M}}^T\boldsymbol{U}\Phi(x).\quad (5)\end{aligned}$$

In the same manner, the space derivative can be obtained as

$$\frac{\partial}{\partial x}u(x,t)=\Phi^T(t)\boldsymbol{U}\tilde{\boldsymbol{M}}\Phi(x).\quad (6)$$

Hence, it is obvious that

$$\frac{\partial^2}{\partial x^2}u(x,t)=\Phi^T(t)\boldsymbol{U}\tilde{\boldsymbol{M}}^2\Phi(x).\quad (7)$$
$$\frac{\partial^2}{\partial t^2}u(x,t)=\Phi^T(t)(\tilde{\boldsymbol{M}}^T)^2\boldsymbol{U}\Phi(x).\quad (8)$$

Also, for $s=0,1,\ldots$, by implementing lemma 1, we obtain

$$\begin{aligned}t^s\Phi(t)&=t^s\Phi\boldsymbol{X}_t=\Phi t^s\boldsymbol{X}_t=\Phi\boldsymbol{N}^s\boldsymbol{X}_t\\&=\Phi\boldsymbol{N}^s\Phi^{-1}\Phi\boldsymbol{X}_t=\boldsymbol{N}_s\Phi(t),\\ \boldsymbol{N}_s&=\Phi\boldsymbol{N}^s\Phi^{-1},\end{aligned}$$

therefore, we have

$$t^s\Phi^T(t)=[t^s\Phi(t)]^T=[\boldsymbol{N}_s\Phi(t)]^T=\Phi^T(t)\boldsymbol{N}_s^T.$$

In the same way, for $r=0,1,\ldots$, we have

$$x^r\Phi(x)=\boldsymbol{N}_r\Phi(x),\quad \boldsymbol{N}_r=\Phi\boldsymbol{N}^r\Phi^{-1}.$$

Thus,

$$\begin{aligned}x^rt^su(x,t)&=(t^s\Phi^T(t))\boldsymbol{U}(x^r\Phi(x))\\&=\Phi^T(t)\boldsymbol{N}_s^T\boldsymbol{U}\boldsymbol{N}_r\Phi(x).\end{aligned}$$

Now, we would like to convert any existing nonlinear term into the operational form.

Since $u(x,t)=\Phi^T(t)\boldsymbol{U}\Phi(x)$ is polynomial, any power of $u(x,t)$ is also polynomial and can then be written based on orthogonal polynomials. Hence, for $s=0,1,\ldots$, we have

$$u^s(x,t)=\Phi^T(t)\boldsymbol{U}_s\Phi(x),\quad (9)$$

where $\boldsymbol{U}_s$ is a matrix in which its elements are a combination of elements of $\boldsymbol{U}$. Because of the space limitations of this Letter, we do not state the structure of its elements. Therefore, using Eqs. (6) and (9), we have

$$\begin{aligned}u^s(x,t)\frac{\partial}{\partial x}u(x,t)&=(\Phi^T(t)\boldsymbol{U}_s\Phi(x))(\Phi^T(t)\boldsymbol{U}\tilde{\boldsymbol{M}}\Phi(x))\\&=\Phi^T(t)\boldsymbol{K}\Phi(x),\\ \boldsymbol{K}&=\boldsymbol{U}_s\Phi(x)\Phi^T(t)\boldsymbol{U}\tilde{\boldsymbol{M}}.\end{aligned}$$

Substituting Eqs. (5)–(9) in Eq. (4), we have the algebraic system

$$\begin{aligned}&\tau\Phi^T(t)(\tilde{\boldsymbol{M}}^T)^2\boldsymbol{U}\Phi(x)+\Phi^T(t)\tilde{\boldsymbol{M}}^T\boldsymbol{U}\Phi(x)\\&\quad+r\Phi^T(t)\boldsymbol{K}\Phi(x)-\Phi^T(t)\boldsymbol{U}\tilde{\boldsymbol{M}}^2\Phi(x)=0.\end{aligned}$$

Thus, the residual $R(x,t)$ for Eq. (4) can be written as

$$\begin{aligned}R(x,t)&=\Phi^T(t)[\tau(\tilde{\boldsymbol{M}}^T)^2\boldsymbol{U}+\tilde{\boldsymbol{M}}^T\boldsymbol{U}\\&\quad+r\boldsymbol{K}-\boldsymbol{U}\tilde{\boldsymbol{M}}^2]\Phi(x)\\&=\Phi^T(t)\boldsymbol{R}\Phi(x),\end{aligned}$$

where

$$\boldsymbol{R}=\tau(\tilde{\boldsymbol{M}}^T)^2\boldsymbol{U}+\tilde{\boldsymbol{M}}^T\boldsymbol{U}+r\boldsymbol{K}-\boldsymbol{U}\tilde{\boldsymbol{M}}^2.$$

The initial conditions are also converted to

$$u(x,0) = \Phi^T(0)\boldsymbol{U}\,\Phi(x) = \boldsymbol{F}\,\Phi(x),$$
$$\boldsymbol{F} = [f_0, f_1, \ldots],$$
$$\frac{\partial u(x,0)}{\partial t} = \Phi^T(0)\tilde{\boldsymbol{M}}^T\boldsymbol{U}\,\Phi(x) = \boldsymbol{G}\,\Phi(x),$$
$$\boldsymbol{G} = [g_0, g_1, \ldots].$$

Thus the residuals $E_1(x,0)$ and $E_2(x,0)$ for the initial conditions can be written as

$$E_1(x,0) = (\Phi^T(0)\boldsymbol{U} - \boldsymbol{F})\,\Phi(x) = \boldsymbol{E}_1\,\Phi(x),$$
$$\boldsymbol{E}_1 = \Phi^T(0)\boldsymbol{U} - \boldsymbol{F}.$$
$$E_2(x,0) = (\Phi^T(0)\tilde{\boldsymbol{M}}^T\boldsymbol{U} - \boldsymbol{G})\,\Phi(x) = \boldsymbol{E}_2\,\Phi(x),$$
$$\boldsymbol{E}_2 = \Phi^T(0)\tilde{\boldsymbol{M}}^T\boldsymbol{U} - \boldsymbol{G}.$$

Now, we set the residual matrices $\boldsymbol{R} = 0$ and $\boldsymbol{E}_1 = \boldsymbol{E}_2 = 0$ or use the following inner products

$$\langle R(x,t), \phi_i(x)\phi_j(t)\rangle = \int_0^\infty\int_0^\infty R(x,t)\phi_i(x)\phi_j(t)e^{-(x+t)}dxdt = 0,$$
$$\langle E_1(x,0), \phi_i(x)\rangle = \int_0^\infty E_1(x,0)\phi_i(x)e^{-x}dx = 0,$$
$$\langle E_2(x,0), \phi_i(x)\rangle = \int_0^\infty E_2(x,0)\phi_i(x)e^{-x}dx = 0.$$

Therefore a system of algebraic equations is obtained. By solving this system the unknown matrix $\boldsymbol{U}$ is obtained. This is the so-called operational Tau method and is independent of chosen points in the domain and is applicable for both the finite and infinite domains.

Since somewhere we need finite terms of approximation, we must truncate the series in finite terms. Therefore, the following system is obtained as

$$\boldsymbol{R}_{i,j} = 0, \quad i = 0,1,\ldots,n, \quad j = 0,1,\ldots,n-2,$$
$$\boldsymbol{E}_{1i} = 0, \qquad i = 0,1,\ldots,n,$$
$$\boldsymbol{E}_{2i} = 0, \qquad i = 0,1,\ldots,n,$$

or

$$\langle R(x,t), \phi_i(x)\phi_j(t)\rangle = 0,\ i = 0,\ldots,n, j = 0,\ldots,n-2,$$
$$\langle E_1(x,0),\ \phi_i(x)\rangle = 0,\ i = 0,1,\ldots,n,$$
$$\langle E_2(x,0), \phi_i(x)\rangle = 0,\ i = 0,1,\ldots,n.$$

Hence, a system with $(n+1)^2$ unknowns and equations is obtained and must be solved to find the unknown matrix $\boldsymbol{U}$.

To show applications of the proposed method, we give two numerical experiments including nonlinear forms of the problem (4). Some comparisons are made between the results of the proposed method and the homotopy perturbation method (HPM), the Adomian decomposition method (ADM) and the variational iteration method (VIM).[5] In all the experiments the exact solution u_{exact} refers to the closed form series solution of the aforementioned methods. The computations associated with these experiments were performed in Maple 14 on a PC with a CPU of 2.4 GHz.

Experiment 1: Consider the following nonlinear DBE

$$\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + ru^2\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t > 0,$$

under the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2.$$

The exact solution of this experiment is not known. We have solved this experiment using the OTM for $n = 16$ with different r and τ and have compared it with the HPM, ADM and VIM. This comparison is given in Table 1.

Table 1. Comparison of the solutions of the OTM with the HPM, ADM and VIM for $n = 16$ with different r and τ of experiment 1.

		$\tau = 5,\ r = 5$		$\tau = 0.5,\ r = 0.5$	
t	x	u_{OTM}	u_{exact}	u_{OTM}	u_{exact}
0.2	0.25	0.261433	0.261433	0.263262	0.263262
	0.50	0.543914	0.543914	0.540363	0.540363
	0.75	0.847402	0.847402	0.835014	0.835014
	1.00	1.171307	1.171307	1.146734	1.146734
0.4	0.25	0.272128	0.272128	0.288007	0.288007
	0.50	0.575087	0.575087	0.574039	0.574036
	0.75	0.908323	0.908319	0.881168	0.881136
	1.00	1.267184	1.267156	1.206345	1.206162
0.6	0.25	0.283312	0.283312	0.326062	0.326024
	0.50	0.592277	0.592261	0.604975	0.604554
	0.75	0.925695	0.925456	0.897853	0.894971
	1.00	1.270757	1.269070	1.197909	1.184163

The obtained results in Table 1 demonstrate the approximate solutions obtained using the OTM and they are in good agreement with the exact solutions. The main idea behind the Tau method is to employ the orthogonal basis which implies a reduction in the size of computational domain and the runtime of the algorithm of the method, because orthogonal bases produce a system of equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. This advantage gives this method a wider applicability.

Experiment 2: Consider the following nonlinear DBE

$$\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + ru^2\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t > 0,$$

under the initial conditions

$$u(x,0) = -\frac{\partial u(x,0)}{\partial t} = x.$$

The exact solution of this experiment is not known. We have solved this experiment using an OTM for $n = 15$ with different r and τ and have compared it with the HPM, ADM and VIM. This comparison is given in Table 2.

Table 2. Comparison of the solutions of the OTM with the HPM, ADM and VIM for $n = 15$ with different r and τ of experiment 2.

		$\tau = 10,\ r = 10$		$\tau = 1,\ r = 1$	
t	x	u_{OTM}	u_{exact}	u_{OTM}	u_{exact}
0.2	0.25	0.199470	0.199470	0.203611	0.203611
	0.50	0.396933	0.396933	0.405419	0.405419
	0.75	0.592394	0.592394	0.605329	0.605329
	1.00	0.785870	0.785870	0.803357	0.803357
0.4	0.25	0.148509	0.148509	0.163152	0.163152
	0.50	0.290705	0.290705	0.321877	0.321878
	0.75	0.426593	0.426594	0.474882	0.474885
	1.00	0.556336	0.556341	0.622320	0.622331
0.6	0.25	0.097605	0.097605	0.125920	0.125924
	0.50	0.184185	0.184196	0.247709	0.247757
	0.75	0.259534	0.259613	0.359746	0.360011
	1.00	0.324139	0.324470	0.462494	0.463452

The results obtained in Table 2 demonstrate that the approximate solutions obtained using the OTM are again in good agreement with the solutions of the HPM, ADM and VIM for all values of t and x. The capability of the proposed method can also be seen for a nonlinear BDE. Table 2 also verifies all advantages of the method discussed in the previous experiments; that the proposed method is efficient and valid for solving nonlinear DBE. This is the main advantage of the OTM which solves nonlinear problems without any need for discretizing the variables. It should be noted that it is also quite straightforward to write computer code for the OTM.

In summary, the main idea of the proposed method is to convert the problem to an algebraic system using orthogonal Laguerre polynomials in order to decrease the computations and runtime of the algorithm of the method. The Laguerre polynomials also produce a system of equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. All the numerical results obtained using the OTM described earlier show very good agreement with the analytical methods such as the VIM, ADM and HPM. It is also quite straightforward to write computer code for this method. In addition, this method is simple and easy to use, and solves nonlinear problems without any need for discretizing the variables. These facts illustrate the OTM as a fast, reliable, valid and powerful tool for solving DBE.

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